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PALEY-WIENER TYPE THEOREMS FOR RANK-1 SEMISIMPLE LIE GROUPS

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ABSTRACT. Let G be a connected, semisimple, rank-1 Lie group with finite center. Let K be a maximal compact sub-group of G.
Using a generalization of S. Helgason's technique for the spherical case, we prove Paley-Wiener theorems for the space of left and right K-finite, compactly-supported, infinitely differentiable functions on G.

INTRODUCTION. Let G be a connected, semisimple, rank-1 Lie group with finite center. Let K be a maximal compact subgroup of G.
The purpose of this exposition is to give a (preliminary) description of the image, under the complex Fourier transform, of the space of left and right K-finite, compactly-supported, infinitely differentiable functions on G (the Paley-Wiener problem).
Historically, L. Ehrenpreis and F. Mautner [2],[3] and [4] were the first to prove Paley-Wiener theorems for G = Sl(2,R).
S.Helgason [8] proved a Paley-Wiener theorem for K-biinvariant functions on a rank-1 semisimple Lie group, later on extended by R. Gangolli [5] to K-biinvariant functions on a general-rank semisimple Lie group. Using Helgasons's technique, K. Johnson [9] proved a Paley-Wiener-type theorem for general K-types on a rank-1 group, although his conditions were difficult to check in practice.
Helgasons's technique was also used by Gupta (Thesis, University of Washington) to prove a Paley-Wiener theorem for one-dimensional K-types of SU(2,1) where the only conditions were, as expected, symmetry and growth conditions. His result was extended in Wallach [14] to the case of one dimensional K-types of SU(n,1) and its finite coverings, again with the same conditions of growth and symmetry. K. Johnson's work consists, essentially, in reducing the Paley-Wiener problem to the analysis of a certain combination of residues at the singular points in the asymptotic expansion of generalized spherical functions

* This is essentially our Ph. D. Dissertation at Rutger's University, in 1976 and we express our gratitude to professor Nolan R. Wallach, our advisor.

(see theorem 2.2.2). Our work carries on the study of that combination of residues.

In chapter 1 we introduce some needed notation and results on the asymptotic expansion of generalized spherical functions.

The first section of chapter 2 is devoted to the introduction of various Paley-Wiener spaces and in section 2 we give a proof of the already mentioned theorem 2.2.2.

In the last section of chapter 2 we state and prove the main theorem (theorem 2.3.1.) which gives a necessary and sufficient condition for a function to be the (vectorial) complex Fourier transform of an element of $C_c^\infty(G;\tau)$ (see 2.1.5. for notation). Also in the last section of chapter 2, we prove theorem 2.3.3. which gives a family of functions (namely, $\phi(\chi_{\xi,\nu}(Z_1),\ldots,\chi_{\xi,\nu}(Z_n))\cdot E_\tau$ where ϕ is an entire function, $\chi_{\xi,\nu}$ is the infinitesimal character of the principal series and $Z_1,\ldots,Z_n$ are elements of the center of universal enveloping algebra) for which the combination of residues in theorem 2.2.2. is a finite linear combination of matrix entries of the discrete series for G and hence it has zero Fourier transform (Harish-Chandra [6]).
In chapter 3 we restrict our attention to the case of M-multiplicity 1 K-types. We begin by introducing the spherical trace functions associated with the principal series representations and the various Paley-Wiener spaces associated with the complex (scalar) Fourier transform.

The main results of section 2 are theorem 3.2.1., which is the analogue of theorem 2.3.1. in the scalar case, and theorem 3.2.3. which corresponds to 2.3.3. Theorem 2.3.3. can be applied, assuming suitable growth conditions, to prove that certain functions are Fourier transform of elements of certain corresponding L^p-spaces of G (for one such application, see R. Miatello, Thesis, Rutgers University).

In section 3 we prove a Paley-Wiener theorem in the case of an M-irreducible K-type which generalizes the results of Helgason and Wallach with the same type of conditions, i.e., symmetry and growth.

In the last section of chapter 3 we give an example of a Paley-Wiener theorem for a two-dimensional K-type of SU(2,1) where, besides the growth and symmetry conditions, there is another extra condition, related to the existence of partially-defined intertwining operators between the principal series representations. Also in the last section of chapter 3 we give an example of a four-dimensional K-type of SU(2,1) which shows that in our proof of theorem 3.2.1. the derivatives in definition 3.1.5. are necessary.

The basic references throughout are Wallach [12] and Warner [15] and [16].

We should emphasize the fact that the nature of the results is preliminary since it is not clear the group theoretic significance of condition ii) of 2.1.5. and 3.1.5.

NOTE. Recently P.C. Trombi has apparently worked out the details of the application of theorem 2.3.3. mentioned above in "L^p-Harmonic Analysis on Real semisimple Lie Groups: The Split rank one case" (preprint).

1. THE ASYMPTOTIC EXPANSION OF GENERALIZED SPHERICAL FUNCTIONS.

1.1. NOTATION. Let G be a connected semisimple Lie group with finite center. Let $G = KAN$ be an Iwasawa decomposition with $\dim A = 1$ (i.e., G has split-rank 1). Let θ be the Cartan involution on G such that θ/K = identity.

We denote by M the centralizer of A in K. $\underline{g}$, $\underline{k}$, $\underline{a}$, $\underline{n}$, $\underline{m}$ will denote the respective Lie algebras of G, K, A, N and M. We will denote by $\hat{K}$ a set of representatives of equivalence classes of (finite dimensional, unitary) irreducible representations of K.

If (τ, V_τ) is a finite dimensional (not necessarily irreducible) representation of K, we set $V_\tau^M = \mathrm{Hom}_M(V_\tau, V_\tau)$ = space of endomorphisms A of V_τ such that $A \circ \tau(m) = \tau(m) \circ A$, for every m in M. If (ξ, H_ξ) is a representation of M, we write $[\tau | M:\xi] = \dim(\mathrm{Hom}_M(V_\tau, H_\xi))$.

Let $\underline{a} = R.H$ where H is such that $\mathrm{ad}(H)|\underline{n}$ has eigenvalues 1 and possibly 2. If p is the dimension of the eigenspace of eigenvalue 1, q the dimension of the eigenspace of eigenvalue 2, set $\rho = \frac{1}{2}(p+2q)$.

If $g \in G$, $g = k(g)a(g)n(g)$ with $k(g) \in K$, $n(g) \in N$, $a(g) \in A$, $a(g) = \exp(H(g).H) = a_{H(g)}$, $H(g) \in R$; and the measure dg on G is normalized so that $dg = e^{2t\rho}\, dk\, dt\, dn$.

Also, for $\nu \in C$, $(\xi, H_\xi) \in \hat{M}$, let H^ξ denote the Hilbert space completion of the space of continuous functions $f: K \longrightarrow H_\xi$ such that $f(km) = \xi(m)^{-1}(f(k))$, $k \in K$, $m \in M$, with respect to $\|f\|^2 = \int_K \|f(k)\|^2\, dk$.

If $f \in H^\xi$ let

$$(\pi_{\xi,\nu}(g)(f))(k) = \exp[(-\sqrt{-1}\nu - \rho)H(g^{-1}k)].f(k(g^{-1}.k))$$

for all $g \in G$, $k \in K$ (note that $\pi_{\xi,\nu|K}$ is independent of ν).

Then $(\pi_{\xi,\nu}, H^\xi)$ is a representation of G and if $\nu \in R$, it is unitary. These are the, so called, principal series representations of G.

1.2. GENERALIZED SPHERICAL FUNCTIONS. Let $(\xi, H_\xi) \in \hat{M}$, $\nu \in C$, and let (τ, V_τ) be a finite dimensional representation of K. Suppose $(\tau_1, V_{\tau_1}), \ldots, (\tau_n, V_{\tau_n})$ in $\hat{K}$ are its irreducible constituents.

Set $\chi_\tau(k) = \sum_{i=1}^{n} \dim(V_{\tau_i}) \overline{\mathrm{tr}(\tau_i(k))}$, all $k \in K$.

We denote by H_τ^ξ the space of functions f in H^ξ such that

$$f = E_\tau(f) = \int_K \chi_\tau(k)\, \pi_{\xi,\nu}(k)(f)\,dk$$

Note that, by Frobenius Reciprocity (c.f. Warner [15]), $\dim(H_\tau^\xi) = [\tau|M:\xi]$.

Let us denote by (γ, V_γ) the representation $(\pi_{\xi,\nu}|_K, H_\tau^\xi)$.

