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ELEMENTARY GEOMETRY OF THE UNSYMMETRIC MINKOWSKI PLANE

H. Guggenheimer

Dedicated to Luis A. Santaló

1.

Plane unsymmetric Minkowski geometry is given by a proper convex body Z in the affine plane and a point $0 \in \operatorname{int} Z$. Z is called the *indicatrix* of the geometry; it defines a pseudonorm for any vector x: Write $x = OX$, then

$$\|x\| = \inf \lambda \mid X \subset \lambda Z. \qquad (1)$$

Then $\|x\| \geqslant 0, \quad \|x\| = 0$ only if $x = 0$,

$$\|\alpha x\| = \alpha\|x\| \quad \text{if} \quad \alpha \geqslant 0 \ ,$$

$$\|x + x'\| \leqslant \|x\| + \|x'\| \ .$$

The pseudonorm is a norm, $\|\alpha x\| = |\alpha| \ \|x\|$ for all x and all $\alpha \in R$, if and only if Z is symmetric of center O, $Z = -Z$. (All operations of vector algebra will be taken for the origin at O). The elementary geometry of the symmetric Minkowski plane was studied in detail by C.M.Petty [9], we are interested in the unsymmetric case. Although we are going to use trigonometry and analytic geometry, no smoothness conditions will be imposed on Z.

We shall parametrize the convex curve ∂Z by t, two times its polar area function in the sense of polar coordinates, relative to a fixed polar axis. (For an arbitrary monotone and continuous parameter τ on ∂Z, t is a Stieltjes integral $t = \int_{\tau_0}^{\tau} \det(z(\sigma),dz(\sigma))$). Let $\hat{Y}$ be the dual of Z [7]. The curve $\partial\hat{Y}$ will be referred to s, two times the area function in the sense of polar coordinates computed from the polar axis. In a homothety of ratio c, the area of Z is multiplied by c^2 and that of $\hat{Y}$ by c^{-2}. Since (1) is affine and the dual is an affine covariant of Z, we may *normalize* the geometry by requiring

$$\text{Area } (Z) = \text{Area } (\hat{Y}) \ .$$

We shall assume from now on that we work with a normalized geometry (or, that we have defined the unit area so as to make the original body Z of area equal to that of its dual). One can show that $3\sqrt{3}/2 \leqslant \Pi \leqslant \pi$

in general, where Π denotes the area common to Z and $\hat{Y}$ [5] . For symmetric geometries, $2\sqrt{2} \leqslant \Pi \leqslant \pi$.

To any vector z(t) from 0 to ∂Z there correspond all vectors y(s) from 0 to $\partial\hat{Y}$ for which $z(t).y(s) = 1$. Therefore, if Y is the image of $\hat{Y}$ in the rotation of angle $-\frac{\pi}{2}$ and center 0, the relation between the *isoperimetrix* Y [2] and the indicatrix is given by

$$\det(y(s),z(t)) = 1 \ . \tag{2}$$

Since Y also is a proper convex body, it defines a norm $\| x \|$. The relation (2) defines a map Φ of the circle $S = R/2\Pi$ onto arcs of S:

$$\Phi(t) = \{a(t) \leqslant s \leqslant b(t)\}$$

by: $-y[\varphi(t)]$ is the oriented direction of a support line of Z at z(t),

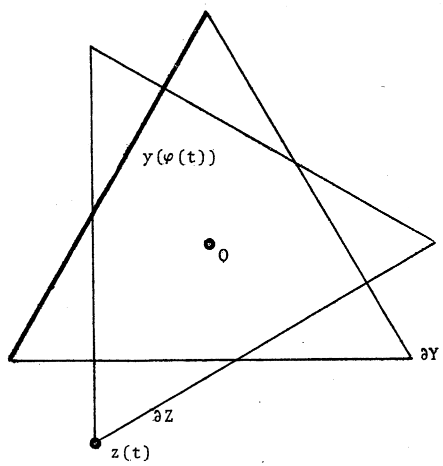

Fig. 1

where we denote by $\varphi(t)$ any $s \in \Phi(t)$. The map Φ satisfies

a) $a(t) = b(t)$ if Z has a unique line of support at z(t)

b) $\mathrm{int}\Phi(t_1) \cap \mathrm{in}\Phi(t_2) \neq \emptyset$ implies $\Phi(t_1) = \Phi(t_2)$

c) $\bigcup\limits_{t\in S} \Phi(t) = S$.

