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PROJECTORS ON CONVEX SETS IN REFLEXIVE BANACH SPACES

Eduardo H. Zarantonello

Dedicated to Professor Luis A. Santaló

Selfadjoint operators in Hilbert space can be synthetized out of orthogonal projectors by the process of forming the integrals of numerical functions with respect to an increasing one-parameter family of projectors. To be viable such a mechanism - known as spectral synthesis - requires from projectors a certain number of algebraic properties. Not long ago I have shown [7,8,9] that these properties subsist if the class of linear projectors is enlarged so as to include projectors on closed convex cones, conceived as nearest point mappings, and thus I was able to synthetize a new class of operators, mostly nonlinear. But then, having freed the spectral theory from its original confinement I was faced with the question of how far one can go on extending it. For instance, would it be valid in spaces other than Hilbert space?. It is precisely to this question that I am addressing myself in this paper, beginning with the study of projectors in reflexive Banach spaces. A first basic question is to decide what projectors on convex sets should be. Nearest point mappings certainly do not qualify, as they form an unruly class devoid of any algebraic structure, nor does any class of operators mapping the space into itself, since for these many of the required properties do not even make sense. This realized, one is led to the view that projectors must be mappings, perhaps multivalued, acting from the dual into the space, view which in Hilbert space is thoroughly concealed by the standard identification of the space with its dual. At this stage a choice offers itself in a most natural way: The projector on a closed convex set K in a real reflexive Banach space X is the mapping $P_K: X^* \to 2^X$ assigning to each $x^* \in X^*$ the set of points minimizing $\frac{1}{2}\|x^*\|^2 + \frac{1}{2}\|x\|^2 - \langle x^*,x\rangle$ over K. A series of familiar looking results soon brings out the certainty of being on the right track. So reassured, I have proceeded to investigate these new mathematical objects, not so much on their own right but rather as possible instruments for the spectral theory. My results are inconclu-

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sive as they failed to prove or disprove a couple of essential points. It is however apparent that the very existence of an increasing family of projectors requires from the space a good deal of Hilbert space structure, and therefore that there is not much occasion for the spectral theory to take place in a reflexive space chosen at random.

§1. PROJECTORS ON CONVEX SETS.

All throughout this article we shall be working in a real reflexive Banach space X, whose dual we shall denote X*. As usual the double bar indicates the norm in either space, and the angular brackets the bilinear form effecting the pairing of X and X*. We shall let $J: X \to 2^{X^*}$ denote the duality mapping:

$$Jx = \{x^* \mid \langle x^*,x\rangle = \|x\|^2 = \|x^*\|^2\}$$

of X onto X*, and $J^{-1}: X^* \to 2^X$,

$$J^{-1}x^* = \{x \mid \langle x^*,x\rangle = \|x\|^2 = \|x^*\|^2\},$$

the duality mapping of X* onto X. Let us recall that $Jx = \partial\frac{1}{2}\|x\|^2$, and $J^{-1}x^* = \partial\frac{1}{2}\|x^*\|^2$, and that the relation

$$\frac{1}{2}\|x^*\|^2 + \frac{1}{2}\|x\|^2 - \langle x^*,x\rangle = 0$$

is equivalent to $x^* \in Jx$ and to $x \in J^{-1}x^*$. Mappings, even when singlevalued, are considered here in the context of multivalued mappings, and so the inverses always exist. The conjugate of a proper lower semicontinuous function $f: X \to (-\infty,+\infty]$ is denoted f*. We shall often use the letter Q for the function $x \to \frac{1}{2}\|x\|^2$, and Q* for its adjoint $x^* \to \frac{1}{2}\|x^*\|^2$. If K is a closed convex set ψ_K denotes its indicator function. The infraconvolution of convex functions is indicated by the symbol □.

To bring out the analogy with projectors in Hilbert space we shall follow closely our discussion of the Hilbert space theory expounded in [9]; the reader is invited to compare the results step by step.

DEFINITION 1. The projector on a closed convex set K in X is the mapping $P_K: X^* \to 2^X$ assigning to each x* the set of points minimizing the function

$$\frac{1}{2}\|x^*\|^2 + \frac{1}{2}\|x\|^2 - \langle x^*,x\rangle$$

over K, that is

(1) $$P_Kx^* = \{x \in K \mid \frac{1}{2}\|x\|^2 - \langle x^*,x\rangle \leq \frac{1}{2}\|y\|^2 - \langle x^*,y\rangle,\ \forall\, y \in K\}$$

Since $\|x\|^2 - \langle x^*,x\rangle$ is l.s.c. convex function of x tending to $+\infty$ with

$\|x\|$ the infimum is always attained and $P_K x^*$ is never empty. In Hilbert space P_K is simply the nearest point mapping on K. If K = X then $P_K = J^{-1}$, whereas if $K = \{tz\}_{t\geq 0}$ then $P_K x^* = \langle x^*, \frac{z}{\|z\|}\rangle^+ \frac{z}{\|z\|}$. In the latter case we recognize $P_K x^*$ as the ordinary projection of x^* on a half-line.

THEOREM 1.

$$P_K x^* = \{x \mid (Q+\psi_K)x + (Q+\psi_K)^* x^* = \langle x^*,x\rangle\} = (J+\partial\psi_K)^{-1}x^*. \tag{2}$$

Proof. From (1) we obtain

$$\{x \in P_K x^*\} \iff \{\langle x^*,x\rangle - (\tfrac{1}{2}\|x\|^2 + \psi_K(x)) = \sup_y [\langle x^*,y\rangle - \tfrac{1}{2}(\|y\|^2+\psi_K(y))]\}$$

$$\iff \{(Q+\psi_K)(x) + (Q+\psi_K)^*(x)^* = \langle x^*,x\rangle\} \iff \{x \in \partial(Q+\psi_K)^* = (J+\partial\psi_K)^{-1}x^*\}$$

COROLLARY 1. P_K is a subdifferential.

COROLLARY 2. The function $\frac{1}{2}\|x\|^2 - \langle x^*,x\rangle$ remains constant over $P_K x^*$.

This corollary justifies the notation $\langle x^*,P_K x^*\rangle - \frac{1}{2}\|P_K x^*\|^2$ for the common value of $\langle x^*,x\rangle - \frac{1}{2}\|x\|^2$ on $P_K x^*$.

COROLLARY 3.

$$\langle x^*,P_K x^*\rangle - \frac{1}{2}\|P_K x^*\|^2 = (Q+\psi_K)^* x^*. \tag{3}$$

Proof. The left hand side coincides with the supremum of $\langle x^*,y\rangle - (\frac{\|y\|^2}{2} + \psi_K(y))$, which is $(Q+\psi_K)^* x^*$.

COROLLARY 4. P_K satisfies the subdifferential equation

$$P_K x^* = \partial[\langle x^*,P_K x^*\rangle - \frac{1}{2}\|P_K x^*\|^2]. \tag{4}$$

COROLLARY 5.

$$P_K x^* \cap P_K y^* \subset P_K(tx^* + (1-t)y^*). \tag{5}$$

Proof. This is just another way of saying that $P_K^{-1}x = Jx + \partial\psi_K x$ is convex. On the other hand convexity follows from the maximal monotonicity of $J + \partial\psi_K$.