We are going to use the following

1.2.1. LEMMA. *Let* T_0 *be the element of* $\mathrm{Hom}_C(H_\tau^\xi, H_\tau^\xi)$ *defined by*

$$\langle T_0\phi, \psi\rangle = \langle \phi(e), \psi(e)\rangle \quad \textit{for all } \phi, \psi \textit{ in } H_\tau^\xi \textit{ and set}$$

$$T = \left(\int_M \gamma(m) T_0 \gamma(m)^{-1} dm\right) \in (H_\tau^\xi)^M$$

Then, for all $x \in G$ *we have*

$$E_\tau \pi_{\xi,\nu}(x) E_\tau = \int_K \gamma(k(xk))T\ \gamma(k)^{-1} \exp[-(\sqrt{-1}\nu+\rho)H(xk)]\,dk.$$

Proof. Let ϕ, ψ be in H_τ^ξ. Using that $k(xk)m = k(xkm)$, $H(xk) = H(xkm)$ for $m \in M$ and the invariance of the integral we see that

$$\left\langle \int_K \gamma(k(xk))T\ \gamma(k)^{-1}\exp[-(\sqrt{-1}\nu+\rho)H(xk)]\,dk(\phi), \psi\right\rangle =$$

$$= \left\langle \int_K \gamma(k(xk))T_0\ \gamma(k)^{-1}\exp[-(\sqrt{-1}\nu+\rho)H(xk)]\,dk(\phi), \psi\right\rangle =$$

$$= \int_K \exp[-(\sqrt{-1}\nu+\rho)H(xk)]\ \langle (T_0\gamma(k)^{-1})\phi,\ \gamma(k(xk))^{-1}\psi\rangle\ dk =$$

$$= \int_K \exp[-(\sqrt{-1}\nu+\rho)H(xk)]\ \langle (\gamma(k)^{-1}\phi)(e), (\gamma(k(xk))^{-1}\psi)(e)\rangle\ dk =$$

$$= \int_K \langle \phi(k), (\pi_{\xi,-\bar{\nu}}(x^{-1})(\psi))(k)\rangle\ dk = \langle \phi, \pi_{\xi,-\bar{\nu}}(x^{-1})(\psi)\rangle$$

Since $\pi_{\xi,-\bar{\nu}}(x^{-1})^* = \pi_{\xi,\nu}(x)$, the lemma is proved.

In Warner [16] it is shown that there exist $\mathrm{Hom}_C((H^\xi_\tau)^M,(H^\xi_\tau)^M)$-valued meromorphic functions c_1, c_{-1} and a function $\phi(\nu: a_t)$ for $\nu \in C$, $t > 0$ in R, with values in $\mathrm{Hom}_C((H^\xi_\tau)^M,(H^\xi_\tau)^M)$, meromorphic in ν, real analytic in t, continuous in both variables simultaneously as a map into the extended plane, such that

$$(1.2.2.)\qquad e^{\rho t}\int_K \gamma(k(a_t k))T\,\gamma(k)^{-1}\exp[(-\sqrt{-1}\nu-\rho)H(a_t k)]\,dk =$$

$$= \Phi(\nu: a_t)(c_1(\nu)(T)) + \Phi(-\nu: a_t)(c_{-1}(\nu)(T))$$

(the functions Φ and $c_{\pm 1}$ depend on ξ and τ).

Furthermore, if $\bar N = \theta(N)$, also from Warner [16], we know that

$$(1.2.3.)\qquad c_1(\nu)(T) = T\circ\left(\int_{\bar N}\gamma(k(\bar n))^{-1}\exp[-(\sqrt{-1}\nu+\rho)H(\bar n)]\,d\bar n\right) =$$

$$= T\circ c^1(\nu)$$

for $\mathrm{Im}(\nu) < 0$.

1.2.4. NOTE. It is not difficult to check that the integral on the right-hand side of 1.2.3. defines an element of $(H^\xi_\tau)^M$ when it is absolutely convergent and this is true for $\mathrm{Im}(\nu) < 0$.

Combining 1.2.1. and 1.2.2. we obtain

$$(1.2.5.)\quad e^{\rho t}E_\tau\pi_{\xi,\nu}(a_t)E_\tau = \Phi(\nu:a_t)(c_1(\nu)(T)) + \Phi(-\nu:a_t)(c_{-1}(\nu)(T)) .$$

The functions $g \longrightarrow E_\tau\pi_{\xi,\nu}(g)E_\tau$ are generalized spherical functions. In Wallach [13] (see also Schiffmann [10]) there is a proof of the fact that the matrix entries of $c^1(\nu)$ are expressible as linear combinations of functions of the form

$$B(m,(\sqrt{-1}\nu/r) + 2s-2n-m)\;B(n,(\sqrt{-1}\nu/r) + s-n)$$

where m,n,s are nonnegative integers or half integers, r=1 or 2 and B denotes the classical beta function (c.f. Whittaker and Watson [17]). Elementary properties of the beta function and a minor modification of the argument used in Warner [16] to prove lemma 9.1.7.4. shows that we can find constants $C,N,R > 0$ such that for $\mathrm{Im}(\nu) \leqslant -R$, the matrix entries of $c^1(\nu)^{-1}$ are bounded, in absolute value, by $C(1+|\nu|)^N$.

1.2.6. NOTE. From the expression of the matrix entries of $c^1(\nu)$ in terms of beta functions we obtain that $\det(c^1(\nu))$ is holomorphic for $\mathrm{Im}(\nu) < 0$ and the poles form a discrete subset of the imaginary axis. On the other hand (Harish-Chandra [7])

$$\mu_\xi(\nu)^{-1} I = c^1(\nu) c^1(\bar{\nu})^*$$

and the poles of $\mu_\xi(\nu)$ form a discrete subset of the imaginary axis (c.f. Warner [16]). Therefore, a zero of $\det(c^1(\nu))$ which is not a pole of $\mu_\xi(\nu)$ must be a pole of $\det(c^1(\bar{\nu})^*)$ and hence must also be on the imaginary axis.

From the preceding discussion we obtain a number $R > 0$ such that all the zeros of $\det(c^1(\nu))$ are on $\mathrm{Im}(\nu) \geqslant -R$. We thus conclude that for any real number α, there are finitely many singularities of $c^1(\nu)^{-1}$ on $\mathrm{Im}(\nu) \leqslant \alpha$, a fact that we are going to need.

Finally, it is not difficult to see (c.f. Wallach [12]) that if M^* denotes the normalizer of A in K and $m^* \in M^*-M$, then for $\mathrm{Im}(\nu) > 0$

$$\text{(1.2.7.)} \qquad c_{-1}(\nu)(T) = \gamma(m^*)^{-1} \circ c^1(\bar{\nu})^* \circ T \circ \gamma(m^*)$$

1.2.8. NOTE. Both sides of 1.2.3. and 1.2.7. are meromorphic functions and hence the equalities hold for the meromorphic extensions.

1.3. THE RATIONAL FUNCTIONS $\Gamma_k(\nu)$. Let $\underline{g}_C$ denote the complexification of $\underline{g}$, Δ the set of roots of $\underline{g}_C$ with respect to a Cartan subalgebra containing $\underline{a}_C$ and contained in $\underline{a}_C \oplus \underline{m}_C$, Δ^+ a set of positive roots in Δ, $\underline{g}_C^\alpha$ the root subspace of $\underline{g}_C$ corresponding to each $\alpha \in \Delta$, $B(\ ,\ )$ the Killing form of $\underline{g}_C$ and for each $\alpha \in \Delta^+$ let us choose elements $X_\alpha \in \underline{g}_C^\alpha$, $X_{-\alpha} \in \underline{g}_C^{-\alpha}$ such that $B(X_\alpha, X_{-\alpha}) = 1$. If $\underline{p}$ is the subspace of $\underline{g}$ of elements X such that $\theta(X) = -X$, write

$$X_{\pm\alpha} = Y_{\pm\alpha} + Z_{\pm\alpha}$$

where $Y_{\pm\alpha} \in \underline{k}_C$, $Z_{\pm\alpha} \in \underline{p}_C$, for all $\alpha \in \Delta^+$.

