Revista de la
Unión Matemática Argentina
Volumen 29, 1984.

SOME PROXIMITY RELATIONS IN A PROBABILISTIC METRIC SPACE(*)

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Dedicated to Professor Luis A. Santaló

0. INTRODUCTION.

Proximities in a probabilistic metric space have been studied previously by R. Fritsche [3], Gh. Constantin and V. Radu [2] and A.Leonte [4]. In this paper we introduce, using some results concerning order and weak convergences [1], a family of semi-proximities $\{\delta_\varphi;\ \varphi \in \Delta^+\}$ analyzing when they are Efremovič-proximities and relating the induced closure operators $\{C_{\delta_\varphi};\ \varphi \in \Delta^+\}$ to those of R. Tardiff [8] and B. Schweizer [7]. In the last section we exhibit a uniform topology where the neighborhood of a point p is precisely the closure of {p} in the topology generated by C_{δ_φ}.

1. PRELIMINARIES.

Let Δ^+ be the set of all one-dimensional positive distribution functions, i.e., let

$\Delta^+ = \{F: R \to [0,1];\ F(0) = 0,\ F$ is non-decreasing and left-continuous$\}$.

Δ^+ has a partial order, namely, $F \geqslant G$ iff $F(x) \geqslant G(x)$, for every x.

$(\Delta^+, \leqslant)$ is a complete lattice with minimum element $\varepsilon_\infty(x) = 0$, for every x, and maximum element the step function given by

$$\varepsilon_0(x) = \begin{cases} 0\ , & \text{for } x \leqslant 0\ , \\ 1\ , & \text{for } x > 0\ . \end{cases} \tag{1.1}$$

It is well known that weak convergence $(\text{w-}\lim_{n\to\infty} F_n)$ in Δ^+ is metrizable by the modified Lévy metric $\mathcal{L}$ introduced by Sibley [6].

(*) Presented at the INTERNATIONAL CONGRESS OF MATHEMATICIANS, Helsinki, Finland 1978.

DEFINITION 1.1. A *triangle function* is a two-place function τ from $\Delta^+ \times \Delta^+$ into Δ^+ such that, for all F, G and H in Δ^+,

i) $\tau(F,\varepsilon_0) = F$,

ii) $\tau(F,G) \geqslant \tau(F,H)$ whenever $G \geqslant H$,

iii) $\tau(F,\tau(G,H)) = \tau(\tau(F,G),H)$,

iv) $\tau(F,G) = \tau(G,F)$.

A triangle function τ is *continuous* if it is a continuous function from $\Delta^+ \times \Delta^+$ into Δ^+, where Δ^+ is indowed with the $\mathcal{L}$-metric topology and $\Delta^+ \times \Delta^+$ with the product topology. For a complete study of the fundamental topological semigroups (Δ^+,τ) see [6].

DEFINITION 1.2. A *probabilistic metric space* (briefly, a PM-space) is an ordered pair $(S,\mathcal{F})$, where S is a set, and $\mathcal{F}$ is a mapping from $S \times S$ into Δ^+ such that for all $p,q,r \in S$:

I) $\mathcal{F}(p,q) = \varepsilon_0$ iff $p=q$,

II) $\mathcal{F}(p,q) = \mathcal{F}(q,p)$,

III) $\tau(\mathcal{F}(p,q),\mathcal{F}(q,r)) \leqslant \mathcal{F}(p,r)$.

If $\mathcal{F}$ satisfies just (I) and (II) we say that $(S,\mathcal{F})$ is a *semi-PM space*. The function $\mathcal{F}(p,q)$ is denoted by F_{pq}, and $F_{pq}(x)$, for $x > 0$, is interpreted as the probability that the distance between p and q is less than x.

We collect some definitions about proximities which will be used in the sequel. For a complete survey of proximities see [5].

DEFINITION 1.3. Let X be a set and δ a binary relation on $P(X)$, the power set of X. δ is a *semi-proximity* if satisfies, for A, B and C subsets of X, the following conditions:

1) $\emptyset \not\delta A$,

2) If $A \cap B \neq \emptyset$ then $A \delta B$,

3) $A \delta B$ implies $B \delta A$,

4) $A \delta (B \cup C)$, if and only if $A \delta B$ or $A \delta C$.

A semi-proximity δ is called an *Efremovič proximity* if verifies the additional axiom:

5) $A \not\delta B$ implies there exists $E \subset X$ such that $E \not\delta B$ and $(X-E) \not\delta A$.

A semi-proximity δ is said to be *separated* if

6) $a \delta b$ implies $a=b$.