COROLLARY 6.

$$\{x \in P_K x^*\} \iff \{\exists\, \bar{x}^* \in Jx \mid \langle x^*-\bar{x}^*,x-y\rangle \geqslant 0, \quad \forall\, y \in K\}. \tag{6}$$

Proof. $\langle x \in P_K x^*\rangle \iff \{x^* \in Jx + \partial\psi_K(x)\} \iff \{\exists\, \bar{x}^* \in Jx \mid x^*-\bar{x}^* \in \partial\psi_K x\}$

$$\iff \{x \in K,\ \bar{x}^* \in Jx \mid \langle x^*-\bar{x}^*,x-y\rangle \geqslant 0,\ y \in K\}$$

Let us recall a few basic notions. A vector $u^* \in X^*$ is said to be normal to a closed convex set K at a point $x \in K$ if

$$\langle u^*, x-y \rangle \geq 0 \ , \quad y \in K \ ;$$

such vectors are called *normals*. It is evident that $\partial\psi_K(x)$ is the set of all normals to K at x.

A hyperplane is said to support a convex set K if it bounds a minimal halfspace containing K. If K is closed the intersections of a supporting hyperplane with K is called a face of K; if the face is not empty the hyperplane is said to support K at any point of this face, otherwise it supports K at infinity. As intersections of closed convex sets faces are closed convex sets. The equation of any hyperplane supporting K at finite distance can be written in the form: $\langle u^*, x \rangle - r = 0$, with u^* normal to K, and $r = \sup_{y \in K} \langle u^*, y \rangle$. It follows that a K-face is the set of points having a common nonvanishing normal. To also include the case $u^* = 0$, K itself is considered to be a face, if only an improper one. In this context it is important to bear in mind that Jx is the set of normals at x to the ball of radius $\|x\|$ centered at the origin with norms all equal to $\|x\|$, and also the face of the ball of radius $\|x\|$ in X^* having x as normal.

THEOREM 2. *Any* $P_K x^*$ *is the intersection of a* K*-face with a face of a ball centered at the origin, and conversely. The* K*-face is proper if* $x^* \notin JK$.

Proof. For fixed u^* and v^* we have

$$\{x \mid u^* \in Jx\} \cap \{x \mid v^* \in \partial\psi_K(x)\} \subset \{x \mid u^*+v^* \in Jx+\partial\psi_K x\} = P_K(u^*+v^*).$$

Moreover, by definition of P_K,

$$\{x_1 \in P_K(u^*+v^*)\} \iff \{u^*+v^* = u_1^* + v_1^*,\ u_1^* \in Jx_1,\ v_1^* \in \partial\psi_K(x_1)\}$$

and if x belongs to the intersection set on the left in the previous equation,

$$\{x_1 \in P_K(u^*+v^*)\} \Rightarrow \{0 = \langle u^*-u_1^*, x-x_1 \rangle + \langle v^*-v_1^*, x-x_1 \rangle\}$$

and by the monotonicity of J and $\partial\psi_K$,

$$0 = \langle u^*-u_1^*, x-x_1 \rangle = \langle v^*-v_1^*, x-x_1 \rangle$$

But $0 = \langle u^*-u_1^*, x-x_1 \rangle = \langle u^*, x \rangle + \langle u_1^*, x_1 \rangle - \langle u^*, x_1 \rangle - \langle u_1^*, x \rangle =$

$$= [\tfrac{1}{2} \|u^*\|^2 + \tfrac{1}{2} \|x_1\|^2 - \langle u^*, x_1 \rangle] + [\tfrac{1}{2} \|x\|^2 + \tfrac{1}{2} \|u_1^*\|^2 - \langle u_1^*, x \rangle],$$

and since both terms on the right are nonnegative, they vanish, implying that $u^* \in Jx_1$, $u_1^* \in Jx$. Furthermore, from $0 = \langle v^*-v_1^*, x-x_1 \rangle$ we deduce for any $z \in K$,

$$\langle v^*, x_1-z \rangle = \langle v^*, x-z \rangle + \langle v^*, x_1-x \rangle = \langle v^*, x-z \rangle + \langle v_1^*, x_1-x \rangle + \langle v^*-v_1^*, x-x_1 \rangle$$

$$= \langle v^*, x-z \rangle + \langle v_1^*, x_1-x \rangle \geq 0.$$

whence $v^* \in \partial\psi_K(x_1)$. In conclusion,

$$\{x_1 \in P_K(u^*+v^*)\} \Rightarrow \{u^* \in Jx_1,\ v^* \in \partial\psi_K(x_1)\},$$

and therefore

$$P_K(u^*+v^*) = \{x \mid u^* \in Jx\} \cap \{x \mid v^* \in \partial\psi_K(x)\}.$$

Of these two last sets the former is the face of the ball through x having u^* as normal and the latter the K-face perpendicular to v^*. This concludes the proof because any x^* can be written in the form $x^* = u^*+v^*$, with v^* normal to K at a point x, and u^* normal at x to the ball through x. It is clear that if $x^* \notin JK$ then $u^* \neq 0$, and the corresponding K-face is proper.

COROLLARY 1. If J^{-1} is single valued so is P_K for any K.

This corollary can also be stated by saying that if the unit ball in X^* is smooth then P_K is singlevalued.

COROLLARY 2. The functions $\frac{1}{2}\|x\|^2$ and $\langle x^*,x\rangle$ take constant values for $x \in P_K x^*$.

We can now use the notation $\frac{1}{2}\|P_K x^*\|^2$, $\langle x^*,P_K x^*\rangle$ without any ambiguity, because the results do not depend on the representative point in $P_K x^*$ used to calculate them.

COROLLARY 3. $P_K x^*$ is a bounded closed convex set for every $x^* \in X^*$.

THEOREM 3. $x^* \in JK$ *if and only if* $P_K x^* = J^{-1}x^* \cap K$.

Proof. It is obvious that if $P_K x^* = J^{-1}x^* \cap K$ then $x^* \in JK$. Conversely, if $x \in K$ and $x^* \in Jx$, then for each $y \in P_K x^*$ there is a $y^* \in Jy$ and a $u^* \in \partial\psi_K(y)$ such that $x^* = y^*+u^*$, and so

$$\langle x^*-y^*,x-y\rangle + \langle u^*,y-x\rangle = 0.$$

The two terms on the left are nonnegative, the first by monotonicity, and the second because u^* is normal to K at y. Hence both vanish. From $\langle x^*-y^*,x-y\rangle = 0$ it follows that $y \in J^{-1}x^*$, and hence, since this holds for every y in $P_K x^*$, that $P_K x^* \subset J^{-1}x^* \cap K$. The opposite inclusion being obvious, the theorem is proved.

COROLLARY 1. $R(P_K) = K$.

Proof. From the definition of projector $R(P_K) \subset K$, and from the above theorem $P_K(JK) \supset K$, so $R(P_K) = K$.

COROLLARY 2.

$$P_K x^* \subset P_K J P_K x^* = J^{-1} J P_K x^* \cap K \tag{8}$$

COROLLARY 3.

$$P_K x^* \subset P_K(tx^* + (1-t)\, J\, P_K x^*)\ , \quad 0 \leqslant t \leqslant 1. \tag{9}$$

Proof. From Theorem 1, Corollary 5 and Corollary 2 above.