Finally, let $X_1, \ldots, X_m$ be a basis of $\underline{m}_C$, $(g_{i,j})_{1\leq i,j\leq m}$ the inverse of the matrix $(B(X_i,X_j))_{1\leq i,j\leq m}$ and set

$$w_0 = \sum_{i,j} g_{i,j} X_i X_j$$

Then in Warner [16] it is shown that (for a given finite dimensional representation (τ, V_τ) of K) if (γ, V_γ) denotes $(\pi_{\xi,\nu|K}, H_\tau^\xi)$ then

$$(1.3.1.)\qquad \Phi(\nu :\ a_t) = \exp(\sqrt{-1}\nu t)\sum_{k=0}^{\infty} e^{-kt}\,\Gamma_k(\sqrt{-1}\nu-\rho)$$

where the functions $\Gamma_k(\nu)$ are defined by the following relations:

$$(1.3.2.)\qquad \Gamma_k(\nu) \equiv 0 \quad \text{for } k \text{ a negative integer}$$

$$\Gamma_0(\nu) \equiv 1$$

and for $k > 0$ it satisfies

$$B(H,H)^{-1}k(2\nu-k+2\rho)\Gamma_k(\nu)(T)-[\gamma(w_0)\circ\Gamma_k(\nu)(T)-\Gamma_k(\nu)(T\circ\gamma(w_0))] =$$

$$= 2p\ B(H,H)^{-1}\sum_{n\geq 1}(\nu-k+2n)\ \Gamma_{k-2n}(\nu)(T) +$$

$$+ 4q\ B(H,H)^{-1}\sum_{n\geq 1}(\nu-k+4n)\ \Gamma_{k-4n}(\nu)(T) +$$

$$+ 8\sum_{n\geq 1}\sum_{\substack{\alpha\in\Delta^+\\ \alpha(H)=1}}(2n-1)\ \gamma(Y_\alpha)\circ\Gamma_{k+1-2n}(\nu)(T)\circ\gamma(Y_{-\alpha}) +$$

$$+ 8\sum_{n\geq 1}\sum_{\substack{\alpha\in\Delta^+\\ \alpha(H)=2}}(2n-1)\ \gamma(Y_\alpha)\circ\Gamma_{k+2-4n}(\nu)(T)\circ\gamma(Y_{-\alpha}) -$$

$$- 8\sum_{n\geq 1}\sum_{\substack{\alpha\in\Delta^+\\ \alpha(H)=1}} n[\gamma(Y_\alpha Y_{-\alpha})\circ\Gamma_{k-2n}(\nu)(T)+\Gamma_{k-2n}(\nu)(T)\circ\gamma(Y_\alpha Y_{-\alpha})] -$$

$$- 8\sum_{n\geq 1}\sum_{\substack{\alpha\in\Delta^+\\ \alpha(H)=2}} n[\gamma(Y_\alpha Y_{-\alpha})\circ\Gamma_{k-4n}(\nu)(T)+\Gamma_{k-4n}(\nu)(T)\circ\gamma(Y_\alpha Y_{-\alpha})]$$

for all $T \in V_\gamma^M$.

Furthermore, (see Helgason [8] or Johnson [9]) there exists a polynomial $P(\nu)$ and constants $A,B > 0$ such that for $\mathrm{Im}(\nu) \geqslant 0$ we have

$$(1.3.3.)\qquad \|P(\nu)\ \Gamma_k(\sqrt{-1}\nu-\rho)\| \leqslant A\exp(kB)\ , \quad \text{for all integers } k\ .$$

1.3.4. Note that 1.3.3. implies, in particular, that $\Phi(\nu;\ a_t)$ has finitely many singularities for $\mathrm{Im}(\nu) \geqslant 0$.
Also, it is not difficult to see that they are all on the imaginary axis.

2. THE PALEY-WIENER PROBLEM FOR RANK-1 GROUPS.

2.1. THE PALEY-WIENER SPACE. For each $f \in C_c^\infty(G)$, $\phi \in H^\xi$, let

$$F_f(ma_t)(\phi)(k) = e^{\rho t} \int_{K\times N} f(kma_t nk_1^{-1})\phi(k_1)dk_1\, dn \quad ,$$

for $m \in M$; $t \in R$, $k \in K$.

$$\pi_{\xi,\nu}(f) = \int_G f(x)\ \pi_{\xi,\nu}(x)dx$$

Then we know (c.f. Wallach [12]) that

$$\text{(2.1.1.)} \qquad \pi_{\xi,\nu}(f)(\phi)(k) = \int_M \xi(m) \int_{-\infty}^{\infty} F_f(ma_t)(\phi)(k)e^{\sqrt{-1}\nu t}\, dt\, dm$$

If $f \in C^\infty(G)$, $h \in C^\infty(K)$, we write

$$(h*_K f)(g) = \int_K h(k)f(k^{-1}g)\ dk$$

$$(f*_K h)(g) = \int_K f(gk)h(k)\ dk \quad , \quad \text{all} \quad g \in G.$$

Let τ be a finite dimensional representation of K.

2.1.2. NOTATION. If $f \in C^\infty(G)$ satisfies that $f = \overline{\chi}_\tau *_K f *_K \chi_\tau$ (see 1.2.), we write $f \in C^\infty(G;\tau)$. Then for $f \in C_c^\infty(G;\tau)$ we have that

$$\text{(2.1.3.)} \qquad \pi_{\xi,\nu}(f) = E_\tau\ \pi_{\xi,\nu}(f)E_\tau \in \mathrm{Hom}_C(H_\tau^\xi, H_\tau^\xi)$$

and the following

2.1.4. THEOREM. *Let* $f \in C_c^\infty(G;\tau)$ *and set* $F(\xi,\nu) = \pi_{\xi,\nu}(f)$. *Then for each* $\xi \in \hat{M}$, $F(\xi,\nu)$ *is an entire function of* ν *such that*

i) $F(\xi,\nu) \equiv 0$ *whenever* $[\tau|M:\ \xi] = 0$.

ii) *Let* $\xi_1,\ldots,\xi_r$ *be the* M-*types of* τ *and for each* $i = 1,\ldots,r$, *let* $\{v_{i,j}\}_j$ *be an orthonormal basis of* $H_\tau^{\xi_i}$. *Let* $\nu_1,\ldots,\nu_s$ *be in* C.

If $\sum a_{i,j,k}^{\ell,m} \left(\frac{d^m}{d\nu^m}\Big|_{\nu=\nu_\ell}\right) \langle\pi_{\xi_i,\nu}(x)(v_{i,j}),v_{i,j}\rangle = 0$ *for all* $x \in G$, *with* $a_{i,j,k}^{\ell,m}$ *in* C, *then*

$$\sum a_{i,j,k}^{\ell,m} \left(\frac{d^m}{d\nu^m}\Big|_{\nu=\nu_\ell}\right) \langle F(\xi_i,\nu)(v_{i,j}),v_{i,k}\rangle = 0\ .$$

iii) *There exists a constant* $A > 0$ *so that for any integer* $N \geq 0$ *we can find a constant* $C_N > 0$ *for which*

$$\| F(\xi,\nu)\| \leq C_N(1+|\nu|)^{-N} \exp(A|\mathrm{Im}(\nu)|).$$

Proof. i) Follows from 2.1.3. and Frobenius Reciprocity (c.f. Warner [15]).

ii) Is immediate from the definition of $\pi_{\xi,\nu}(f)$.

iii) Is a consequence of 2.1.1. and the classical Paley-Wiener theorem noting that F_f is a C_c^∞-function on A.

The Paley-Wiener problem consists in finding a converse for this theorem.

2.1.5. DEFINITION. A function F defined on $\hat{M} \times C$ such that for every $\xi \in \hat{M}$, $F(\xi, \ ): C \longrightarrow \mathrm{Hom}_C(H_\tau^\xi, H_\tau^\xi)$ is an entire function satisfying i), ii) and iii) of theorem 2.1.4. is said to be in the τ-Paley-Wiener space for G and we write $F \in P\text{-}W(\tau)$.

2.2. THE TECHNIQUE OF S.HELGASON. Let $f \in C_c^\infty(G)$. Then the Plancherel theorem says that, if f is K-finite,

$$f(g) - \sum_{\xi\in\hat{M}} \int_{-\infty}^{\infty} \mathrm{tr}(\pi_{\xi,\nu}(L_g f))\mu_\xi(\nu)d\nu$$

is a finite linear combination of matrix entries of the discrete series of G, where $\mu_\xi(\nu)d\nu$ denotes the Plancherel measure on G and $L_g(f)(x) = f(g.x)$, all g,x in G. But $\pi_{\xi,\nu}(L_g f) = \pi_{\xi,\nu}(g^{-1})\pi_{\xi,\nu}(f)$. This suggests that given $F \in P\text{-}W(\tau)$, we should study the expression

$$f(g) = \sum_{\xi\in\hat{M}} \int_{-\infty}^{\infty} \mathrm{tr}(E_\tau \pi_{\xi,\nu}(g^{-1})F(\xi,\nu)E_\tau)\mu_\xi(\nu)d\nu. \tag{2.2.1.}$$

In this direction we have the following

2.2.2. THEOREM (JOHNSON). *Let* f *be defined as above. Then* $f \in C^\infty(G;\tau)$ *and there exists* $A > 0$ *such that for* $t > A$ *we have*

$$f(a_t) = 2\pi\sqrt{-1}\ e^{-\rho t} \ . \tag{2.2.3.}$$

$$\cdot \sum_{\substack{\xi\in\hat{M}\\ [\tau|M:\xi]\neq 0}} \Big\{ \sum_{\mathrm{Im}(\nu)>0} \mathrm{Res}\ \mathrm{tr}[\Phi(\nu:a_t)(T\circ c^1(\bar{\nu})^{*-1})\tilde{F}(\xi,\nu)] -$$

$$- \sum_{\mathrm{Im}(\nu)\leq 0} \mathrm{Res}\ \mathrm{tr}[\Phi(-\nu:a_t)(\gamma(m^*)\circ c^1(\nu)^{-1}\circ T\circ\gamma(m^*)^{-1}\tilde{F}(\xi,\nu)]\Big\}$$

where $m^* \in M^*\text{-}M$ *as before, and* $\tilde{F}(\xi,\nu) = \gamma(m^*)F(\xi,\nu)\gamma(m^*)^{-1}$.