Since the dual of the dual is the original convex body, (2) defines in the same way a map $\Psi(s) = \{A(s) \leqslant t \leqslant B(s)\}$, with properties a)-c) for s. Clearly, $s \in \Phi \circ \Psi(s)$, $t \in \Psi \circ \Phi(t)$. For the endpoints of the intervals we shall write $a(t) = \varphi_-(t)$, $b(t) = \varphi_+(t)$. Similarly, the interval $\Psi(s)$ is written $\Psi_-(s) \leqslant \Psi(s) \leqslant \Psi_+(s)$.

We shall pair vectors in a frame

$$\begin{bmatrix} y(s) \\ z(t) \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -z(t) \\ y(s) \end{bmatrix}$$

only if $s = \varphi(t)$, $t = \Psi(s)$. Since the frames are unimodular, so is the matrix in

$$\begin{bmatrix} y(s) \\ z(t) \end{bmatrix} = \begin{bmatrix} cm_s(s,s_o) & st(t_o,s) \\ -sm(s_o,t) & cm_t(t_o,t) \end{bmatrix} \begin{bmatrix} y(s_o) \\ z(t_o) \end{bmatrix} \tag{3}$$

whose elements are the *trigonometric functions* of the geometry. It follows that the "cosine" function cm is "even",

$$cm_s(s,s_o) = cm_t(t,t_o)$$

and the "sine" functions sm and st are odd:

$$sm(s_o,t) = -sm(s,t_o)$$
$$st(t_o,s) = -st(t,s_o) \ .$$

As a consequence, we drop the indices of the cm-functions since the arguments alone identify the functions. These functions have been studied in detail in [5] for smooth indicatrices; here we note only that

$$s,s_o \in \Phi(t) \text{ implies } cm(s,s_o) = 1 \ , \ sm(s_o,t) = sm(s,t) = 0$$
$$t,t_o \in \Psi(s) \text{ implies } cm(t_o,t) = 1 \ , \ st(t,s) = st(t_o,s) = 0 \ .$$

Let t^*, s^* be the values of t and s, respectively, for which $z(t^*)$ has the direction of $-z(t)$, $y(s^*)$ the direction of $-y(s)$. Then

$$sm(s_o,t) = 0 \text{ implies } s_o = \varphi(t) \text{ or } s_o = \varphi(t^*)$$
$$st(t_o,s) = 0 \text{ implies } t_o = \Psi(s) \text{ or } t_o = \Psi(s^*) \ .$$

2.

All triangles will be *oriented*. For a triangle ABC, the leg a is the vector $a = \overrightarrow{BC}$. All notations allow for cyclic permutations. We write $a = \|a\| Z(t_a) = \|a| Y(s_a)$; this defines the angle variables. We have to distinguish several notions of orthogonality (really, transversality in the sense of the Calculus of Variations).

A vector v is *orthogonal to* a vector $x = \|x\| Z(t)$ if $v = |v| Y(s)$, $s = \varphi(t)$. The orthogonal direction is unique only if $\Phi(t)$ is a single point. A vector v is *orthogonal from* $x = |x| Y(s)$ if $v = \|v\| Z(t)$, $t = \Psi(s)$. An *altitude* h_a is a vector orthogonal *from* A *to* a. The height is $|h_a|$. The area Δ of the triangle is

$$\Delta = \frac{1}{2} \det(h_a,a) = \frac{1}{2} \|a\| |h_a| \det(Y(s_{h_a}),Z(t_a)) = \frac{1}{2} \|a\| |h_a| \ . \tag{4}$$

This is the Minkowski form of the area formula; it allows for cyclic permutation. Without the norm defined by Y, $\|a\| \|h_a\| = \|b\| \|h_b\| =$

$= \|c\|\|h_c\|$ holds only if Z and Y are homothetic, i.e., if they are Radon curves; this is a theorem of Tamâssy [10].