THEOREM 4. *A subdifferential operator* $P: X^* \longrightarrow 2^X$ *is a projector if and only if it satisfies*

$$Px^* = \partial[\langle x^*, Px^*\rangle - \tfrac{1}{2}\,\|Px^*\|^2], \tag{10}$$

where the notation is construed to mean that $\langle x^*,x\rangle - \frac{1}{2}\,\|x\|^2$ *takes a constant value for* $x \in Px^*$, *and that the resulting function, assumed equal to* $+\infty$ *when* Px^* *is empty, is a proper l.s.c. convex function of* x^*.

Proof. Necessity is the content of Theorem 1, Corollary 4. As for sufficiency start out by remarking that $\mathcal{D}(P)$ is convex because by hypothesis it coincides with the domain of a l.s.c. convex function. We claim that P is locally bounded about each point in space. Indeed, if it were not there would be a point x^* and a sequence $\{x_n^*\}_1^\infty \subset \mathcal{D}(P)$ such that $x_n^* \longrightarrow x^*$, $\|Px_n^*\| \uparrow +\infty$, and then $\langle x_n^*, Px_n^*\rangle - \frac{1}{2}\,\|Px_n^*\|^2 \longrightarrow -\infty$, implying, by lower semicontinuity, that $\langle x^*,Px^*\rangle - \frac{1}{2}\,\|Px^*\|^2 = -\infty$, which is impossible. Then, local boundedness coupled with demicontinuity (itself a consequence of maximal monotonicity) require that $\mathcal{D}(P)$ be closed. Now, if u is normal to $\mathcal{D}(P)$ at x^* then, by maximal monotonicity again, $Px^* + tu \in Px^*$, $t \geqslant 0$, and $u = 0$, since Px^* is a bounded set. Having no nonvanishing normal $\mathcal{D}(P)$ is the whole space. (The foregoing argument is a particular case of the theorem that says that a maximal monotone operator is surjective if and only if its inverse is locally bounded [4]).

Next we observe that (10) amounts to

$$[\langle x^*,Px^*\rangle - \tfrac{1}{2}\,\|Px^*\|^2] - [\langle y^*,Py^*\rangle - \tfrac{1}{2}\,\|Py^*\|^2] \geqslant \langle x^*-y^*,y\rangle\ ,$$

$\forall\ x^*,y^* \in X^*$, $\forall\ y \in Py^*$, that is, to

$$\langle x^*,Px^*\rangle - \tfrac{1}{2}\,\|Px^*\|^2 \geqslant \langle x^*,y\rangle - \tfrac{1}{2}\,\|y\|^2,\ \forall\ x^*,y^* \in X^*\ ,\ \forall\ y \in Py^*.$$

Hence, since for $y \in Px^*$ the right hand member of this inequality coincides with the one on the left,

$$\langle x^*,Px^*\rangle - \tfrac{1}{2}\,\|Px^*\|^2 = \sup_{y \in \overline{R(P)}} \{\langle x^*,y\rangle - \tfrac{1}{2}\,\|y\|^2\}.$$

As the closure of the range of a maximal monotone operator $\overline{R(P)}$ is convex [cf.5], and the supremum above is $(Q + \psi_{\overline{R(P)}})^*(x^*) =$
$= \langle x^*, P_{\overline{R(P)}}\, x^*\rangle - \frac{1}{2}\,\|P_{\overline{R(P)}}\, x^*\|^2$. Finally,

$$Px^* = \partial[\langle x^*,Px^*\rangle - \tfrac{1}{2}\,\|Px^*\|^2] = \partial[\langle x^*,P_{\overline{R(P)}}\, x^*\rangle - \tfrac{1}{2}\,\|P_{\overline{R(P)}}\, x^*\|^2] =$$
$= P_{\overline{R(P)}}\, x^*$. Q.E.D.

THEOREM 6. $\sum_1^n P_{K_i}$ *is a projector if and only if*

(11) $$\sum_1^n \|P_{K_i}x^*\|^2 - \|\sum_1^n P_{K_i}x^*\|^2 = \text{const.}$$

In such a case $\sum_1^n P_{K_i} = P_{\sum_1^n K_i}$.

Proof. If $\sum_1^n P_{K_i}$ is a projector then the subdifferential of $\langle x^*, (\sum_1^n P_{K_i})x^*\rangle - \frac{1}{2}\|(\sum_1^n P_{K_i})x^*\|^2$, namely $\sum_1^n P_{K_i}x^*$, is contained in that of $\sum_1^n [\langle x^*, P_{K_i}x^*\rangle - \frac{1}{2}\|P_{K_i}x^*\|^2]$, and in consequence both convex functions coincide up to an additive constant, that is, (11) holds. Conversely, if (11) holds, then

$$\langle x^*, (\sum_1^n P_{K_i})x^*\rangle - \frac{1}{2}\|(\sum_1^n P_{K_i})x^*\|^2 = \sum_1^n [\langle x^*, P_{K_i}x^*\rangle - \frac{1}{2}\|P_{K_i}x^*\|^2] + \text{const},$$

and

$$\partial[\langle x^*, (\sum_1^n P_{K_i})x^*\rangle - \frac{1}{2}\|(\sum_1^n P_{K_i})x^*\|^2] \supset \sum_1^n \partial[\langle x^*, P_{K_i}x^*\rangle - \frac{1}{2}\|P_{K_i}x^*\|^2] =$$
$$= (\sum_1^n P_{K_i})x^*.$$

Since the subdifferential of a convex function is monotone, and $\sum_1^n P_{K_i}$ maximal monotone [6], the above inclusion is in fact an equality, and $\sum_1^n P_{K_i}$ is a projector because it satisfies relation (10). Thus the first part of the theorem is proved. As to the last, note first that if $f_i(x) = \frac{1}{2}\|x\|^2 + \psi_{K_i}(x)$, $i = 1,2,\dots,n$ then $\sum_1^n P_{K_i} = \sum_1^n \partial f_i^* = \partial \sum_1^n f_i^*$ because the f_i^*'s are continuous [6]. Hence $\overline{R(\sum_1^n P_{K_i})} = \overline{R(\partial \sum_1^n f_i^*)} = \overline{D(\partial(\sum_1^n f_i^*)^*} = \overline{D(\partial(f_1 \Box f_2 \Box \dots \Box f_n))}$, and, as the domain of the subdifferential of a l.s.c. convex function is dense in the domain of the function [1],

$$\overline{D(\partial(f_1 \Box f_2 \Box \dots \Box f_n)} = \overline{D(f_1 \Box f_2 \Box \dots \Box f_n)} = \overline{D(f_1)+D(f_2)+\dots+D(f_n)} =$$
$$= \overline{K_1+K_2+\dots+K_n}.$$

Therefore, $\overline{R(\sum_1^n P_{K_i})} = \overline{\sum_1^n K_i}$. Now, if $\sum_1^n P_{K_i}$ is a projector its range is closed and $\overline{\sum_1^n K_i} = R(\sum_1^n P_{K_i}) \subset \sum_1^n K_i$, whence $R(\sum_1^n P_{K_i}) = \sum_1^n K_i$. The proof concludes by remarking that any projector is the projector on its range.