Proof: We know that (c.f. Warner [16]) $\mu_\xi(\nu)$ is analytic on an open strip of C containing R and that there exist constants C, N, $R_1 > 0$ such that if $\nu \in R$, $|\nu| \geq R_1$, then

$$|\mu_\xi(\nu)| \leq C(1+|\nu|)^N$$

Therefore, for each $g \in G$, there exists $C_1 > 0$ such that if $\nu \in R$, $|\nu| \geq R_1$

$$\|E_\tau \, \pi_{\xi,\nu}(g^{-1})F(\xi,\nu)E_\tau \, \mu_\xi(\nu)\| \leq C_1(1+|\nu|)^N \|F(\xi,\nu)\|$$

Since $F \in P\text{-}W(\tau)$ we see that f is well defined and differentiable on G.

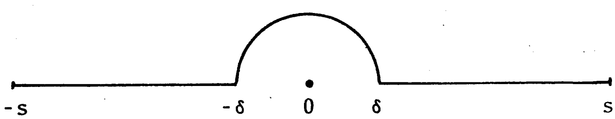
That f belongs in fact to $C^\infty(G;\tau)$ is a consequence of the transformation rules of generalized spherical functions and the orthogonality relations in K.

Since μ_ξ is analytic on an open strip of C containing R, using 1.2. and 1.3. we can find a number $\varepsilon > 0$ such that for $0 < |\mathrm{Im}(\nu)| \leq \varepsilon$ there are no singularities of the functions $\Phi(\nu;a_t)(T\circ c^1(\nu))\mu_\xi(\nu)$, $\phi(-\nu:a_t)(\gamma(m^*)c^1(\bar{\nu})^*T\gamma(m^*)^{-1})\mu_\xi(\nu)$, for $[\tau|M:\xi] \neq 0$. Now

$$f(a_t) = \int_{-\infty}^{\infty} tr(E_\tau \, \pi_{\xi,\nu}(a_t^{-1})F(\xi,\nu)E_\tau)\mu_\xi(\nu)d\nu =$$

$$= \lim_{s\to+\infty} \int_{-s}^{s} tr(E_\tau \, \pi_{\xi,\nu}(a_t^{-1})F(\xi,\nu)E_\tau)\mu_\xi(\nu)d\nu.$$

Choose a $0 < \delta < \varepsilon$ and let σ_s denote the curve below.



Then by analyticity,

$$\int_{-s}^{s} tr(E_\tau \, \pi_{\xi,\nu}(a_t^{-1})F(\xi,\nu)E_\tau)\mu_\xi(\nu)d\nu =$$

$$= \int_{\sigma_s} tr(E_\tau \, \pi_{\xi,\nu}(a_t^{-1})F(\xi,\nu)E_\tau)\mu_\xi(\nu)d\nu.$$

Now, $E_\tau \, \pi_{\xi,\nu}(a_t^{-1})E_\tau = E_\tau \, \pi_{\xi,\nu}(m^* a_t m^{*-1})E_\tau = \gamma(m^*)E_\tau \pi_{\xi,\nu}(a_t)E_\tau \gamma(m^*)^{-1} =$

$$= \gamma(m^*)\cdot\Phi(\nu:a_t)(T\circ c^1(\nu))\cdot\gamma(m^*)^{-1} +$$
$$+ \gamma(m^*)\cdot\Phi(-\nu:a_t)(\gamma(m^*)\circ c^1(\bar{\nu})^*\circ T\circ\gamma(m^*)^{-1})\cdot\gamma(m^*)^{-1}$$

and we know that (Harish-Chandra [7]) $\mu_\xi(\nu)\cdot I = (c^1(\nu)c^1(\bar{\nu})^*)^{-1}$ where I is the identity of H^ξ_τ, thus

$$\mathrm{tr}(E_\tau\, \pi_{\xi,\nu}(a_t^{-1})F(\xi,\nu)E_\tau)\mu_\xi(\nu) =$$
$$= \mathrm{tr}\{[\Phi(\nu:a_t)(T\circ c^1(\bar{\nu})^{*-1}) +$$
$$+ \Phi(-\nu:a_t)(\gamma(m^*)\circ c^1(\nu)^{-1}\circ T\circ\gamma(m^*)^{-1})]\tilde{F}(\xi,\nu)\}$$

Hence

(2.2.4.) $$\int_{\sigma_s} \mathrm{tr}(E_\tau\, \pi_{\xi,\nu}(a_t^{-1})F(\xi,\nu)E_\tau)\mu_\xi(\nu)d\nu =$$
$$= \int_{\sigma_s} \mathrm{tr}[\phi(\nu:a_t)(T\circ c^1(\bar{\nu})^{*-1})\tilde{F}(\xi,\nu)]d\nu +$$
$$+ \int_{\sigma_s} \mathrm{tr}[\Phi(-\nu:a_t)(\gamma(m^*)\circ c^1(\nu)^{-1}\circ T\circ\gamma(m^*)^{-1})F(\xi,\nu)]d\nu$$

By the results in 1.3., we can find A_1, B_1, C_1, N_1, R_1 so that the series defining $\Phi(\nu:a_t)$ is uniformly and absolutely convergent for $t > A_1$, $\mathrm{Im}(\nu) \geqslant R_1$ and

$$\|\Gamma_k(\sqrt{-1}\nu-\rho)\| \leqslant C_1(1+|\nu|)^{-N_1} e^{kB_1}, \text{ for all } k.$$

From the discussion in 1.2., we can find C_2, N_2, R_2 so that $c^1(\bar{\nu})^{*-1}$ is analytic for $\mathrm{Im}(\nu) \geqslant R_2$ and

$$\|c^1(\bar{\nu})^{*-1}\| \leqslant C_2\,(1+|\nu|)^{N_2}.$$

Let $R = \max(R_1,R_2)$. Then for $\mathrm{Im}(\nu) \geqslant R$, we can find a $C_3 > 0$ so that

$$|\mathrm{tr}[\Phi(\nu:a_t)(T\circ c^1(\bar{\nu})^{*-1})\tilde{F}(\xi,\nu)]| \leqslant$$
$$\leqslant C_3(1+|\nu|)^{N_2-N_1}\|F(\xi,\nu)\|e^{-\mathrm{Im}(\nu)t} \sum_{k=0}^{\infty}[\exp(B_1-t)]^k$$

Since $F \in P\text{-}W(\nu)$, we can find $B_2, C_4 > 0$ so that

$$\|F(\xi,\nu)\| \leqslant C_4(1+|\nu|)^{-N_2+N_1-2} e^{B_2|\operatorname{Im}(\nu)|}.$$

Let $A_2 = \max(A_1,B_1)$. Therefore, for $\operatorname{Im}(\nu) \geqslant R$, $t > A_2$, we have a $C_5 > 0$ so that

$$|\operatorname{tr}[\Phi(\nu:a_t)(T\circ c^1(\bar{\nu})^{*-1}\tilde{F}(\xi,\nu)]| \leqslant C_5(1+|\nu|)^{-2} e^{(B_2-t)\operatorname{Im}(\nu)}.$$

and hence if $A = \max(A_2,B_2)$, we can find a $C > 0$ so that for $t > A$, $\operatorname{Im}(\nu) \geqslant R$ we have

$$|\operatorname{tr}[\Phi(\nu:a_t)(T\circ c^1(\bar{\nu})^{*-1})\tilde{F}(\xi,\nu)]| < C(1+|\nu|)^{-2}.$$

Then, for $t > A$, $\operatorname{Im}(\nu) \geqslant R$, we have

$$\left| \int_{\pm s}^{\pm s+\sqrt{-1}R} \operatorname{tr}[\Phi(\nu:a_t)(T\circ c^1(\bar{\nu})^{*-1})\tilde{F}(\xi,\nu)]\,d\nu \right| \leqslant$$

$$\leqslant C\left|\int_0^R [1+(s^2+u^2)^{1/2}]^{-2}\,du\right| \xrightarrow[s\to\infty]{} 0$$

Also, by analyticity,

$$\int_{-s+\sqrt{-1}R}^{s+\sqrt{-1}R} \operatorname{tr}[\Phi(\nu:a_t)(T\circ c^1(\bar{\nu})^{*-1}\tilde{F}(\xi,\nu)]\,d\nu =$$

$$\int_{\substack{|\nu-\sqrt{-1}R|=s\\ \operatorname{Im}(\nu)\geq R}} \operatorname{tr}[\Phi(\nu:a_t)(T\circ c^1(\bar{\nu})^{*-1})\tilde{F}(\xi,\nu)]\,d\nu \quad \text{and hence}$$

$$\left|\int_{-s+\sqrt{-1}R}^{s+\sqrt{-1}R} \operatorname{tr}[\Phi(\nu:a_t)(T\circ c^1(\nu)^{*-1})F(\xi,\nu)]\,d\nu\right| \leqslant$$

$$\leqslant C\int_0^\pi s(1+[s^2\cos^2\theta+(R+s\sin\theta)^2]^{1/2})^{-2}\,d\theta \leqslant$$