Any semi-proximity δ induces a mapping C_δ from $P(X)$ into itself defined by $C_\delta(A) = \{x \in X;\ x \delta A\}$. C_δ satisfies the conditions

a) $C_\delta(\emptyset) = \emptyset$,

b) $C_\delta(A) \supset A$, for every $A \in P(X)$,

c) $C_\delta(A \cup B) = C_\delta(A) \cup C_\delta(B)$ for all $A,B \in P(X)$,

i.e., C_δ is a Cech closure operator which is a Kuratowski closure ($C_\delta(C_\delta(A)) = C_\delta(A)$ for every $A \in P(X)$) whenever δ is an Efremovič proximity. So δ provides a topology on X called the *topology induced* by δ. The topological spaces whose topologies can be derived in this way from proximities are called *proximizable*.

Finally, we summarize some definitions and theorems about order and weak convergences (see [1]).

The supremums and infimums of two functions $F,G \in \Delta^+$, in the lattice $(\Delta^+,\leqslant)$ will be denoted, respectively, by $F \vee G$ and $F \wedge G$.

DEFINITION 1.4. (a) A non-decreasing (resp., non-increasing) sequence (G_n) in Δ^+ is *order* convergent to $G \in \Delta^+$, if and only if $G = \bigvee_{n=1}^{\infty} G_n$ (resp., $G = \bigwedge_{n=1}^{\infty} G_n$). (b) A sequence (F_n) in Δ^+ is *order* convergent to $F \in \Delta^+$ ($F = \text{o-}\lim_{n\to\infty} F_n$), if and only if there exist two sequences (G_n) and (H_n) such that (G_n) is non-decreasing with $\bigvee_{n=1}^{\infty} G_n = F$, (H_n) is non-increasing with $\bigwedge_{n=1}^{\infty} H_n = F$, and for all $n \in N$ is $G_n \leqslant F_n \leqslant H_n$. The order limit is unique.

THEOREM 1.1. *Let* (F_n) *be a sequence in* Δ^+ *and* $F \in \Delta^+$. *Then we have:*

i) $F = \text{o-}\lim_{n\to\infty} F_n$ *iff* $\lim_{n\to\infty} F_n(x) = F(x)$, *for all* $x \in R^+$ (*pointwise convergence*);

ii) *If* $F = \text{o-}\lim_{n\to\infty} F_n$ *then* $F = \text{w-}\lim_{n\to\infty} F_n = \mathcal{L}\text{-}\lim_{n\to\infty} F_n$, *but the reciprocal does not hold in general*;

iii) *If* $F = \text{w-}\lim_{n\to\infty} F_n$ *and* F *is continuous or* (F_n) *is non-decreasing then* $F = \text{o-}\lim_{n\to\infty} F_n$.

THEOREM 1.2. (Weak version of Everett diagonal condition in Δ^+). *Let* $(F_k^n)_{(n,k)\in N\times N}$ *be a collection of sequences in* Δ^+, *let* (F_n) *be a sequence in* Δ^+ *and* $F \in \Delta^+$. *If* F *has at most a finite set of discontinuities,* $F = \text{o-}\lim_{n\to\infty} F_n$, *and for each* $n \in N$, $F_n = \text{o-}\lim_{k\to\infty} F_k^n$, *then there exists a strictly increasing sequence of integers* $k_1 < k_2 < \dots < k_n < \dots$ *in* N, *such that* $F = \text{o-}\lim_{n\to\infty} F_{k_n}^n$.

2. A FAMILY OF PROXIMITIES IN A PM-SPACE.

Let (S,F) be a semi-PM space. For each $\varphi \in \Delta^+$ we define a binary relation δ_φ on $P(S)$ in the following way, for $A,B \in P(S)$,
"$A\ \delta_\varphi\ B$ iff there exists a sequence $((a_n,b_n))_{n\in N}$ in $A \times B$ such that $\varphi = \operatorname{o-lim}_{n\to\infty} (\varphi \wedge F_{a_n b_n})$".

When $A\ \delta_\varphi\ B$ we will say that A and B have a φ-proximity.