§2. CONICAL PROJECTORS.

Projectors on closed convex cones with vertex at the origin are called conical projectors. It is clear that a projector on a convex set is positive homogeneous when the set is a cone with vertex at 0, and only then, so that the class of conical projectors coincides with that of positive homogeneous projectors. The letter C will be reserved to designate the above type of cones, so that P_C will always indicate a conical projector.

The dual of a cone $C \subset X$ in the cone in X^*

(12) $$C^{\perp} = \{x^* \in X^* \mid \langle x^*,x\rangle \leqslant 0,\ x \in C\}.$$

$C^{\perp}$ is nonempty, closed and convex. The operation of taking duals has the following properties:

(13) $$C^{\perp\perp} = C\ ,\quad \{C_1 \subset C_2\} \Longleftrightarrow \{C_1^{\perp} \supset C_2^{\perp}\}\ ,\quad (\bigcap_i C_2)^{\perp} = \overline{\mathrm{co} \cup C_i^{\perp}}.$$

For linear spaces $\perp$ coincides with the operation of taking anihilators. The indicator functions of dual cones are conjugate of each other. We leave to the reader the verification of these facts.

The original definition (1) acquires a special form in the case of projectors on cones:

THEOREM 6.

(14) $$P_C x^* = \{x \in C \mid \langle x^*,x\rangle = \|x\|^2 = [\sup_{u\in C,\|u\|\le 1} \langle x^*,u\rangle]^2\}$$

Proof. If x minimizes $\frac{1}{2}\|y\|^2 - \langle x^*,y\rangle$ over C, then, for any $x \in C$, $\frac{1}{2}t^2\|x\|^2 - t\langle x^*,x\rangle$ as a function of t attains its minimum on the positive real axis at $t = 1$, and hence $\|x\|^2 = \langle x^*,x\rangle$. Therefore $x \in P_C x^*$ if and only if $\|x\|^2 = \langle x^*,x\rangle$ and

$$-\frac{\|x\|^2}{2} = \frac{\|x\|^2}{2} - \langle x^*,x\rangle = \inf_{y\in C} \frac{1}{2}\{\|y\|^2 - \langle x^*,y\rangle\} =$$

$$= \inf_{y\in C}\inf_{t\ge 0} \{\frac{1}{2}t^2\|y\|^2 - t\langle x^*,y\rangle\} =$$

$$= \inf_{y\in C}\begin{cases} 0, & \text{if } \langle x^*,y\rangle \leqslant 0 \\ -\frac{1}{2}\langle x^*,\frac{y}{\|y\|}\rangle^2, & \text{if } \langle x^*,y\rangle > 0 \end{cases} = -\frac{1}{2}[\sup_{u\in C,\|u\|\le 1}\langle x^*,u\rangle]^2 .$$

Q.E.D.

It is worth remarking that any $x \neq 0$ in $P_C x^*$ is of the form $\langle x^*,u\rangle^+ u$, where u is a vector in C maximizing $\langle x^*,v\rangle^+$, so that $P_C x^*$ is simply obtained by looking for the directions in C making the smallest angle with x* and projecting on them in the ordinary sense. This geometrical definition may very well be taken as the point of departure for the theory of conical projectors. It is indeed the idea of "least angle mapping" what lies at the roots of projectors. J.P. Aubin has used this idea to define projectors on linear spaces [1].

THEOREM 7.

$$\|P_C x^*\|^2 = \langle x^*, P_C x^*\rangle = [\sup_{u\in C,\, \|u\|\leq 1} \langle x^*, u\rangle]^2 = \delta^2_{C^\perp}(x^*), \tag{15}$$

where $\delta_{C^\perp}(x^*)$ *denotes the distance from* x^* *to* $C^\perp$.

Proof. Only the last equality requires a proof. By Theorem 1, Corollary 3,

$$\langle x^*, P_C x^*\rangle - \frac{1}{2}\|P_C x^*\|^2 = (Q+\psi_C)^*(x^*) = (Q^* \square \psi_C^*)(x^*) = (Q^* \square \psi_{C^\perp})(x^*) =$$

$$= \inf_{y^*\in C^\perp} \frac{1}{2}\|x^*-y^*\|^2 = \frac{1}{2}\delta^2_{C^\perp}(x^*).$$

Since $\langle x^*, P_C x^*\rangle - \frac{1}{2}\|P_C x^*\|^2$ is equal to both $\frac{1}{2}\|P_C x^*\|^2$ and $\frac{1}{2}\langle x^*, P_C x^*\rangle$, the theorem is proved.

COROLLARY 1. $\eta(P_C) = C^\perp$.

COROLLARY 2.

$$P_C x^* = \partial \frac{1}{2}\|P_C x^*\|^2 = \partial\frac{1}{2}\delta^2_{C^\perp}(x^*). \tag{16}$$

Next theorem establishes a relation between projectors and nearest point mappings.

THEOREM 8. $(I^*-JP_C)x^* \cap C^\perp$ *is the set of points in* $C^\perp$ *closest to* x^*. (I^* *denotes the identity map in* X^*).

Proof. If $z^* \in (I^*-JP_C)x^* \cap C^\perp$ then $x^*-z^* \in JP_C x^*$ and $\|x^*-z^*\| = \|JP_C x^*\| = \|P_C x^*\| = \delta_{C^\perp}(x^*)$, which shows that z^* minimizes the distance from x^* to points in $C^\perp$.

Conversely, if $z^* \in C^\perp$ realizes the distance from x^* to $C^\perp$, then $\frac{1}{2}\delta^2_{C^\perp}(x^*) = \frac{1}{2}\|x^*-z^*\|^2$. Since on the other hand $\frac{1}{2}\delta^2_{C^\perp}(y^*) \leqslant \frac{1}{2}\|y^*-z^*\|^2$ for all $y^* \in X^*$, and since $\partial\frac{1}{2}\delta^2_{C^\perp}(x^*) = P_C x^*$ (Corollary above),

$$\frac{1}{2}\|y^*-z^*\|^2 - \frac{1}{2}\|x^*-z^*\|^2 \geqslant \frac{1}{2}\delta^2_{C^\perp}(y^*) - \frac{1}{2}\delta^2_{C^\perp}(x^*) \geqslant \langle y^*-x^*, P_C x^*\rangle,\ y^*\in X^*,$$

whence by definition of subgradient,

$$P_C x^* \subset \partial\frac{1}{2}\|x^*-z^*\|^2 = J^{-1}(x^*-z^*),$$

that is, $z^* \in x^* - J\,P_C x^*$, completing the proof.

If we let $\Pi_{C^\perp}: X^* \to 2^{X^*}$ denote the nearest point mapping on $C^\perp$ we can give this theorem a form suggestive of Moreau's decomposition of a vector in Hilbert space along orthogonal directions in dual cones [3].

COROLLARY. For any $x^* \in X^*$ there are vectors u and v^* such that

(17) $$x^* \in Ju+v^*,\ u \in C,\ v^* \in C^\perp,\ \langle v^*,u\rangle = 0.$$

Moreover, if (17) holds then $u \in P_C x^*$ and $v^* \in \Pi_{C^\perp} x^*$.