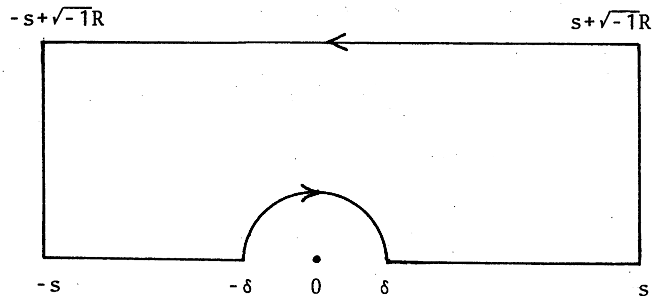
$$\leqslant \frac{C\pi}{s} \xrightarrow[s\to+\infty]{} 0$$

Similarly we can see that

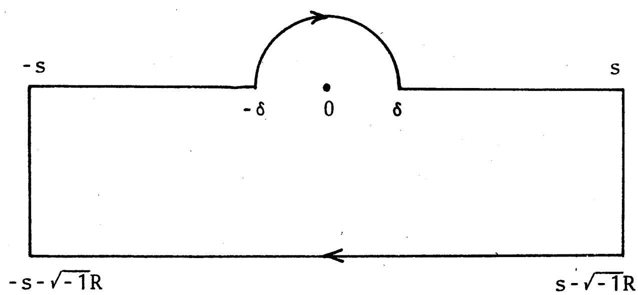
$$\lim_{s\to+\infty} \int_{\pm s}^{\pm s-\sqrt{-1}R} \operatorname{tr}[\Phi(-\nu:a_t)(\gamma(m^*)\circ c^1(\nu)^{-1}\circ T\circ\gamma(m^*)^{-1})\tilde{F}(\xi,\nu)]\,d\nu =$$

$$= \lim_{s\to+\infty} \int_{-s-\sqrt{-1}R}^{s-\sqrt{-1}R} \mathrm{tr}[\Phi(-\nu:a_t)(\gamma(m^*)\circ c^1(\nu)^{-1}\circ T\circ\gamma(m^*)^{-1})\tilde{F}(\xi,\nu)]\,d\nu = 0$$

Therefore, computing the first integral on the right-hand side of 2.2.4 along the contour



the second along the contour



and taking limits for $s \to +\infty$, the theorem follows.

2.2.5. We would have an answer to the Paley-Wiener problem if we were able to show that the residues in theorem 2.2.2. are a finite linear combination of matrix entries of the discrete series for G (Harish-Chandra [6]). In what follows we are going to deal with this question.

2.3. A PALEY-WIENER THEOREM FOR RANK-1 GROUPS.

2.3.1. THEOREM. *Let the notation be as before. Then* $F \in P\text{-}W(\tau)$ *if and only if there exists* $f \in C_c^\infty(G;\tau)$ *such that* $F(\xi,\nu) = \pi_{\xi,\nu}(f)$.

Proof: From 2.1.4. and the definitions it is clear that if $f \in C_c^\infty(G;\tau)$ then $F(\xi,\nu) = \pi_{\xi,\nu}(f)$ is in $P\text{-}W(\tau)$.

For the converse, let $\xi_1,\ldots,\xi_r$ be the M-types of τ. For each $i = 1, \ldots, r$, let $\{v_{i,j}\}_j$ be an orthonormal basis of $H_\tau^{\xi_i}$, $\nu_{i,1},\ldots,\nu_{i,s(i)}$

the poles of $\Phi(\nu:a_t)(T\circ c^1(\bar{\nu})^{*-1})$ on $Im(\nu) > 0$ and $\nu_{i,s(i)+1},\ldots,$ $\nu_{i,n(i)}$ the poles of $\Phi(-\nu:a_t)(\gamma(m^*)\circ c^1(\nu)^{-1}\circ T\circ\gamma(m^*)^{-1})$ on $Im(\nu) \leq 0$. Say N is the maximum among the orders of the poles.

Set $\phi^{\ell,m}_{i,j,k}(g) = \left(\frac{d^m}{d\nu^m}\Big|_{\nu=\nu_{i,\ell}}\right)\langle\pi_{\xi_i,\nu}(g)(v_{i,j}),v_{i,k}\rangle$ for all $g \in G$.

Let $\{\phi_1,\ldots,\phi_n\}$ be a maximal linearly independent subset of $\{\phi^{\ell,m}_{i,j,k}\}_{i,j,k,\ell,m\leq N-1}$. Say

$$\phi^{\ell,m}_{i,j,k} = \sum_{\alpha=1}^{n} a^{\ell,m,\alpha}_{i,j,k}\,\phi_\alpha$$

Let $1 \leq \alpha \leq n$ be fixed. Then there exists h_α in $C^\infty_c(G;\tau)$ such that

$$\int_G h_\alpha(x)\phi_\beta(x)\,dx = \delta_{\alpha,\beta}$$

Then $F_\alpha(\xi,\nu) = \pi_{\xi,\nu}(h_\alpha)$ is in $P\text{-}W(\tau)$. Define f_α as in 2.2.1. using F_α and then, applying 2.2.2., we find an $A > 0$ such that for $t > A$

$$f_\alpha(a_t) = 2\pi\sqrt{-1}e^{-\rho t}\sum_{\substack{\xi\in M\\ [\tau|M:\xi]\neq 0}}\{\sum_{Im(\nu)>0}\operatorname{Res}\, tr[\phi(\nu:a_t)(T\circ c^1(\bar{\nu})^{*-1})\tilde{F}_\alpha(\xi,\nu)] -$$

$$-\sum_{Im(\nu)\leq 0}\operatorname{Res}\, tr[\Phi(-\nu:a_t)(\gamma(m^*)\circ c^1(\nu)^{-1}\circ T\circ\gamma(m^*)^{-1})\tilde{F}_\alpha(\xi,\nu)]\} =$$

$$= 2\pi\sqrt{-1}\,e^{-\rho t}\sum_{i=1}^{r}\{\sum_{\ell=1}^{s(i)}\operatorname*{Res}_{\nu=\nu_{i,\ell}}\, tr[\Phi(\nu:a_t)(T\circ c^1(\bar{\nu})^{*-1})\tilde{F}_\alpha(\xi_i,\nu)] -$$

$$-\sum_{\ell=s(i)+1}^{n(i)}\operatorname*{Res}_{\nu=\nu_{i,\ell}}\, tr[\Phi(-\nu;a_t)(\gamma(m^*)\circ c^1(\nu)^{-1}\circ T\circ\gamma(m^*)^{-1})\tilde{F}_\alpha(\xi_i,\nu)]\}\ .$$

Now, for $1 \leq \ell \leq s(i)$

$$tr[\Phi(\nu:a_t)(T\circ c^1(\bar{\nu})^{*-1})\tilde{F}_\alpha(\xi,\nu)] =$$

$$= \sum_{j,k}\langle\gamma(m^*)^{-1}\circ\Phi(\nu:a_t)(T\circ c^1(\bar{\nu})^{*-1})\circ\gamma(m^*)(v_{i,k}),v_{i,j}\rangle\cdot$$

$$\cdot\langle F_\alpha(\xi_i,\nu)(v_{i,j}),v_{i,k}\rangle$$

Therefore, being F_α analytic in ν,

$$\operatorname*{Res}_{\nu=\nu_{i,\ell}}\, tr[\Phi(\nu:a_t)(T\circ c^1(\bar{\nu})^{*-1})\tilde{F}_\alpha(\xi_i,\nu)] =$$

$$= \sum_{j,k,m} u^{\ell,m}_{i,j,k}(a_t)\Big(\frac{d^m}{d\nu^m}\Big|_{\nu=\nu_{i,\ell}}\Big) \langle F_\alpha(\xi_i,\nu)(v_{i,j}),v_{i,k}\rangle \quad \text{where}$$

$u^{\ell,m}_{i,j,k}(a_t)$ is $m!$ times the $(-m-1)^{th}$-Laurent coefficient of $\langle\gamma(m^*)^{-1}\circ\Phi(\nu:a_t)(T\circ c^1(\bar\nu)^{*-1})\circ\gamma(m^*)(v_{i,k}),v_{i,j}\rangle$ at $\nu=\nu_{i,\ell}$.