THEOREM 2.1. *δ_φ is a semi-proximity.*

The Čech closure induced by δ_φ will be:

$$C_{\delta_\varphi}(A) = \{x \in S;\ \exists (a_n) \subset A\colon \varphi = \operatorname{o-lim}_{n\to\infty} (\varphi \wedge F_{xa_n})\}.$$

THEOREM 2.2. *If τ is continuous, $\tau(\varphi,\varphi) = \varphi$ and φ is continuous in R^+, then C_{δ_φ} is a Kuratowski closure.*

Proof. If $x \in C_{\delta_\varphi}(C_{\delta_\varphi}(A))$ there is $(x_n) \subset C_{\delta_\varphi}(A)$ such that $\operatorname{o-lim}_{n\to\infty} (\varphi \wedge F_{xx_n}) = \varphi$. For each $n \in N$, $x_n \in C_{\delta_\varphi}(A)$, i.e., there exists a sequence $(a_k^n)_{k\in N} \subset A$ such that $\operatorname{o-lim}_{k\to\infty} (\varphi \wedge F_{x_n a_k^n}) = \varphi$. By Theorem 1.2 there exists an increasing sequence of integers $(k_n)_{n\in N}$ such that $\operatorname{o-lim}_{n\to\infty} (\varphi \wedge F_{x_n a_{k_n}^n}) = \varphi$. Let $H_n = \tau(\varphi \wedge F_{xx_n}, \varphi \wedge F_{x_n a_{k_n}^n})$, for every $n \in N$. Using the continuity of τ and φ, we have $\operatorname{o-lim}_{n\to\infty} H_n = \tau(\varphi,\varphi) = \varphi$, and by the triangle inequality $H_n \leqslant F_{xa_{k_n}^n}$ and $H_n \leqslant \tau(\varphi,\varphi) = \varphi$, we will obtain $H_n \leqslant \varphi \wedge F_{xa_{k_n}^n} \leqslant \varphi$ which in turn implies $\operatorname{o-lim}_{n\to\infty} (\varphi \wedge F_{xa_{k_n}^n}) = \varphi$, i.e., $x \in C_{\delta_\varphi}(A)$.

The following example shows that the strong hypothesis $\varphi = \tau(\varphi,\varphi)$ assumed above, is really necessary.

EXAMPLE 2.1. Consider the PM-space $(R^+, \varepsilon_{|x-y|}, *)$ and $\varphi = U_{\frac{1}{2},1}$. The convolution $*$ is continuous [6] and has no idempotents different from ε_0 and ε_∞. It is easy to see that

$$C_{\delta_{U_{\frac{1}{2},1}}}(0) = [0,1/2] \subsetneq C_{\delta_{U_{\frac{1}{2},1}}}(C_{\delta_{U_{\frac{1}{2},1}}}(0)) \text{ because } [0,1] \subset C_{\delta_{U_{\frac{1}{2},1}}}(C_{\delta_{U_{\frac{1}{2},1}}}(0)).$$

In order to analyse the special case $\varphi = \varepsilon_0$ we recall the following lemma.

LEMMA 2.1. *Let* I *be any set of indices and let* $\{F_i;\ i \in I\}$ *be in* Δ^+. *The following statements are equivalent:*

i) $\bigvee_{i \in I} F_i = \varepsilon_0$;

ii) *For any* $\varepsilon \in (0,1)$ *there exists* $i \in I$ *such that* $F_i(\varepsilon) > 1-\varepsilon$;

iii) *For any* $\varepsilon \in (0,1)$ *there exists* $i \in I$ *such that* $\mathcal{L}(F_i,\varepsilon_0) < \varepsilon$;

iv) *There is a countable subset* J *of* I *such that* $\bigvee_{i \in J} F_i = \varepsilon_0$.

Then the Efremovič proximity δ_{ε_0} can be presented in the following ways:

"A δ_{ε_0} B iff $\bigvee_{(a,b)\in A\times B} F_{ab} = \varepsilon_0$ iff for every $\varepsilon,\lambda > 0$ there is $(a,b) \in A\times B$ such that $F_{ab}(\varepsilon) > 1-\lambda$"

and

$$C_{\delta_{\varepsilon_0}}(A) = \{x \in S;\ N_x(\varepsilon,\lambda) \cap A \neq \emptyset,\ \varepsilon,\lambda > 0\},$$

where $N_x(\varepsilon,\lambda) = \{y \in S;\ F_{xy}(\varepsilon) > 1-\lambda\}$ are the neigborhood of the classical ε,λ-topology for these spaces, i.e., the ε,λ-topology is proximizable by δ_{ε_0}.

THEOREM 2.3. *Under the hypothesis of Theorem 2.2, the topological space* (S,C_{δ_φ}) *is completely regular.*

In a PM-space (S,F,τ) and for a fixed $\varphi \in \Delta^+$, Schweizer [7] has introduced the next relation in P(S):

"A I_φ B iff there exists $(a,b) \in A\times B$ such that $F_{ab} \geqslant \varphi$", and when A I_φ B, A and B are said to be *indistinguishable* (*mod.*φ).