Proof. The possibility of decomposition (17) follows from Theorem 1, Corollary 6 and the theorem above. As to the last part notice that if $v^* \in C^\perp$ and $\langle v^*,u\rangle = 0$ then $v^* \in \partial\psi_C(u)$, and apply Theorems 1 and 10. Projectors and nearest point mappings are the same objects in Hilbert space. If the identification of the space with its dual is made explicit this coincidence can be expressed by the equation

(18) $$\Pi_C = P_C J.$$

Now, is this relation characteristic of Hilbert space? We don't know, we only conjecture that it is. The following theorem gives some support to our contention.

THEOREM 9. *Let* X *and* X^* *be dual reflexive Banach spaces. Then if the duality mapping* $J: X \to 2^{X^*}$ *is bijective, and*

(19) $\Pi_C = P_C J$ *for all straight lines and hyperplanes* $C \subset X$,

(20) $\Pi_{C^*} = P_{C^*} J^{-1}$ *for all straight lines and hyperplanes* $C^* \subset X^*$,

X *is a Hilbert space.*

Proof. By Theorem 2, Corollary 1 all projectors are single valued, and on use of Theorem 8, (19) and (20) can be written in the form

$$(I-J^{-1}P_{C^\perp})x = P_C Jx \quad , \quad (I^*-J\,P_C)x^* = P_{C^\perp} J^{-1}x^*.$$

If in the first of these equations $P_{C^\perp}x$ is replaced by its expression derived from the last one obtains

$$(I-P_C J)x = J^{-1}(J-JP_C J)x$$

that is,

$$J(x-P_C Jx) = Jx - JP_C Jx.$$

In a similar manner

$$J^{-1}(x^*-P_{C^*}J^{-1}x^*) = J^{-1}x^* - J^{-1}P_{C^*}J^{-1}x^*.$$

Making in the above equations the following identifications

$$C = \{tu\}_{-\infty<t<+\infty}\ ,\ C^* = \{tJu\}_{-\infty<t<+\infty}\ ,\ x = v,\ x^* = Jv$$

where u and v are any two unit vectors, one gets

$$J(v - \langle Jv,u\rangle\, u) = Jv - \langle Jv,u\rangle\, Ju$$

$$J(v - \langle Ju,v\rangle\, u) = Jv - \langle Ju,v\rangle\, Ju\ .$$

Set $r = v-\beta u$, $s = v-\alpha u$, $\alpha = \langle Ju,v\rangle$, $\beta = \langle Jv,u\rangle$, and on use of these identities proceed to the following calculations:

$$\|r\|^2 = \langle Jr,r\rangle = \langle Jv-\beta Ju, v-\beta u\rangle = 1+\beta^2-\beta^2-\beta\alpha = 1-\alpha\beta$$

$$\|s\|^2 = \langle Js,s\rangle = \langle Jv-\alpha Ju, v-\alpha u\rangle = 1+\beta^2-\alpha\beta-\alpha^2 = 1-\alpha\beta$$

$$\langle Jr,s\rangle = \langle Jv-\beta Ju, v-\alpha u\rangle = 1+\alpha\beta-\alpha\beta-\alpha\beta = 1-\alpha\beta.$$

Therefore, $\langle Jr,s\rangle = \|Jr\|^2 = \|s\|^2$ and by definition of J, Jr = Js. This implies r=s, which in turn yields $\alpha=\beta$, that is, $\langle Ju,v\rangle = \langle Jv,u\rangle$. This equation, valid for unitary u and v, is at once extended to all u's and v's in X by use of the homogeneity of J. But then J is a self adjoint mapping of X onto X*, and as such linear. It follows that $\|x\|^2 = \langle Jx,x\rangle$ is a quadratic form, and the theorem is proved.

Theorem 4 takes a simpler form in the case of conical projectors:

THEOREM 10. *A positive homogeneous-subdifferential operator* $P: X^* \to 2^X$ *is a conical projector if and only if it satisfies*

(21) $$Px^* = \partial \tfrac{1}{2}\, \|Px^*\|^2.$$

Proof. It follows from Theorem 4, and equation (15) that a conical projector satisfies (21). Conversely, if a positive homogeneous subdifferential P satisfies (21), then, since it also satisfies $Px^* = \partial \frac{1}{2} \langle x^*,Px^*\rangle$, [9], $\|Px^*\|^2 = \langle x^*,Px^*\rangle$ (use the fact that $P0^*=0$), that is $\frac{1}{2}\|Px^*\|^2 = \langle x^*,Px^*\rangle - \frac{1}{2}\|Px^*\|$. Hence, (10) holds for P, and P is a projector.

COROLLARY. A positive homogeneous subdifferential operator $P: X^* \to 2^X$ is a conical projector if and only if

(22) $$\|Px^*\|^2 = \langle x^*,Px^*\rangle\ ,\ \forall\ x^* \in \mathcal{D}(P).$$

Proof. Necessity is contained in Theorem 7. If, on the other hand, P is a subdifferential operator satisfying (22), then $Px^* = \partial \frac{1}{2} \langle x^*,Px^*\rangle = \partial \frac{1}{2} \|Px^*\|^2$, and I is a projector by the above theorem.

Now we turn our attention to the important question of when a sum of projectors is a projector.

THEOREM 11. $\sum_1^n P_{C_i}$ *is a conical projector if and only if*

$$(23) \qquad \| \sum_1^n P_{C_i} x^* \|^2 = \sum_1^n \| P_{C_i} x^* \|^2 .$$

In such a case

$$\sum_1^n P_{C_i} = P_{\sum_1^n C_i} .$$

Proof. This is a particular case of Theorem 5. The constant in equation (11) is zero because all P_{C_i}'s vanish at $x^* = 0$.

It may be checked that if all C_i's are rays: $\{tu_i\}_{t \geq 0}$, $\|u_i\| = 1$, (23) simply says that $\|x\|^2$ is quadratic over the n-hedron $\{\sum_1^n t_i u_i\}_{t_i \geq 0}$, and that the u_i's are orthogonal with regard to the induced scalar product, or more briefly, that $\{\sum_1^n C_i, \| \ \|\}$ is a 2^n-tant of an n-dimensional Hilbert space. Based on this remark the system of n cones satisfying the Pythagorean relation (23) may be conceived as a generalization of an orthogonal n-tuple of vectors where the vectors are replaced by cones. Accordingly we shall say that such cones form an orthogonal n-tuple, and shall use the notation $C_1 \perp C_2 \perp \ldots \perp C_n$ or $P_{C_1} \perp P_{C_2} \perp \ldots \perp P_{C_n}$ to denote this fact. It is remarkable how much of the Hilbert space structure is brought into the space by the requirement that a projector should split into the sum of others.

THEOREM 12. $C_1 \perp C_2 \perp \ldots \perp C_n$ *if and only if*

$$(24) \qquad \inf_{\sum_1^n x_i = x,\, x_i \in C_i} \sum_1^n \|x_i\|^2 = \|x\|^2 , \ \forall \ x \in C_1 + C_2 + \ldots + C_n .$$

In such a case the infimum is always attainable.