But, by 2.1.4. and the assumption on h_α,

$$\Big(\frac{d^m}{d\nu^m}\Big|_{\nu=\nu_{i,\ell}}\Big)\langle F_\alpha(\xi_i,\nu)(v_{i,j}),v_{i,k}\rangle = a^{\ell,m,\alpha}_{i,j,k}$$

Therefore

$$\operatorname*{Res}_{\nu=\nu_{i,\ell}} \operatorname{tr}[\Phi(\nu:a_t)(T\circ c^1(\bar\nu)^{*-1})\tilde F_\alpha(\xi_i,\nu)] = \sum_{j,k,m} u^{\ell,m}_{i,j,k}(a_t)\; a^{\ell,m}_{i,j,k}$$

Similarly, if $s(i)+1\leq\ell\leq n(i)$,

$$-\operatorname*{Res}_{\nu=\nu_{i,\ell}} \operatorname{tr}[\Phi(-\nu:a_t)(\gamma(m^*)\circ c^1(\nu)^{-1}\circ T\circ\gamma(m^*)^{-1}\tilde F_\alpha(\xi_i,\nu)] =$$

$$= \sum_{j,k,m} u^{\ell,m}_{i,j,k}(a_t)\; a^{\ell,m,\alpha}_{i,j,k}$$

Thus, for $t > A$ we have

$$f_\alpha(a_t) = 2\pi\sqrt{-1}\; e^{-\rho t} \sum_{i,j,k,\ell,m} u^{\ell,m}_{i,j,k}(a_t)a^{\ell,m,\alpha}_{i,j,k} = Q^\tau_\alpha(a_t) \quad .$$

Applying the Plancherel theorem to h_α , we obtain by analyticity, that Q^τ_α extends to a function Q^τ_α in $C^\infty(G;\tau)$ which is a finite linear combination of matrix entries of the discrete series for G.

Suppose now we are given an F in P-W(τ). Define f as in 2.2.1., apply 2.2.2. and find a $C > 0$ such that for $t > C$,

$$f(a_t) = 2\pi\sqrt{-1}e^{-\rho t}\sum_{i=1}^{r}\Big\{\sum_{i=1}^{s(i)} \operatorname*{Res}_{\nu=\nu_{i,\ell}} \operatorname{tr}[\Phi(\nu:a_t)(T\circ c^1(\bar\nu)^{*-1})\tilde F(\xi_i,\nu)] -$$

$$-\sum_{\ell=s(i)+1}^{n(i)} \operatorname*{Res}_{\nu=\nu_{i,\ell}} \operatorname{tr}[\Phi(-\nu:a_t)(\gamma(m^*)\circ c^1(\nu)^{-1}\circ T\circ\gamma(m^*)^{-1})\tilde F(\xi_i,\nu)]\Big\} .$$

Now again

$$\operatorname*{Res}_{\nu=\nu_{i,\ell}} \operatorname{tr}[\Phi(\nu:a_t)(T\circ c^1(\bar\nu)^{*-1})\tilde F(\xi_i,\nu)] =$$

$$= \sum_{j,k,m} u^{\ell,m}_{i,j,k}(a_t)\left(\frac{d^m}{d\nu^m}\Big|_{\nu=\nu_{i,\ell}}\right) \langle F(\xi_i,\nu)(v_{i,j}),v_{i,k}\rangle$$

and since $F \in P\text{-}W(\tau)$,

$$\left(\frac{d^m}{d\nu^m}\Big|_{\nu=\nu_{i,\ell}}\right) \langle F(\xi_i,\nu)(v_{i,j}),v_{i,k}\rangle = \sum_{\alpha=1}^{n} a^{\ell,m,\alpha}_{i,j,k}\, c_\alpha(F)$$

for some $c_\alpha(F)$ in C , $\alpha = 1,\dots,n$. Hence

$$\operatorname*{Res}_{\nu=\nu_{i,\ell}} \operatorname{tr}[\Phi(\nu:a_t)(T\circ c^1(\bar{\nu})^{*-1})\tilde{F}(\xi_i,\nu)] = \sum_{\alpha=1}^{n} c_\alpha(F) \sum_{j,k,m} u^{\ell,m}_{i,j,k}(a_t)\, a^{\ell,m,\alpha}_{i,j,k}$$

Doing a similar analysis with the rest of the terms we conclude that for $t > C$,

$$f(a_t) = \sum_{\alpha=1}^{n} c_\alpha(F)\, Q^\tau_\alpha(a_t).$$

Therefore, $f - \sum_{\alpha=1}^{n} c_\alpha(F)Q^\tau_\alpha$ is in $C^\infty_c(G;\tau)$.

Finally, using that $\pi_{\xi,\nu}(Q^\tau_\alpha) = 0$ for every ν in R (Harish-Chandra [6]), it follows that $F(\xi,\nu) = \pi_{\xi,\nu}(f - \sum_{\alpha=1}^{n} c_\alpha(F)Q^\tau_\alpha)$ for every ν in R. Being both sides of the equality analytic functions, the theorem is proved.

2.3.2. OBSERVATION. From the proof of theorem 2.3.1. we see that condition ii) of the definition of $P\text{-}W(\tau)$ (2.1.5.) is only needed on a finite number of poles in the asymptotic expansion of finitely many generalized spherical functions and that the number of derivatives to consider is bounded, for each τ, by N-1, where N is the maximum of the orders of such poles.

We conclude this section with a theorem about condition ii) of 2.1.5. itself, but first some notation.

Let z be the center of the universal enveloping algebra of $\underline{g}$ and let $\{Z_1,\dots,Z_n\} \subset z$. Let $\chi_{\xi,\nu}$ denote the infinitesimal character of $(\pi_{\xi,\nu},H^\xi)$ (c.f. Warner [15]).

2.3.3. THEOREM. *Assume* $\phi: C^n \longmapsto C$ *is analytic and set* $F(\xi,\nu) = \phi(\chi_{\xi,\nu}(Z_1),\dots,\chi_{\xi,\nu}(Z_n))E_\tau$. *Then* F *satisfies* ii) *of* 2.1.5. *and the right-hand side of* 2.2.3. *extends to an element of* $C^\infty(G;\tau)$ *which is a finite linear combination of matrix entries of the discrete series for* G.

In particular, if G *has no discrete series, it is zero.*

Proof. Let $\phi(x_1,\ldots,x_n) = \Sigma a_{i_1,\ldots,i_n} x_1^{i_1}\ldots x_n^{i_n}$ and assume we are given (as in 2.1.5. ii)) a relation of the form

$$(2.3.4.)\quad \sum b^{\ell,m}_{i,j,k}\left(\frac{d^m}{d\nu^m}\Big|_{\nu=\nu_\ell}\right)<\pi_{\xi_i,\nu}(g)(v_{i,j}),v_{i,k}> = 0$$

for all $g \in G$, with $b^{\ell,m}_{i,j,k}$ in C.

For each $\alpha = 1,\ldots,n$, let Z_α act on both sides of 2.3.4. as a left-invariant differential operator of the g-variable.

Since $Z_\alpha<\pi_{\xi_i,\nu}(g)(v_{i,j}),v_{i,k}> =$

$= \chi_{\xi_i,\nu}(Z_\alpha)<\pi_{\xi_i,\nu}(g)(v_{i,j}),v_{i,k}>$ we see that

$$0 = \sum b^{\ell,m}_{i,j,k}\left(\frac{d^m}{d\nu^m}\Big|_{\nu=\nu_\ell}\right) a_{i_1,\ldots,i_n}\chi_{\xi_i,\nu}(Z_1)^{i_1}\ldots\chi_{\xi_i,\nu}(Z_n)^{i_n}\;.$$

$$\cdot <\pi_{\xi_i,\nu}(g)(v_{i,j}),v_{i,k}>$$

for all $g \in G$. Therefore, evaluating at $g = e$ and summing on $i_1,\ldots,i_n$ we obtain

$$0 = \sum b^{\ell,m}_{i,j,k}\left(\frac{d^m}{d\nu^m}\Big|_{\nu=\nu_\ell}\right)\phi(\chi_{\xi_i,\nu}(Z_1),\ldots,\chi_{\xi_i,\nu}(Z_n))$$

and hence F satisfies ii) of 2.1.5.

Going through the argument in the proof of 2.3.1., we conclude that the sum of the residues on the right-hand side of 2.2.3. extends to an element of $C^\infty(G;\tau)$ which is a finite linear combination of matrix entries of the discrete series for G.

Therefore, if G has no discrete series, the sum of the residues is zero.

3. THE CASE OF M-MULTIPLICITY 1 K-TYPES.

3.1. THE PALEY-WIENER SPACE. In what follows we assume $(\tau,V_\tau) \in \hat{K}$ and it satisfies $[\tau|M:\xi] = 0$ or 1 for every $(\xi,H_\xi) \in \hat{M}$ (this is the case for every $(\tau,V_\tau) \in \hat{K}$ if $G = SO(n,1)$ or $SU(n,1)$, c.f. Warner [15]).

For $f \in C_c^\infty(G)$ set

$$\Theta_{\xi,\nu}(f) = tr(\pi_{\xi,\nu}(f))$$

(i.e., $\Theta_{\xi,\nu}$ is the character of $(\pi_{\xi,\nu}, H^\xi)$).

If $\phi^\tau_{\xi,\nu}(x) = tr(E_\tau\ \pi_{\xi,\nu}(x)E_\tau)$ for all $x \in G$, then for every f in $C_c^\infty(G;\tau)$,

$$(3.1.1.) \qquad \Theta_{\xi,\nu}(f) = \int_G f(x)\ \phi^\tau_{\xi,\nu}(x)dx$$

(c.f. Wallach [12]).