From a+b+c = 0 we get the cosine theorem

$$\|a\| + \|b\| \operatorname{cm}(t_a,t_b) + \|c\| \operatorname{cm}(t_a,t_c) = 0 \tag{5}$$

for the components in the direction of $Z(t_a)$ and the sine theorem

$$\frac{\|b\|}{\operatorname{sm}(\varphi(t_c),t_a)} = \frac{\|c\|}{\operatorname{sm}(\varphi(t_a),t_b)} = \frac{\|a\|}{\operatorname{sm}(\varphi(t_b),t_c)} \tag{6}$$

for the transversal components. The sm-function was first defined by Busemann [2] who also found the sine theorem (for symmetric metric) from the area formula

$$\Delta = \frac{1}{2}\det(a,b) = \frac{1}{2}\|a\|\|b\| \det(Z(t_a),Z(t_b)) =$$
$$= \frac{1}{2}\|a\|\|b\| \operatorname{sm}(\varphi(t_a),t_b) .$$

In the norm of Y, the formula $\Delta = \frac{1}{2}\mathbf{|}a\mathbf{|}\mathbf{|}b\mathbf{|} \operatorname{st}(\Psi(s_a),s_b)$ which yields

$$\frac{\mathbf{|}a\mathbf{|}}{\operatorname{st}(\Psi(s_b),s_c)} = \frac{\mathbf{|}b\mathbf{|}}{\operatorname{st}(\Psi(s_c),s_a)} = \frac{\mathbf{|}c\mathbf{|}}{\operatorname{st}(\Psi(s_a),s_b)} \tag{7}$$

We say that ABC is a *right triangle* if $b = h_a$. Then

$$-c = \|a\| Z(t_a) + \mathbf{|}b\mathbf{|} Y(\varphi(t_a))$$
$$= \|-c\| [\operatorname{cm}(t_a,t_c^*)Z(t_a) - \operatorname{sm}(\varphi(t_a),t_c^*)Y(\varphi(t_a))]$$

or $\quad \|a\| = \|-c\| \operatorname{cm}(t_a,t_c^*) = \|c\| \, |\operatorname{cm}(t_c,t_c^*)| \operatorname{cm}(t_a,t_c^*)$

$\quad \mathbf{|}b\mathbf{|} = \|-c\| \, |\operatorname{sm}(\varphi(t_a),t_c^*)| = \|c\| \, |\operatorname{cm}(t_c,t_c^*) \operatorname{sm}(\varphi(s_a),t_c^*)|$.

Since the determinant of the matrix in (3) is 1,

$$\|-c\|^2 = \|-c\| \operatorname{cm}(t_a,t_c^*)\|-c\| \operatorname{cm}(t_c^*,t_a) + \|-c\| \operatorname{sm}(\varphi(t_a),t_c^*)\|-c\| \operatorname{st}(t_c^*,\varphi(t_a)) ;$$

we obtain the generalization of the theorem of Pythagoras

$$\|-c\|^2 = \|a\|^2 \frac{\operatorname{cm}(t_c^*,t_a)}{\operatorname{cm}(t_a,t_c^*)} + \mathbf{|}b\mathbf{|}^2 \frac{\operatorname{st}(t_a,\varphi(t_c^*))}{\operatorname{sm}(\varphi(t_a),t_c^*)} . \tag{8}$$

This leads to a characterization of euclidean geometry. We start with

LEMMA 1. *If* $\operatorname{cm}(t_o,t) = \operatorname{cm}(t,t_o)$ *for all* t,t_o *then the geometry is euclidean.*