Proof. $C_1 \perp C_2 \perp \ldots \perp C_n$ is equivalent to

$$\sum_1^n \frac{1}{2} \|P_{C_i} x^*\|^2 = \frac{1}{2} \|P_{\sum_1^n C_i} x^*\|^2 ,$$

which by taking conjugates and recalling that the conjugate of $\frac{1}{2} \|P_C x^*\|^2$ is $\frac{1}{2} \|x\|^2 \perp \psi_C(x)$ (Theorem 1, Corollary 3, and (15)) becomes (24). To see that the infimum is attained take n sequences $\{x_i^{(k)}\}_1^\infty \subset C_i$, such that $\sum_{i=1}^n \|x_i^{(k)}\|^2 \downarrow \|x\|^2$, $\sum_{i=1}^n x_i^{(k)} = x$. Since the sequences are obviously bounded they can be assumed to be weakly convergent to limits x_i in C_i respectively. Then, the limit inferior of the norms being larger than the norm of the weak limit, we must have

$\sum_{i=1}^{n} \|x_i\|^2 \leqslant \|x\|^2$, $\sum_1^n x_i = x$, that is $\sum_{i=1}^{n} \|x_i\|^2 = \|x\|^2$, $\sum_{i=1}^{n} x_i = x$.
(Briefer but less direct: $R(\sum_1^n P_{C_i}) = R(P_{\sum_1^n C_i}) = \sum_1^n C_i$).

For the inversion of the statement: If $C_1 \perp C_2 \perp \ldots \perp C_n$, then $\{x_i \in P_{C_i}x^*,\ i = 1,2,\ldots n\} \Rightarrow \{\|\sum_1^n x_i\|^2 = \sum_1^n \|x_i\|^2\}$, we need a couple of lemmas.

LEMMA 1. *Let* $C_1 \perp C_2 \perp \ldots \perp C_n$. *Then*
$\{J(\sum_1^n x_i) \cap J(\sum_1^n x_i') \neq \emptyset,\ \|\sum_1^n x_i\|^2 = \sum_1^n \|x_i\|^2,\ \|\sum_1^n x_i'\|^2 = \sum_1^n \|x_i'\|^2,$
$x_i, x_i' \in C_i,\ i = 1,2,\ldots,n\}$ *implies* $\{Jx_i \cap Jx_i' \neq \emptyset,\ i = 1,2,\ldots,n\}$.

Proof. From $J(\sum_1^n x_i) \cap J(\sum_1^n x_i') \neq \emptyset$ it follows $\|t(\sum_1^n x_i)+(1-t)(\sum_1^n x_i')\|^2 =$ $=$ const., for $0 \leqslant t \leqslant 1$. Then,

$$\|t(\sum_1^n x_i)+(1-t)(\sum_1^n x_i')\|^2 = t\|\sum_1^n x_i\|^2+(1-t)\ \|\sum_1^n x_i'\|^2 = \sum_1^n (t\|x_i\|^2+(1-t)\|x_i'\|^2) \geqslant$$
$$\geqslant \sum_1^n \| tx_i+(1-t)x_i'\|^2,\quad 0 \leqslant t \leqslant 1\ ,$$

and by Theorem 12, since $tx_i + (1-t)x_i' \in C_i$,

$$\sum_1^n \|tx_i+(1-t)x_i'\|^2 \geqslant \|\sum_1^n (tx_i+(1-t)x_i')\|^2 = \|t \sum_1^n x_i+(1-t) \sum_1^n x_i'\|^2,$$

so

$$\sum_1^n \|tx_i+(1-t)x_i'\|^2 = \|t \sum_1^n x_i+(1-t) \sum_1^n x_i'\|^2 = \text{const}.$$

Now, the sum of the squares of convex functions being constant if and only if the individual terms are constant, we must have, $\|tx_i+(1-t)x_i'\|^2 =$ const., $0 \leqslant t \leqslant 1$, from which it follows $Jx_i \cap Jx_i' \neq \emptyset$, $i = 1,2,\ldots,n$. Q.E.D.

LEMMA 2. *If* $C_1 \perp C_2 \perp \ldots \perp C_n$, *then*
$\{\|\sum_1^n x_i\|^2 = \sum_1^n \|x_i\|^2,\ x_i \in C_i\} \Rightarrow \{P_{C_i}x^* \in J^{-1}Jx_i,\ i = 1,2,\ldots,n\ ,$
$\forall\ x^* \in J(\sum_1^n x_i)\}$.

Proof. By Theorems 3 and 11,

$$P_{\sum_1^n C_i} x^* = J^{-1}x^* \cap (\sum_1^n C_i) = \sum_1^n P_{C_i}x^*, \qquad \forall\ x^* \in J(\sum_1^n x_i).$$

Hence, if $x_1', x_2', \ldots, x_n'$ are any n points in $P_{C_1}x^*, P_{C_2}x^*, \ldots, P_{C_n}x^*$ respectively, $\sum_1^n x_i' \in J^{-1}x^*$, and $x^* \in J(\sum_1^n x_i) \cap J(\sum_1^n x_i')$.

The lemma above then yields $Jx_i \cap Jx_i^! \neq \emptyset$, that is, $x_i^! \in J^{-1}Jx_i$, $i = 1,2,\dots,n$, and since $x_i^!$ is any point in $P_{C_i}x^*$, $P_{C_i}x^* \subset J^{-1}Jx_i$, $i = 1,2,\dots,n$.

THEOREM 13. *If* $C_1 \perp C_2 \perp \dots \perp C_n$, *then*

$$(25)\ \{\|\sum_1^n x_i\|^2 = \sum_1^n \|x_i\|^2,\ x_i \in C_i,\ i = 1,2,\dots,n\} \iff$$
$$\iff \{x_i \in P_{C_i}x^*,\ i = 1,2,\dots,n,\ \forall\ x^* \in J(\sum_1^n x_i)\}$$

Proof. Assume that the proposition on the left holds. Then, by last lemma, $\|P_{C_i}x^*\| = \|x_i\|$, $x^* \in J(\sum_1^n x_i)$, and so, since $\|x_i\| = \|P_{C_i}x^*\| = \sup_{u_i \in C_i, \|u_i\| \leq 1} \langle x^*,u_i\rangle$, $\langle x^*,x_i\rangle - \|x_i\|^2 \leqslant 0$, $i = 1,2,\dots,n$, and adding up these inequalities,

$$\sum_1^n [\langle x^*,x_i\rangle - \|x_i\|^2] = \langle x^*,\sum_1^n x_i\rangle - \sum_1^n \|x_i\|^2 = \|\sum_1^n x_i\|^2 - \|\sum_1^n x_i\|^2 = 0$$

Therefore, $\langle x^*,x_i\rangle = \|x_i\|^2 = \|P_{C_i}x^*\|^2 = [\sup_{u_i \in C_i, \|u_i\| \leq 1} \langle x^*,u_i\rangle]^2$, and by (14) $x_i \in P_{C_i}x^*$, proving the implication from left to right. The opposite implication is but a quantification of (23).