3.1.2. NOTATION. If $f \in C_c^\infty(G;\tau)$ satisfies that $f(g) = f(kgk^{-1})$ for all $g \in G, k \in K$ we write $f \in I_c^\infty(G;\tau)$.
Note that if $f \in C_c^\infty(G;\tau)$ and we set $f_K(g) = \int_K f(kgk^{-1})dk$, then $f_K \in I_c^\infty(G;\tau)$ and $\Theta_{\xi,\nu}(f_K) = \Theta_{\xi,\nu}(f)$.

3.1.3. LEMMA. *Let* $f \in I_c^\infty(G;\tau)$. *Then*

i) $d(\tau)\ \pi_{\xi,\nu}(f) = \Theta_{\xi,\nu}(f)E_\tau$

ii) $d(\tau)\ \Theta_{\xi,\nu}(L_g f) = \Theta_{\xi,\nu}(f)\ \phi^\tau_{\xi,\nu}(g^{-1})$

(here $d(\tau) = \dim V_\tau$).

Proof: It is easy to check that $\pi_{\xi,\nu}(f)$ is in $Hom_K(H^\xi_\tau, H^\xi_\tau)$ and hence, by irreducibility of $(\pi_{\xi,\nu|K}, H^\xi_\tau)$ we conclude i).
Now, $d(\tau)\ \Theta_{\xi,\nu}(L_g f) = d(\tau)\ tr(\pi_{\xi,\nu}(g^{-1})\pi_{\xi,\nu}(f)) =$
$= d(\tau)\ tr(E_\tau\ \pi_{\xi,\nu}(g^{-1})E_\tau\ \pi_{\xi,\nu}(f))$. Therefore, ii) follows from i).

As a direct consequence of 2.1.4. we have

3.1.4. THEOREM. *Let* $f \in I_c^\infty(G;\tau)$ *and set* $F(\xi,\nu) = \Theta_{\xi,\nu}(f)$. *Then* $F: \hat{M} \times C \longmapsto C$ *is an entire function of* ν *such that*

i) $F(\xi,\nu) \equiv 0$ *whenever* $[\tau|M: \xi] = 0$.

ii) *Let* $\xi_1,\ldots,\xi_r$ *be the* M-*types of* τ *and* $\nu_1,\ldots,\nu_s$ *complex numbers.*
If $\sum a_{i,j,k} \left(\frac{d^k}{d\nu^k}\Big|_{\nu=\nu_j}\right) \phi^\tau_{\xi_i,\nu}(x) = 0$ *for all* x *in* G, *with* $a_{i,j,k}$

in C, *then*

$$\sum a_{i,j,k}\left(\frac{d^k}{d\nu^k}\Big|_{\nu=\nu_j}\right)(F(\xi_i,\nu)) = 0.$$

iii) *There exists a constant* $A > 0$ *so that for any integer* $N \geqslant 0$ *we can find a constant* $C_N > 0$ *for which*

$$|F(\xi,\nu)| \leqslant C_N(1+|\nu|)^{-N}\exp(A|\mathrm{Im}(\nu)|).$$

The preceding lemma and theorem motivate, in this case, the following

3.1.5. DEFINITION. Given $F\colon \hat{M}\times C \longmapsto C$ such that F is an entire function of the second variable satisfying conditions i), ii) and iii) of theorem 3.1.4., we say F is in the τ-Paley-Wiener space of G and write $F \in P\text{-}W(\tau)$ (see 2.1.5.).

3.2. A PALEY-WIENER THEOREM. Let the notation and assumptions be as in 3.1.. We can now prove the following

3.2.1. THEOREM. *Given* $F\colon \hat{M}\times C \longmapsto C$; $F \in P\text{-}W(\tau)$ *if and only if there exists* $f \in I_c^\infty(G;\tau)$ *such that* $F(\xi,\nu) = \Theta_{\xi,\nu}(f)$.

Proof. One implication is clear. So, suppose $F \in P\text{-}W(\tau)$ is given. Define

$$f'(g) = \sum_{\substack{\xi\in M\\ [\tau|M:\xi]=1}} d(\tau)^{-1}\int_{-\infty}^{\infty}\phi^{\tau}_{\xi,\nu}(g^{-1})F(\xi,\nu)\mu_\xi(\nu)d\nu$$

An argument similar to the one in 2.2.2. shows that $f' \in I^\infty(G;\tau)$ and there exists $A > 0$ such that for $t > A$

$$f'(a_t) = 2\pi\sqrt{-1}\, e^{-\rho t}\,. \tag{3.2.2.}$$

$$\cdot \sum_{\substack{\xi\in \hat{M}\\ [\tau|M:\xi]\neq 0}}\Big\{\sum_{\mathrm{Im}(\nu)>0}\mathrm{Res}\,(F(\xi,\nu)\ \mathrm{tr}[\Phi(\nu:a_t)(T\circ c^1(\bar{\nu})^{*-1})]) -$$

$$-\sum_{\mathrm{Im}(\nu)\leq 0}\mathrm{Res}\,(F(\xi,\nu)\ \mathrm{tr}[\Phi(-\nu:a_t)(\gamma(m^*)\circ c^1(\nu)^{-1}\circ T\circ\gamma(m^*)^{-1})])\Big\}.$$

Proceeding now as in the proof of 2.3.1., we can show that the right-hand side of 3.2.2. extends to an element of $I^\infty(G;\tau)$ which is a finite linear combination of matrix entries of the discrete series for G and from there that $\exists\, f \in I_c^\infty(G;\tau)/\Theta_{\xi,\nu}(f) = F(\xi,\nu)$.

As in 2.3.3. with the same notation, we have now the following

3.2.3. THEOREM. *Let* $\phi: C^n \longmapsto C$ *be analytic and set* $F(\xi,\nu) = \phi(\chi_{\xi,\nu}(Z_1),\ldots,\chi_{\xi,\nu}(Z_n))$. *Then* F *satisfies* ii) *of* 3.1.5. *and the right-hand side of* 3.2.2. *extends to an element of* $I^\infty(G;\tau)$ *which is a finite linear combination of matrix entries of the discrete series of* G.

Proof. Follows from 2.3.3.

3.2.4. NOTE. This result can be used to prove that certain functions are Fourier transforms of elements in the Schwartz spaces. For one such application see R. Miatello, Thesis, Rutgers University.

3.3. THE CASE $\tau|M$ IRREDUCIBLE. In what follows, assume $(\tau,V_\tau) \in \hat{K}$ is such that $(\tau|M,V_\tau)$ is irreducible. Set $(\tau|M,V_\tau) = (\xi,H_\xi)$.

We can now prove the following

3.3.1. THEOREM. *Let* $F: C \longmapsto C$ *be entire. Then there exists* $f \in I_c^\infty(G;\tau)$ *such that* $\Theta_{\xi,\nu}(f) = F(\nu)$ *if and only if* F *satisfies*

i) $F(\nu) = F(-\nu)$.

ii) *There exists a constant* $A > 0$ *so that for any integer* $N \geqslant 0$ *we can find a constant* $C_N > 0$ *for which*

$$|F(\nu)| \leqslant C_N(1+|\nu|)^{-N} \exp(A|\mathrm{Im}(\nu)|).$$

Proof. Assume $f \in I_c^\infty(G;\tau)$ and $F(\nu) = \Theta_{\xi,\nu}(f)$. Then ii) is immediate from iii) of 3.1.4. Also, from 2.1.1., it is not difficult to see that $\Theta_{\xi,\nu} = \Theta_{\xi,-\nu}$ (c.f. Wallach [12]) and hence i) follows.

For the converse, let Ω denote the Casimir element of $\underline{g}$. Then $\chi_{\xi,\nu}(\Omega) = -\nu^2-\lambda_\xi-\rho^2$ (c.f. Wallach [12],p. 280).

If F is given as above, then

$$F(\nu) = \sum_{n=0}^{\infty} a_n \nu^{2n}.$$

Set $\phi(z) = \sum_{n=0}^{\infty} (-1)^n a_n(z+\lambda_\xi+\rho^2)^n$.

Therefore, ϕ is an entire function and $F(\nu) = \phi(-\nu^2-\lambda_\xi-\rho^2) = \phi(\chi_{\xi,\nu}(\Omega))$.

Using 3.2.2. we see that F satisfies i), ii) and iii) of 3.1.5. and hence 3.3.1. follows from 3.2.1.

3.3.2. NOTE. This result was proved in the case $G = SU(2,1)$, $\dim(\tau) = 1$, by Gupta (Thesis, University of Washington). For a proof in the case $G = SU(n,1)$, $\dim(\tau) = 1$, see Wallach [14].