Denote the matrix in (3) by $M(t,t_o;s,s_o)$. From the composition formula $M(t_1,t;s_1,s)M(t,t_o;s,s_o) = M(t_1,t_o;s_1,s_o)$ and the fact that the diagonal elements in M are equal it follows that M is of the form

$$M = \begin{bmatrix} a & b \\ qb & a \end{bmatrix} .$$

A coordinate transformation of matrix $\mathrm{diag}(1,|q|^{-\frac{1}{2}})$ then transforms all M into the form $\begin{bmatrix} a & b \\ \pm b & a \end{bmatrix}$. Since Z is convex, Z is in the halfplane of the line of direction $y(s_o)$ through $z(t_o)$ that contains 0, therefore

$$cm(t_o,t) \leqslant 1 \quad \text{for all} \quad t_o,t\ . \tag{9}$$

(The inequality in absolute value holds only in symmetric geometries). For t sufficiently close to t_o, the diagonal elements in M are positive and < 1 in absolute value. Since $\det M = 1 = a^2 \mp b^2$ and $a^2 < 1$, it follows that q is negative and M is a rotation matrix. Since q is the same for all M, all M are rotation matrices and both Z and Y are the unit circle.

If $cm(t_c^*,t_a) = \lambda cm(t_a,t_c^*)$ for all directions a and c with λ independent of a and c, it follows by a change of names that $\lambda = 1$ and the geometry is euclidean by the lemma.

If $st(t_a,\varphi(t_c^*)) = \mu sm(\varphi(t_a),t_c^*)$ for all directions with constant μ it follows that the linear dependence of $Y(\varphi(t_o))$ and $Y(\varphi(t))$ implies the linear dependence of $Z(t_o)$ and $Z(t)$: $t_o = \Psi\circ\varphi(t_o)$, $t = \Psi\circ\varphi(t)$. Therefore φ (and Ψ) are point-valued functions, Z and Y are rotund ovals. Also, the map by oppositely oriented parallel lines of support is the map which commutes with φ (and Ψ). Hence, both Z and Y are symmetric ovals. From the composition formula of the matrices M we get the addition formulas

$$\begin{aligned} sm(s_o,t_2) &= sm(s_1,t_2)cm(s_1,s_o)+cm(t_1,t_2)sm(s_o,t_1) \\ st(t_o,s_2) &= cm(s_2,s_1)st(t_o,s_1)+st(t_1,s_2)cm(t_o,t_1). \end{aligned} \tag{10}$$

Under our hypothesis, we either have $\mu = 0$ which is impossible or simultaneously

$$st(t_o,s_2) = st(t_1,s_2)cm(t_1,t_o)+cm(t_1,t_2)st(t_o,s_1)$$

$$st(t_o,s_2) = st(t_1,s_2)cm(t_o,t_1)+cm(t_2,t_1)st(t_o,s_1)$$

i.e., $st(t_1,t_2)[cm(t_1,t_o)-cm(t_o,t_1)] = st(t_o,s_1)[cm(t_2,t_1)-cm(t_1,t_2)]$.

Since the directions t_o,t_1,t_2 are arbitrary, they can be chosen so that $st(t_o,s_1) = 0$, $st(t_1,s_2) \neq 0$. Therefore, the cm-function is symmetric and the geometry is euclidean by Lemma 1. We have proved:

PROPOSITION 1. *If there exists a constant* λ *or a constant* μ *such that*

either $\quad \|-c\|^2 = \lambda\|a\|^2 + f(a,b)\|b\|^2$

or $\quad \|-c\|^2 = g(a,b)\|a\|^2 + \mu\|b\|^2$

holds for all right triangles, then the geometry is euclidean.

3.

One of the definitions of an angle bisector in euclidean geometry is the set of centers of the circles that touch both legs of an angle. For the isoperimetric inequality, the Minkowski analog of a euclidean circle is the isoperimetrix [3,5] and its homothetic images. The center of the circle is the image of O in the homothety. The radius of the circle is the ratio of homothety, or the $\| \ \|$-norm of any vector from the center to the circumference.

DEFINITION. A Y-*bisector* of ABC is the set of centers of the circles that touch two of the lines that carry the legs of the triangle in one fixed angle domain. An *interior* bisector is a bisector that contains interior points of the triangle and a vertex as endpoint. A bisector that is not interior is exterior. All bisectors are continua that have a vertex as only relative boundary point.