COROLLARY.

$$(26)\ \{\sum_1^n P_{C_i} = P_{\sum_1^n C_i}\} \Rightarrow \{P_{C_i}x^* \subset P_{C_i}JP_{\sum_1^n C_j}x^* \subset J^{-1}JP_{C_i}x^* \cap C_i\ ,$$
$$i = 1,2,\dots,n,\ \ \forall\ x^* \in X^*\}.$$

Proof. Let $x_1,x_2,\dots,x_n$ be points in $P_{C_1}x^*,P_{C_2}x^*,\dots,P_{C_n}x^*$ respectively. Then by Lemma 2 and the theorem above $x_i \in P_{C_i}J(\sum_1^n x_i) \subset$ $\subset J^{-1}Jx_i \cap C_i$, $i = 1,2,\dots,n$, whence (26) follows from the fact that when the x_i's range over the sets $P_{C_i}x^*$, $\sum_1^n x_i$ ranges over $P_{\sum_1^n C_i}x^*$.

REMARK. By (8) $J^{-1}JP_{C_i}x^* \cap C_i = P_{C_i}JP_{C_i}x^*$, so that the right member of (26) can be written in the form $P_{C_i}x^* \subset P_{C_i}JP_{\sum_1^n C_j}x^* \subset P_{C_i}JP_{C_i}x^*$.

Comparison with (8) prompts the conjecture that the last inclusion is not proper, that is, that $P_{C_i}JP_{\sum_1^n C_k} = P_{C_i}JP_{C_i}$. However, this is not true in general. Consider the following example:

Let X and X* be the dual two dimensional Banach space with norms:

$$\|x\| = \begin{cases} (|\xi_1|^2 + |\xi_2|^2)^{1/2}, & \xi_1\xi_2 \geq 0 \\ |\xi_1| + |\xi_2|, & \xi_1\xi_2 \leq 0 \end{cases}, \quad \|x^*\| = \begin{cases} (|\xi_1^*| + |\xi_2^*|)^{1/2}, & \xi_1^*\xi_2^* \geq 0 \\ \max(|\xi_1^*|,|\xi_2^*|), & \xi_1^*\xi_2^* \leq 0. \end{cases}$$

The second and fourth quadrants in X, wich we call C_1 and C_2 respectively, form an orthogonal couple, and $J^{-1} = P_{C_1}+P_{C_2}$. For any $x^* \in X^*$ in the first quadrant and away from the axes $JP_{C_1+C_2}x^* = x^*$, and $JP_{C_1}x^* = x_i^*$, where x_i is the Euclidean projection of x^* on the i-axis. Moreover, $P_{C_1} J P_{C_1+C_2}x^* = P_{C_1}x^*$ is a singleton x_1 on the 2-axis, whereas $P_{C_1}JP_{C_1}x^* = J^{-1}x^* \cap C_1$ is a straight line segment through x_1 across C_1 parallel to the first quadrant bisector. Obviously $P_{C_1} J P_{C_1+C_2}x^* \neq P_{C_1} J P_{C_1}x^*$.

All that has been said of conical projections from Theorem 11 on applies also to projections on general convex sets, the only difference being the presence of an additive constant all throughout.

THEOREM 14. *If* $C_1 \perp C_2 \perp \ldots \perp C_n$ *then*

$$P_{C_i}(tI^*+(1-t)\, J\, P_{\sum_1^n C_j}) = P_{C_i}, \quad 0 < t \leq 1,\ i=1,2,\ldots,n \tag{27}$$

Proof. For $x^* \in X^*$ and $y^* \in J\, P_{\sum_1^n C_j} x^*$ set $z^*(t) = tx^* + (1-t)y^*$, $0 < t \leq 1$. Now

$$\sup_{u_i\in C_i,\|u_i\|\leq 1} \langle z^*(t),u_i\rangle \leq t \sup_{u_i\in C_i,\|u_i\|\leq 1} \langle x^*,u_i\rangle + (1-t) \sup_{u_i\in C_i,\|u_i\|\leq 1} \langle y^*,u_i\rangle =$$

$$= t\|P_{C_i}x^*\| + (1-t)\|P_{C_i}y^*\|.$$

By (26) $\|P_{C_i}y^*\| = \|P_{C_i}x^*\|$ so,

$$\sup_{u_i\in C_i,\|u_i\|\leq 1} \langle z^*,u_i\rangle \leq \|P_{C_i}x^*\|.$$

Moreover, by hypothesis and choice of y^* there are points $x_i \in P_{C_i}x^*$, $i = 1,2,\ldots,n$ such that $y^* \in J \sum_1^n x_i$. Since $\langle x^*,x_i\rangle = \|x_i\|^2$ by (14), and $\langle y^*,x_i\rangle = \|x_i\|^2$ by (25), we have $\langle z^*(t),x_i\rangle = \|x_i\|^2 = \|P_{C_i}x^*\|^2$, $t = 1,2,\ldots,n$. In view of what has already been proved these equations mean that the suprema of $\langle z^*(t),u_i\rangle$, $\langle x^*,u_i\rangle$, $\langle y^*,u_i\rangle$ over the u_i's in C_i with $\|u_i\| \leq 1$ are attained simultaneously and are equal to $\|P_{C_i}x^*\|$. Then,

$$\{v_i \in P_{C_i}z^*(t)\} \iff \{\|v_i\| = \sup_{u_i\in C_i,\|u_i\|\leq 1} \langle z^*,u_i\rangle = \|P_{C_i}x^*\|,\ \langle z^*(t),v_i\rangle = \|P_{C_i}x^*\|\}$$

$$\Longleftrightarrow \{\|v_i\| = \sup_{u_i\in C_i,\|u_i\|\le 1} \langle x^*,u_i\rangle = \sup_{u_i\in C_i,\|u_i\|\le 1} \langle y^*,u_i\rangle = \|P_{C_i}x^*\|,\ \langle x^*,v_i\rangle = \|P_{C_i}x^*\|^2 =$$

$$= \langle y^*,v_i\rangle = \|v_i\|^2\} \Longleftrightarrow \{v_i \in P_{C_i}x^*,\ v_i \in P_{C_i}y^*\}$$

and hence

$$P_{C_i}(tx^*+(1-t)y^*) = P_{C_i}x^* \cap P_{C_i}y^*,\ 0 < t \leqslant 1\ ,\ i = 1,2,\dots,n.$$

Since y^* is any point in $J\,P_{\sum_1^n C_j}x^*$,

$$P_{C_i}(t\,x^* + (1-t)J\,P_{\sum_1^n C_j}x^*) = P_{C_i}x^* \cap P_{C_i}\,J\,P_{\sum_1^n C_j}x^*\ ,$$

and an appeal to the previous theorem concludes the proof.

COROLLARY. For any conical projector,

(28) $$P_C(tI^* + (1-t)J\,P_C) = P_C\ ,\quad 0 < t \leqslant 1.$$

Proof. Set in (27) $C_1 = C$, $C_2 = C_3 = \dots = C_n = \{0\}$.

The geometrical meaning of the relation $C_1 \perp C_2 \perp \dots \perp C_n$ is not sufficiently clear from defining Pythagorean relation (23), nor from (24). In Hilbert space each cone is the dual of the sum of the others relatively to the total sum [9, Equation 2.10]. A similar result holds in reflexive Banach spaces.