3.3.3. NOTE. In the case of G = Spin (4,1), we can take $K = SU(2)\times SU(2)$ and $M = \{(A,A) \mid A \in SU(2)\} \subset K$. Then every $\tau \in \hat{K}$ of the form $\tau = \gamma \otimes 1$ or $\tau = 1 \otimes \gamma$, where $\gamma \in \hat{SU}(2)$ and 1 is the trivial representation of $SU(2)$, restricts to M irreducibly and hence 3.3.1. applies.

Therefore, 3.3.1. includes results in Shimuzu [11].

3.4. EXAMPLES IN THE CASE OF $G = SU(2,1)$. Let us now consider $G = SU(2,1)$ = group of 3x3 complex matrices A, with determinant 1, such that $JA^*J = A^{-1}$, where

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} .$$

On G, we consider the Iwasawa decomposition given by

$$K = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & \det(A)^{-1} \end{array} \right) \;\middle|\; A \in U(2) \right\}$$

$$A = \left\{ a_t = \exp(tH) \;\middle|\; H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} ,\ t \in \mathbf{R} \right\}$$

and N is the connected Lie subgroup of G with Lie algebra

$$\underline{n} = \left\{ \begin{pmatrix} -\sqrt{-1}u & \bar{z} & \sqrt{-1}u \\ -z & 0 & z \\ -\sqrt{-1}u & \bar{z} & \sqrt{-1}u \end{pmatrix} \;\middle|\; u \in R ,\ z \in C \right\}$$

Then

$$M = \left\{ m(\theta) = \begin{pmatrix} e^{\sqrt{-1}\theta} & 0 & 0 \\ 0 & e^{-2\sqrt{-1}\theta} & \\ 0 & 0 & e^{\sqrt{-1}\theta} \end{pmatrix} \;\middle|\; \theta \in R \right\}$$

and the Cartan involution is given by $\theta(g) = (g^*)^{-1}$, for every $g \in G$. Correspondingly, $\underline{g} = su(2,1)$ = space of 3x3 complex matrices A such that the trace of A is zero and $JA^*J = -A$,

$$\underline{k} = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & -\mathrm{tr}(A) \end{array} \right) \;\middle|\; A \in u(2) \right\}$$

and $\underline{g}_C = sl(3,C)$.

If we take $X_0 = \begin{bmatrix} \sqrt{-1} & 0 & 0 \\ 0 & -2\sqrt{-1} & \\ 0 & 0 & \sqrt{-1} \end{bmatrix}$, noting that $B(X,Y) =$

$= 6\,\mathrm{Re}(\mathrm{tr}(XY))$, we have $\omega_0 = (-1/36)X_0^2$.
We also note that $\rho = 2$.
Let V_u = space of polynomials in two complex variables, homogeneous of degree u. For each integer ℓ, let $\tau_{\ell,u}$ be the representation of K on V_u given by

$$\tau_{\ell,u}\left(\left(\begin{array}{c|c} A & 0 \\ \hline 0 & \det(A)^{-1} \end{array} \right)\right)(f)(x) = \det(A)^{\ell}.f(A^{-1}.x)$$

for every $f \in V_u$, $x \in C^2$.

Then $\{(\tau_{\ell,u},V_u)\}_{u\geq 0,\ell\in Z} = \hat{K}$ (c.f. Wallach [12]).

If we set $v_i(z_1,z_2) = z_1^{u-i}\, z_2^i$, then $\{v_0,\ldots,v_u\}$ is a basis of V_u such that if $A \in (V_u)^M_{\tau_{\ell,u}}$, then $A(v_i) = a_i\, v_i$, $a_i \in C$, $0 \leq i \leq u$.

Hence, if $P_i(v_j) = \delta_{i,j}\, v_j$, $\{P_i\}_{i=0}^u$ is a basis of $(V_u)^M_{\tau_{\ell,u}}$.

Let $\tau = \tau_{\ell,u}$. An easy computation shows that $\tau(\omega_0)(v_i) =$
$= [(3i-\ell-u)/6]^2.v_i$, $0 \leq i \leq u$, and hence, from 1.3.2. we see that for each $i = 0,\ldots,u$ the singularities of $\Phi(\nu:a_t)(P_i)$ are at the points

$$\nu = -\sqrt{-1}\,\left[\frac{k}{2} - \frac{(i-j)[3(i+j)-2(u+\ell)]}{2k}\right]$$

for $k = 1,2,\ldots$, $j = 0,\ldots,u$.

EXAMPLE 1. Let $\tau = \tau_{1,1}$, i.e., τ is the two-dimensional representation of K such that

$$\tau(m(\theta))(v_0) = e^{-2\sqrt{-1}\theta}.v_0 = \xi_0(m(\theta)).v_0$$

$$\tau(m(\theta))(v_1) = e^{\sqrt{-1}\theta}.v_1 = \xi_1(m(\theta)).v_1 .$$

Then the only singularity of $\Phi(\nu:a_t)(P_0)$ on $\mathrm{Im}(\nu) \geq 0$ is a simple pole at $\nu = 0$ and $\Phi(\nu:a_t)(P_1)$ is analytic on $\mathrm{Im}(\nu) \geq 0$.

Using an argument as in Wallach [12], 9.11.9., (see also Cohn [1]), we obtain

$$c^1(\nu)(v_0) = c_0^1(\nu).v_0 = c.\frac{\Gamma((\sqrt{-1}\nu+1)/2)}{\Gamma((\sqrt{-1}\nu+4)/2)} . v_0$$

$$c^1(\nu)(v_1) = c_1^1(\nu)\cdot v_1 = c\left(\frac{1-\sqrt{-1}\ \nu}{1+\sqrt{-1}\ \nu}\right) \frac{\Gamma(\sqrt{-1}\nu/2)}{\Gamma((\sqrt{-1}\nu+3)/2)} \cdot v_1$$

where c is a real constant and Γ is the classical gamma function (c.f. Whittaker and Watson [17]).

Therefore $c^1(\bar{\nu})^* = c^1(-\nu)$, the only singularity of $c_1^1(\nu)^{-1}$ on $\mathrm{Im}(\nu) \leq 0$ is a simple pole at $\nu = -\sqrt{-1}$ and $c_0^1(\nu)^{-1}$ is analytic on $\mathrm{Im}(\nu) \leq 0$.

3.4.1. NOTE. The singularity of $\Phi(\nu:a_t)(P_0)$ at $\nu = 0$ shows why we have to avoid 0 in the proof of 2.2.2.

In this example, since the singularities appearing on the right-hand side of 3.2.2. are simple poles (see 2.3.2.) and spherical trace functions are linearly independent or equal (c.f. Wallach [12]), condition ii) of 3.1.5. reduces to

ii') $\phi^\tau_{\xi,\nu} = \phi^\tau_{\xi,\nu'}$ implies $F(\xi,\nu) = F(\xi',\nu')$.

From the results in Wallach [14], we can see that

$$\phi^\tau_{\xi,\nu} = \phi^\tau_{\xi',\nu'}$$

if and only if $\xi = \xi'$ and $\nu = \pm\nu'$ or $(\xi,\nu) = (\xi_0,0)$, $(\xi',\nu') =$ $= (\xi_1,\sqrt{-1})$.

Therefore, we have the following

3.4.2. THEOREM. *Let* $F: \hat{M} \times \mathbb{C} \longmapsto \mathbb{C}$ *be an entire function of the second variable. Then there exists* $f \in I_c^\infty(G;\tau)$ *with* $\Theta_{\xi,\nu}(f) = F(\xi,\nu)$ *if and only if*

i) $F(\xi,\nu) \equiv 0$ *for* $\xi \neq \xi_i$, $i = 0,1$.

ii) $F(\xi,\nu) = F(\xi,-\nu)$, $F(\xi_0,0) = F(\xi_1,\sqrt{-1})$.

iii) *There exists a constant* $A > 0$ *so that for any* $N \geq 0$ *we can find* $C_N > 0$ *for which*

$$|F(\xi,\nu)| \leq C_N(1+|\nu|)^{-N} \exp(A|\mathrm{Im}(\nu)|) .$$

EXAMPLE 2. Let $\tau = \tau_{6,3}$. In this case it can be seen that

$$c_0^1(\nu) = \langle c^1(\nu)(v_0), v_0\rangle = \left[\prod_{j=1}^{3}\left(\frac{-9+2j-\sqrt{-1}\nu}{-9+2j+\sqrt{-1}\nu}\right)\right] \cdot \frac{\Gamma(\sqrt{-1}\nu/2)\Gamma((1+\sqrt{-1}\nu)/2)}{\Gamma((11+\sqrt{-1}\nu)/2)\Gamma((-7+\sqrt{-1}\nu)/2)}$$

Therefore, $\nu = -\sqrt{-1}$ is a pole of $c_0^1(\nu)^{-1}$ which is also a pole of $\Phi(-\nu:a_t)(P_0)$.

This example shows that multiple poles do appear on the right-hand side of 3.2.2. and hence the necessity of the derivatives in condition ii) of 3.1.5. (and also in ii) of 2.1.5.) for our proof of theorem 3.2.1.

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