Since all circles that touch two concurrent rays are homothetic images of one another in homotheties centered at the vertex, each bisector is a straight ray through that vertex. Two concurrent straight lines define four concurrent-bisectors, they form two straight lines only in symmetric geometry. By construction, the intersection of two Y-bisectors of a triangle is the center of a tritangent circle :

PROPOSITION 2. *The interior* Y-*bisectors of a triangle are concurrent. There are three triples consisting of two exterior and one interior* T-*bisector each.*

The existence of the points of intersection is an easy consequence of Pasch's axiom.

The point of concurrence of the three interior bisectors is the center I of the *incircle,* the homothetic image of Y tangent to the three sides of the triangle. The contact is oriented if both ABC and ∂Y are positively oriented. Let I_a be a point of contact of the incircle and a. Then a is orthogonal *to* II_a and the area of IBC is $\frac{1}{2} r \|a\|$. Therefore, for $p = \frac{1}{2} [\|a\| + \|b\| + \|c\|]$, we have

$$\Delta = pr \tag{11}$$

just as in euclidean geometry.

Another definition of the bisector of an angle is as axis of symmetry or as line making equal but opposite angles with the legs:

DEFINITION. An sm-*bisector* of a and b at C is a line of direction parameter t directed towards C for which $sm(\varphi(t_a),t) = sm(\varphi(t_b),t)$.

PROPOSITION 3. *The* sm-*bisectors are the* Y-*bisectors.*

If ∂Y is not strictly convex then the incircle and a will have a segment in common and for some $\varphi(t_a)$ and some $\varphi(t_b)$ the condition is satisfied. Let I_a be the point on $\partial Y \cap a$ for which $s(II_a) = \varphi(t_a)$. Then

$$\| II \|_a = \det(II_a, Z(t_a)) = \det(IC, Z(t_a)) = \| IC \| \det(Z(t), Z(t_a)) =$$
$$= -\| IC \| \operatorname{sm}(\varphi(t_a), t) .$$

Therefore, there exists I_b such that $II_a = II_b$ and I is the center of a circle of radius II_a which touches both legs (and this holds for every point on the bisector, not just the incenter).
A similar theory holds for Z - and st-bisectors.

4.

DEFINITION. The *perpendicular Z-bisector* of a segment AB is the set of all points P for which $\| PA \| = \| PB \|$. The *perpendicular* Y-*bisector* is defined by $\| PA \| = \| PB \|$. A Z-(Y-) *midpoint* of AB is a point of the intersection of AB and its Z-(Y-) perpendicular bisector.

The Z-bisector of a segment may have nonzero twodimensional measure. For example, in the normalized geometry for which Z is the square of side length $2^{3/4}$ parallel the axes with center at O, let AB be a segment parallel the y-axis of length $2a < 2^{3/4}$. From the endpoints A,B we draw the lines parallel the diagonals of Z and get the diamond ACBD. Then for any P in one of the exterior vertical angles at C and D (the shaded domains in fig.2), $\| PA \| = \| PB \|$. All bisectors are zero-dimensional if Z and Y are rotund.