LEMMA 3.

(29) $$C_1 \perp C_2 \perp \dots \perp C_n \Rightarrow \{JC_j \subset (\sum_{i\neq j} C_i)^\perp\ ,\ j=1,2,\dots,n\}.$$

Proof. Let $x_j \in C_j$, $y_j^* \in Jx_j$. Then, since by (8) $x_j \in P_{C_j}y_j^* \subset J^{-1}Jx_j$,

$$\|x_j + \sum_{i\neq j} P_{C_i}y_j^*\|^2 = \sum_{i=1}^n \|P_{C_i}y_j^*\|^2 = \|x_j\|^2 + \sum_{i\neq j} \|P_{C_i}y_j^*\|^2$$

and by definition of J,

$$\sum_{i\neq j} \|P_{C_i}y_j^*\|^2 = \|x_j + \sum_{i\neq j} P_{C_i}y_j^*\|^2 - \|x_j\|^2 \geqslant 2\langle y_j^*, \sum_{i\neq j} P_{C_i}y_j^*\rangle = 2\sum_{i\neq j} \|P_{C_i}y_j^*\|^2.$$

Hence, $P_{C_i}y_j^* = 0$, that is, $y_j^* \subset C_i^\perp$, $i\neq j$, and $J\,C_j \subset \bigcap_{i\neq j} C_i^\perp = (\sum_{i\neq j} C_i)^\perp$.

THEOREM 15.

(30) $$C_1 \perp C_2 \perp \dots \perp C_n \Rightarrow C_j = J^{-1}[(\sum_{i\neq j} C_i)^\perp] \cap (\sum_{k=1}^n C_k)\ ,\ j = 1,2,\dots,n.$$

Proof. By Lemma 3, $C_j \subset J^{-1}[(\sum_{i\neq j} C_i)^\perp]$, and since $C_j \subset \sum_1^n C_k$,

$$C_j \subset J^{-1}[(\sum_{i\neq j} C_i)^\perp] \cap (\sum_1^n C_k).$$

This is half of (30). To prove the other half start with an x_j in $J^{-1}[(\sum_{i\neq j} C_i)^{\perp}] \cap (\sum_1^n C_k)$, and then observe that

$x_j \in C_1 + C_2 + \ldots + C_n$, $x_j \in J^{-1}x_j^*$, for some $x_j^* \in (\sum_{i\neq j} C_i)^{\perp} = \bigcap_{i\neq j} C_i^{\perp}$.

So

$$x_j^* \in Jx_j \subset J(C_1 + C_2 + \ldots + C_n)$$

and by Theorem 3,

$$x_j \in J^{-1}x_j^* \cap (C_1 + C_2 + \ldots + C_n) = P_{\sum_1^n C_k} x_j^* = \sum_1^n P_{C_k} x_j^* = P_{C_j} x_j^* \subset C_j,$$

and since x_j was any point in $J^{-1}[(\sum_{i\neq j} C_i)] \cap (\sum_1^n C_k)$,

$$J^{-1}[(\sum_{i\neq j} C_i)] \cap (\sum_1^n C_k) \subset C_j,$$

concluding the proof.

COROLLARY. In the relation $P_C = \sum_1^n P_{C_k}$ any n projectors determine the remaining one.

We do not know if the arrow in (30) can be reversed. The most that we can say is that this is the case in Hilbert spaces of dimension not larger than three.

§3. CONCLUSION AND COMMENTS.

The material set forth in the preceeding pages is essentially all we know about projectors in reflexive Banach spaces. No doubt the discussion can be carried further still, and we hope that it will be, for, as it stands the extent of our knowledge is insufficient for the proper development of a spectral theory. Let us point out here to some of the most visible shortcomings.

In the first place it is not known if the relation $P_{C_1} > P_{C_2}$, defined as meaning that $P_{C_1} - P_{C_2}$ is a projector, is a partial ordering for projectors. Indeed, there is no proof of it being transitive.

Important as transitivity is, spectral theory requires something stronger still, namely that any sub k-tuple of an orthogonal n-tuple of cones be again orthogonal. This is necessary if the spectral measure built out of a spectral resolution is to be projector-valued. In Hilbert space this is a consequence of $P_{C_1} > P_{C_2}$ being equivalent to $P_{C_2}JP_{C_1} = P_{C_2}$. No such equivalence has been established in reflexive Banach spaces, we only know that if J^{-1} is single valued $P_{C_1} > P_{C_2}$ implies $P_{C_2}JP_{C_1} = P_{C_2}$ (Corollary, Theorem 13).

Another important property, which in Hilbert space lies buried under the homogenity of orthogonality, is the following:

If $C_1 \perp C_2 \perp \ldots \perp C_n$, and $x_i \in C_i$, $i = 1,2,\ldots,n$, then

$$(\|\sum_1^n x_i\|^2 = \sum_1^n \|x_i\|^2) \Rightarrow \{\|\sum_1^n \alpha_i x_i\|^2 = \sum_1^n \alpha_i^2 \|x_i\|^2 , \alpha_i \geqslant 0\}.$$

The whole of functional calculus is based on it. Needless to say that we have no evidence that it holds in reflexive Banach spaces.

These examples should suffice to show the need of further research. Maybe some of the sought properties are not valid in general. If so, we anticipate serious difficulties in bringing such facts to light, for the construction of counterexamples is a hard task in this field.

REFERENCES

[1] AUBIN,J.P., *Optimal approximation and characterization of the error and stability function in Banach spaces*, J.Approx. Th.3 (1970) 430-444.

[2] BROENDSTED,A. and ROCKAFELLAR,R.T., *On the subdifferentiability of convex functions*, Proc.Am.Math.Soc., 16(1966), 605-611.

[3] MOREAU,J.J., *Décomposition orthogonale dans un espace hilbertien selon deux cônes mutuellement polaires*, C.R.Acad.Sci.Paris, 255 (1962), 233-240.

[4] ROCKAFELLAR,R.T., *Local boundedness of monotone operators*, Mich. Math.J., 16(1969), 397-407.

[5] ROCKAFELLAR,R.T., *On the virtual convexity of the domain and range of nonlinear maximal monotone operators*, Math.Ann. 185 (1970), 81-90.

[6] ROCKAFELLAR,R.T., *Extension of Fenchel's duality theorem for convex functions*, Duke Math.J., 33(1966), 81-90.

[7] ZARANTONELLO,E.H., *Projections on convex sets and spectral theory*, in Contributions to Nonlinear Functional Analysis, Acad. Press, New York, 1971, pp. 237-424.

[8] ZARANTONELLO,E.H., *The product of commuting projections is a projection*, Proc.Am.Math.Soc., 38(1973), 591-593.

[9] ZARANTONELLO,E.H., *L'algèbre des projecteurs coniques*, in Analyse convexe et ses applications, pp. 232-243. Lecture notes in Economics and Mathematical Systems, No.102, Springer Verlag, 1974.

[10] ZARANTONELLO,E.H., *The meaning of the Cauchy-Schwarz-Buniakovsky inequality*, Proc.Am.Math.Soc., 58(1976), 133-138.

Mathematics Research Center
University of Wisconsin
U.S.A.

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