We note that I_φ is a semi-proximity weaker than δ_φ, in the sense that A I_φ B implies A δ_φ B, i.e., indistinguishability (mod.φ) yields φ-proximity. The reciprocal does not hold, in general.

EXAMPLE 2.2. Consider the PM-space $(R^+,\varepsilon,*)$, where $\varepsilon_{pq} = \varepsilon_{|p-q|}$ for all $p,q \in R^+$. Let $k > 0$ and $\varphi = \varepsilon_k$. Take $A = [0,1)$ and $B = (1+k,+\infty)$. Taking for each $n \in N$, $a_n = 1-1/n \in A$ and $b_n = 1+k+1/n \in B$, we have

$$\text{o-}\lim_{n\to\infty} (\varepsilon_k \wedge \varepsilon_{|a_n-b_n|}) = \text{o-}\lim_{n\to\infty} \varepsilon_{k+\frac{2}{n}} = \varepsilon_k,$$

i.e., A δ_{ε_k} B but A $\not{I}_{\varepsilon_k}$ B because for all $(a,b) \in A\times B$ we have $\varepsilon_{|a-b|} < \varepsilon_k$.

Recently, Tardiff has introduced [8] for $\varphi \in \Delta^+$ a closure operator de fined by

$C_\varphi(A) = \{x \in S;\ (\forall\ h \in (0,1])(\exists\ a = a(h) \in A)$ such that $F^h_{xa} \geqslant \varphi\}$,

being

$$F^h_{xa}(t) = \begin{cases} 0 & , \text{ if } t \leqslant 0, \\ \min(F_{xa}(t+h)+h,1), & \text{if } t \in (0,1/h], \\ 1 & , \text{ if } t > 1/h . \end{cases}$$

The semi-proximity T_φ defined by

"$A\ T_\varphi\ B$ iff $C_\varphi(A) \cap C_\varphi(B) \neq \emptyset$",

is stronger than I_φ because if $A\ I_\varphi\ B$ then there is $(a,b) \in A \times B$ such that $F_{ab} \geqslant \varphi$ and for all $h > 0$, $F^h_{ab} \geqslant F_{ab} \geqslant \varphi$, i.e., $A\ T_\varphi\ B$. The reciprocal does not hold, in general.

EXAMPLE 2.3. Consider the PM-space of example 2.1, and the same $\varphi = \varepsilon_k$, $k > 0$. Let $A = [0,1)$. Then $C_{I_{\varepsilon_k}}(A) = [0,1+k) \subset C_{\varepsilon_k}(A)$. But $1+k \in C_{\varepsilon_k}(A)$ because, for any $h \in (0,1]$, taking $1-h \in A$ we have $\varepsilon^h_{1+k-1+h} = \varepsilon^h_{k+h} \geqslant \geqslant \varepsilon_k$, so $\{1+k\}\ T_{\varepsilon_k}\ A$ but $\{1+k\}\ \not I_{\varepsilon_k}\ A$.

Finally we remark that for $\varphi = \varepsilon_0$, $T_{\varepsilon_0} = \delta_{\varepsilon_0}$ is the ε,λ-proximity and for any φ and $p \in S$: $C_{\delta_\varphi}(\{p\}) = C_{I_\varphi}(\{p\}) = C_\varphi(\{p\}) = \{q \in S;\ F_{pq} \geqslant \varphi\}$, and this set is exactly the class of p in the partition of S induced by the equivalence relation of indistinguishability (mod.φ) introduced in [7].

3. A PROXIMITY INDUCED BY AN UNIFORMITY.

DEFINITION 3.1. A triangular function τ is said to be *radical* if for any $F \in \Delta^+ - \{\varepsilon_0\}$ there exists $G \in \Delta^+ - \{\varepsilon_0\}$ such that $F < \tau(G,G) < \varepsilon_0$.