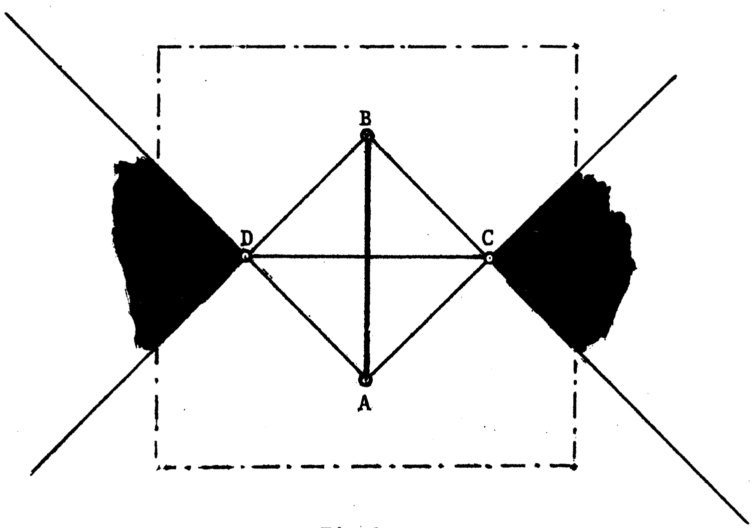

Fig.2

PROPOSITION 4. *The Z-(or Y-) midpoint of any segment is unique.*

For $P \in AB$, the ratio of division $\lambda = \|PA\| / \|PB\|$ is strictly monotone increasing and continuous, by the first and third properties of the pseudonorm. It increases from 0 to ∞, therefore it is = 1 at exactly one point.

In unsymmetric geometry there is no direct connection between the bisector sets in the two halfplanes defined by the line AB. For example, in the geometry defined by a triangle Z and a point $O \in \operatorname{int} Z$, the Z-bisectors p, p' of the segment PB, $P = BO \cap b$, are the rays complementary to the segments OC,OA. For any other segment P_1Q_1, $P_1 \in b$, $Q_1 \in a$, $O \in P_1Q_1$, the bisector p_1 in the halfplane opposite C is p but the bisector p_1' in the halfplane of C is the union of a segment OS on OC and a ray parallel p'.

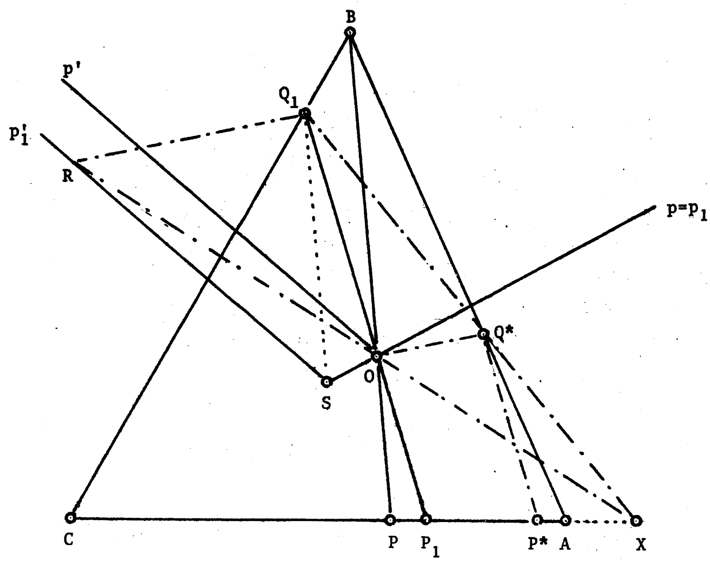

Fig. 3

A point R on that ray can be found as follows: Since $\|RQ_1\| = \|RP_1\|$, RQ_1P_1 is homothetic to a triangle OQ*P*; there is a one-to-one correspondence between $Q^* \in BA$ and R. The center X of the homothety is $b \cap Q_1Q^*$; R is the intersection of XO and the line through Q_1 parallel Q^*O. The locus of R is a conic as intersection of two projectively related pencils of lines (through O and Q_1). The line OQ_1 corresponds to itself in the projectivity. Hence, the projectivity is a perspectivity and the conic is a double line. Since $Q^* \to A$ implies $X \to A$, AO is an asymptote, i.e. $SR \parallel AO$. S is found by $Q_1S \parallel BO$.

Busemann [3] has shown that in any symmetric G-space the perpendicular bisectors are flat only if the geometry is Klein. The bisector of a symmetric Minkowski geometry are flat only if the geometry is euclidean.

The theorem can be extended to unsymmetric Minkowski geometry using an argument of Blaschke.

