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THE NUMBER OF DIAMETERS THROUGH A POINT INSIDE AN OVAL

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Dedicated with greatest admiration and respect to Professor L. A. Santaló

1. INTRODUCTION.

In [6], Professor Santaló raised the question of determining bounds on the expected number of normals that can be drawn from a random point inside a convex body to its boundary. If the body has constant width this is equivalent to determining bounds on the expected number of diameters passing through a random point inside the body, since in this case the expected number of normals is just twice the expected number of diameters.

Let K be a plane convex body. Then a *diameter* is a chord of K whose endpoints lie on parallel supporting lines of K. For each $(x,y) \in K$, let $n(x,y)$ be the number of diameters of K passing through (x,y) (note that $n(x,y)$ might take the value $+\infty$). We are interested in the functional $I(K)$ given by

$$I(K) = \iint_K n(x,y)\ dx\ dy\ .$$

If we denote by $n(K)$ the expected number of diameters passing through a random point of K, then we have

$$n(K) = I(K)/A(K)\ ,$$

where $A(K)$ is the area of K.

Let $DK = K + (-K)$ be the *difference body* of K. In case the boundary of K is sufficiently regular, we shall prove that

$$\frac{1}{4}\,A(DK) \leq I(K) \leq \frac{1}{2}\,A(DK)\ . \tag{1.1}$$

For any plane convex body K, the difference body satisfies the inequalities (see Bonnesen and Fenchel [1])

(1.2) $$4A(K) \leqslant A(DK) \leqslant 6A(K) .$$

Combining this with (1.1) gives

$$A(K) \leqslant I(K) \leqslant 3A(K) .$$

As a consequence we have

(1.3) $$1 \leqslant n(K) \leqslant 3 .$$

The lower bound is not surprising, since a theorem of Hammer [3] guarantees that $n(x,y) \geqslant 1$ for all $(x,y) \in K$.

In Section 3 we shall prove (1.1), which leads to (1.3). We shall also show that the given bounds are sharp, in that $n(K) = 1$ iff K is centrally symmetric, and there exist K satisfying the regularity conditions we shall impose for which n(K) is as close to 3 as we please.

Our proofs will depend on transforming I(K) to an integral involving the length of a variable diameter and the instantaneous radius of rotation of that diameter. Indeed, let $D(\theta)$ be the length of a diameter making angle θ with the horizontal and $\rho(\theta)$ the distance from the instantaneous center of rotation to one endpoint. Then we shall show in Section 2 that

(1.4) $$I(K) = \frac{1}{2}\int_0^{2\pi}[\rho^2(\theta)-\rho(\theta)D(\theta) + \frac{1}{2}D^2(\theta)]\,d\theta .$$

It will follow from this that

(1.5) $$I(K) = \frac{1}{2}\int_0^{2\pi} \rho^2(\theta)\,d\theta .$$

The latter expression is geometrically plausible when we think of K as covered by the infinitesimal sectors of area swept out by diameters rotating through an angle $d\theta$ about their instantaneous centers of rotation (see Fig. 2).

Let $R(\varphi)$ be the radius of curvature at a boundary point of K where the supporting line makes angle φ with the horizontal, and let $w(\varphi)$ be the *width* of K in direction φ, that is, the distance between the parallel supporting lines making angle φ with the horizontal. In section 4 we shall derive from (1.5) the expression

(1.6) $$I(K) = \frac{1}{2}\int_0^{2\pi} \frac{R^2(\varphi)w(\varphi)}{R(\varphi) + R(\varphi+\pi)}\,d\varphi .$$

In case K has constant width $w(\varphi) \equiv b$ we have in addition $R(\varphi) + R(\varphi+\pi) \equiv b$, so (1.6) gives

(1.7) $$I(K) = \frac{1}{2}\int_0^{2\pi} R^2(\varphi)\,d\varphi .$$

This latter expression also follows from (1.5), since for sets of constant width we have $\rho(\theta) = R(\varphi)$ and $d\theta = d\varphi$ (where θ and φ are as in Fig. 1).

Since K has constant width b iff DK is a circular disk of radius b, we obtain from (1.1)

$$\frac{\pi}{4} b^2 \leqslant I(K) \leqslant \frac{\pi}{2} b^2 . \tag{1.8}$$

The area of a plane set K of constant width b satisfies

$$\frac{\pi-\sqrt{3}}{2} b^2 \leqslant A(K) \leqslant \frac{\pi}{4} b^2 , \tag{1.9}$$

with equality on the lefthand side for a Reuleaux triangle and on the righthand side for a circular disk. Using this in (1.8) yields

$$1 \leqslant n(K) \leqslant \frac{\pi}{\pi-\sqrt{3}} , \tag{1.10}$$

for plane sets of constant width. The upper bound corresponds to that given in [6] for the expected number of normals that can be drawn to the boundary from a random point inside a set of constant width. The lower bound is achieved precisely when K is a circular disk, and the upper bound when K is a Reuleaux triangle. Our methods give (1.10) only for sets of constant width satisfying our regularity assumptions, and among such K there are those (approximating Reuleaux triangles) for which n(K) is arbitrarily close to the upper bound in (1.10).