THEOREM 3.1. *If* $\tau \geqslant *$ *then* τ *is radical.*

Proof. We need to show that for any $F < \varepsilon_0$ there is $G < \varepsilon_0$ such that $G*G > F$. In effect, if $F = \varepsilon_k$ for some $k > 0$ then taking $G = \varepsilon_{k/4}$ we have $G*G = \varepsilon_{k/2} > \varepsilon_k$. If $F < \varepsilon_k$ for some $k > 0$ then the same G yields the same conclusion. So we can suppose that there is an interval $(0,k)$ such that $F(x) > 0$ for $x \in (0,k)$. Let

$$H(x) = \begin{cases} 0 & , \text{ if } x \leqslant 0, \\ +\sqrt{F(2x)}, & \text{if } 0 < x. \end{cases}$$

Obviously $F(x) \leqslant +\sqrt{F(x)} \leqslant +\sqrt{F(2x)}$ for $x > 0$, and consequently $F \leqslant H$. If $F(x) = H(x)$ for all $x > 0$ then $F(x) = \sqrt{F(x)}$ and $F(x)(F(x)-1) = 0$, i.e., there would exist $k' > 0$ such that $F(x) = \varepsilon_{k'}$, which is a contradiction. So $F < H$ and there is a $t > 0$ such that $0 < F(t) < H(t) < 1$. Let

$$G(x) = \begin{cases} H(x) & , \text{ if } x \leqslant t, \\ 1 & , \text{ if } x > t. \end{cases}$$

$G > H$ and a straightforward computation shows that $F \leqslant H*H$. By the strict isotony of $*$, $F \leqslant H*H < G*G < \varepsilon_0$.

Let (S,F,τ) be a PM-space. For any $F \in \Delta^+-\{\varepsilon_0\}$, let $U(F) = \{(p,q) \in S \times S;\ F_{pq} > F\}$.

THEOREM 3.2. *If* τ *is radical then the collection* $\{U(F);\ F \in \Delta^+-\{\varepsilon_0\}\}$ *is a basis for a diagonal separated uniformity* $\mathcal{U}$ *on* S.

Proof. Obviously $\Delta_S = \{(p,p);\ p \in S\} \subset U(F)$ and $U(F) = U(F)^{-1}$, for any $F < \varepsilon_0$. If $F,G < \varepsilon_0$ and being τ radical there is $G < \varepsilon_0$ such that $F < \tau(G,G) < \varepsilon_0$. Then $U(G) \circ U(G) \subset U(F)$ because if $(p,r) \in U(G) \circ U(G)$, there is $q \in S$ such that $(p,q) \in U(G)$ and $(q,r) \in U(G)$. By the triangle inequality $F_{pr} \geqslant \tau(F_{pq},F_{qr}) \geqslant \tau(G,G) > F$, so $(p,q) \in U(F)$. Finally note that $\mathcal{U}$ is separated because $\bigcap_{F\in\Delta^+-\{\varepsilon_0\}} U(F) = \Delta_S$.

COROLLARY 3.1. *The topology generated by* $\mathcal{U}$ *is metrizable.*

Proof. Consider the countable family $\{\alpha_{t,t'};\ t,t' \in (0,1) \cap Q\} \subset \Delta^+$, where

$$\alpha_{t,t'}(x) = \begin{cases} 0 & , \text{ if } x \leqslant 0, \\ t' & , \text{ if } 0 < x \leqslant t, \\ 1 & , \text{ if } x > t. \end{cases}$$

If $U \in \mathcal{U}$, there is $F < \varepsilon_0$ such that $U(F) \subset U$. Being $F < \varepsilon_0$ there exists $t \in (0,1) \cap Q$ such that $F(t) < 1$. Let $t' \in (F(t),1)$. Then $F < \alpha_{t,t'}$ and $U(F) \supset U(\alpha_{t,t'})$, i.e., $\{U(\alpha_{t,t'});\ t,t' \in (0,1) \cap Q\}$ is a countable basis for $\mathcal{U}$. We apply then Weyl theorem.

The topology generated by $\mathcal{U}$ can be described by the family $\mathcal{N}(\mathcal{U}) = \{N_p(F);\ F \in \Delta^+-\{\varepsilon_0\},\ p \in S\}$, where each neighborhood $N_p(F)$ is given by

$$N_p(F) = \{q \in S;\ F_{pq} > F\} = C_{I_F}(\{p\}),$$

i.e., $N_p(F)$ is precisely the closure of $\{p\}$ by C_{I_F}, C_{δ_F} or C_F, in other words, if $q \in N_p(F)$ then q is indistinguishable (mod.F) of p.

The uniformity $\mathcal{U}$ induces a proximity $\delta_{\mathcal{U}}$ defined on P(S) by:

"$A \not\delta_{\mathcal{U}} B$ iff for some $F < \varepsilon_0$, $N_p(F) \cap B = \emptyset$, for all $p \in A$".

Applying a well known result of proximity theory we obtain that the topology induced by δ_U is the uniform topology. We remark that this topology is exactly the topology T_F obtained when considering the PM-space as generalized metric space [9].

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Recibido en diciembre de1979.