PROPOSITION 5. *A geometry in which all* Z-(*or* Y-) *perpendicular bisectors are straight lines is euclidean. The same conclusion holds if every* Z- (*or* Y-) *perpendicular bisector is the union of two straight rays and if, in addition, the* Z- (Y-) *perpendicular bisectors of two segments* AB, CD *intersecting at their common midpoint have only that midpoint in common.*

Let M be the midpoint of AB. (We use only Z, the argument for Y is identical). We prove first that the bisectors are straight lines if for any other segment CD with midpoint M the bisectors of AB and CD have only M in common. We may assume without loss of generality that $M=O$, $\|OA\| = \|OB\| = 1$. Let r_1, r_2 be the two rays that form the bisector. For $P \in r_1$, let $OA'B'$ be the homothetic image of PAB in the map that brings P onto O and $A', B' \in \partial Z$. Clearly, $A'B' \parallel AB$. $\|PO\| \longrightarrow \infty$ implies $\|A'B'\| \longrightarrow 0$. Therefore, the line $\overline{r}_1$ defined by r_1 intersects ∂Z at a point P" where a support line is parallel AB. The other point on ∂Z with support line parallel AB is $\overline{r}_2 \cap \partial Z$ by the same argument. By hypothesis, the couples of points of parallel support are in 1-1 order preserving correspondence with the directions through O: no point of ∂Z has more than one support line and no line more than one support point; ∂Z is rotund (strictly convex and smooth). For AB fixed, $A' \longrightarrow B'$ defines an affine relation of axis r_1 in the terminology of Veblen and Young [11] in one halfplane of AB. If $\overline{r}_2 \neq \overline{r}_1$ then at least one of A or B would admit two distinct support lines, since the support line at A cannot be the image of the support line at B in two elations with different axes. Hence, r_1 and r_2 are collinear.

Now let σ_{AB} be the affine extension of the map $A' \longrightarrow B'$. The σ_{AB} generate a group of affine maps that admit O as fixed point and Z as invariant convex body. Therefore, the group is linear and bounded, it is conjugate to an orthogonal group ([7], prop.14-10), Z is an ellipse and the geometry is euclidean.

The Z-midpoint M_a of $a = BC$ is defined by $\|M_aB\| = \|M_aC\|$. Since $\|M_aB\| Z(t_a^*) = \|M_aC\| \mathrm{cm}(t_a, t_a^*) Z(t_a)$, we have

$$\frac{|M_aB|}{|M_aC|} = |\mathrm{cm}(t_a, t_a^*)| \ . \tag{12}$$

The Z-*medians* of ABC are the lines AM_a, BM_b, CM_c. Then we have from Ceva's theorem [6]: The Z-medians of a triangle are concurrent if and only if

$$\mathrm{cm}(t_a, t_a^*)\mathrm{cm}(t_b, t_b^*)\mathrm{cm}(t_c, t_c^*) = -1 \ . \tag{13}$$

The Z-medians are concurrent for all triangles if (13) holds for all triples of directions. For a degenerate triangle we have, for example, $t_b \longrightarrow t_a$, $t_c \longrightarrow t_a^*$ and therefore $cm(t_a^*, t_a) = -1$ for all directions t_a. That means for $s^* \in \Phi(t^*)$, $s \in \Phi(t)$ that

$$\begin{bmatrix} Y(s^*) \\ Z(t^*) \end{bmatrix} = \begin{bmatrix} -1 & st(t,s^*) \\ -sm(s,t^*) & -1 \end{bmatrix} \begin{bmatrix} Y(s) \\ Z(t) \end{bmatrix} .$$

Since the determinant is 1, either $st(t,s^*) = 0$ or $sm(s,t^*) = 0$. It follows from the convexity of Z that st and sm are continuous functions of s and t. Therefore, $0 \leqslant s,t \leqslant 2\Pi$ is the union of a countable set of intervals on which either $Z(t^*) = -Z(t)$ or $Y(s^*) = -Y(s)$. The construction of the dual can be given a local version: If ∂Z is $r = r(\theta)$ in euclidean polar coordinates and $\underline{n}(\theta) = \cos\theta\, \underline{i} + \sin\theta\, \underline{j}$ then the local dual is the envelope of the lines $\underline{n}(\theta).\underline{y} = 1/r$, [7]. Therefore, the local symmetry of ∂Z implies that of ∂Y, $sm(s,t^*) = st(t,s^*) = 0$:

PROPOSITION 6. *The* Z-*medians of a geometry are concurrent for all triangles if and only if the geometry is symmetric. In that case, the* Z-*medians are the affine medians and the* Y-*medians.*

5.

In euclidean geometry, the altitudes are concurrent at the orthocenter. A definition of the orthocenter derived from the euclidean theory of circles was studied by Asplund and Grünbaum [1], their results are valid for unsymmetric metrics and lead to a characterization of the geometries defined by strictly convex, symmetric ovals. Golab and Tamássy [4] proved that the altitudes are concurrent in Radon geometries. The only symmetric Radon curve is the circle, this is a characterization of euclidean geometry.

A triangle is *isosceles* in the Z-norm if $\|a\| = \|b\|$, it is *equilateral* if $\|a\| = \|b\| = \|c\|$. By the sine theorem, a triangle is isosceles if and only if $sm(\varphi(t_c), t_a) = sm(\varphi(t_b), t_c)$. It is not obvious that equilateral triangles exist for all directions of the legs. Without loss of generality, we assume $\|a\| = \|b\| = \|c\| = 1$. For $a = OA$, an equilateral exists if $A \in 2Z^* = -2Z$. For a symmetric metric, $Z = -Z$ and the condition is always satisfied:

PROPOSITION 7. *In symmetric metric, equilateral triangles exist for every direction of the leg* a.

The proposition does not hold for all unsymmetric metrics. Since $Z \cap -Z \neq \emptyset$, we can only say that equilateral triangles exist for a set of directions with positive linear measure. An example is the geo metry given by Z the triangle $(0,1)$, $(\pm 1,-3)$, O at the origin. The admissible directions OA are those for which the y-coordinate of A is $\geq -\frac{1}{2}$.

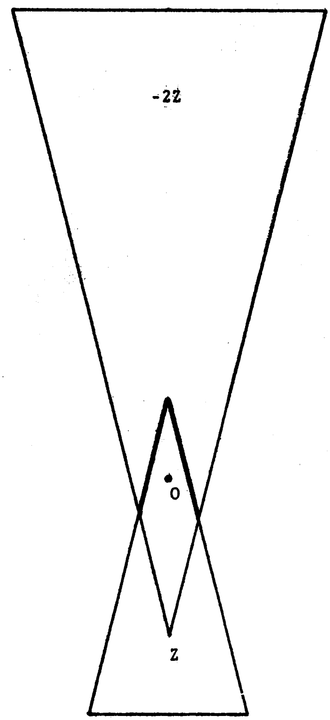

Fig.4

The theory of equilateral triangles can be expected to be simple only for symmetric metrics. A few sample theorems:

PROPOSITION 8. *In symmetric metric, an exterior* Y-*bisector of the equal legs of a* Z-*isosceles triangle is parallel to the basis.*

$\|a\| = \|-b\|$ implies $\mathrm{sm}(\varphi(t_a),t_c) = \mathrm{sm}(\varphi(t_b^*),t_c)$; the direction of the bisector is that of c. By a similar argument, we get:

PROPOSITION 9. *In symmetric metric, an interior* Z-*bisector of the equal legs of a* Y-*isosceles triangle is the altitude* <u>*from*</u> *base to vertex.*

The interior Z-bisector is the st-bisector and satisfies $\mathrm{st}(\Psi(s_a),s)=$ $= \mathrm{st}(\Psi(s_b^*),s)$. For an isosceles triangle,

$$\mathrm{st}(\Psi(s_b),s_c) = \mathrm{st}(\Psi(s_c),s_a) = -\mathrm{st}(\Psi(s_a),s_c) = -\mathrm{st}(\Psi(s_b^*),s_c) .$$

Hence, $s_c = s$ and $\Psi(s_c)$ is the direction of the normal *from* the basis. In this way, many theorems of elementary geometry become valid in an appropriate interplay of the two norms; for theorems of differential and integral geometry see [8].

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Polytechnic Institute of New York
Route 110
Farmingdale, New York 11735
U.S.A.

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