Section 5 contains a discussion of how (1.6) may be viewed as the analogue of (1.7) for a plane convex set K of constant relative width 1 in the relative geometry whose unit disk is DK.

We introduce in Section 2 the background necessary for our development and proceed to the proofs of the formulas (1.4) and (1.5).

2. PROOFS OF (1.4) AND (1.5).

We shall restrict our considerations to plane convex bodies having a certain degree of regularity. In the following, K will be a plane convex body whose boundary, to be denoted C, is a convex curve of class C^3 with nowhere vanishing curvature. We shall refer to such a K as an *oval*. In this case C admits the parametric representation

$$x = x(\varphi) \quad , \quad y = y(\varphi) \quad , \quad 0 \leqslant \varphi \leqslant 2\pi , \tag{2.1}$$

where φ is the angle the tangent line at $P(\varphi) = (x(\varphi), y(\varphi))$ makes with the x-axis (Fig.1).

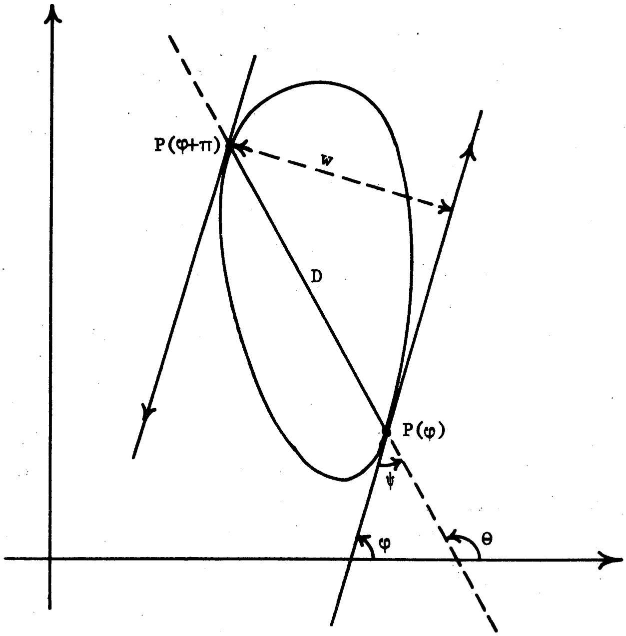

Figure 1

The chord $\overline{P(\varphi)P(\varphi+\pi)}$ is a diameter of K making angle $\theta = \theta(\varphi)$ with the x-axis (as indicated in Fig. 1). Since K is an oval, it is easy to see that θ is a strictly monotonic function of φ, so it is also in fact possible to express $\varphi = \varphi(\theta)$ as a smooth function of θ.

Let $\overline{D}(\varphi)$ denote the length of the diameter $\overline{P(\varphi)P(\varphi+\pi)}$. Then any point (x,y) on this diameter has coordinates of the form

$$(2.2)\qquad \begin{aligned} x &= x(\varphi) + \lambda \cos\theta(\varphi) \\ y &= y(\varphi) + \lambda \sin\theta(\varphi) \end{aligned} \qquad 0 \leqslant \lambda \leqslant \overline{D}(\varphi).$$

If S is the region in the (φ,λ)-plane defined by $S = \{(\varphi,\lambda): 0 \leqslant \lambda \leqslant \overline{D}(\varphi)\ ,\ 0 \leqslant \varphi \leqslant 2\pi\}$, then the equations (2.2) define a smooth mapping of S into K. The theorem of Hammer [3] mentioned in the introduction tells us that in fact this mapping sends S *onto* K. Since (φ,λ) and $(\varphi+\pi,\overline{D}(\varphi)-\lambda)$ always have the same image under this mapping, we see that each $(x,y) \in K$ is the image of $2n(x,y)$ points of S, where $n(x,y)$ is the number of diameters through (x,y). Thus, if $J = J(\varphi,\lambda)$ is the Jacobian determinant of the mapping, we have (see Federer [2, p. 243])

$$(2.3)\qquad 2I(K) = 2\iint_K n(x,y)\,dxdy = \iint_S |J(\varphi,\lambda)|\,d\varphi d\lambda\ .$$

Direct calculation from (2.2) gives

$$(2.4) \qquad J(\varphi,\lambda) = x'(\varphi)\sin\theta - y'(\varphi)\cos\theta - \lambda\theta' ,$$

where $\theta = \theta(\varphi)$, and the prime represents differentiation with respect to φ. But

$$(2.5) \qquad x'(\varphi) = R(\varphi)\cos\varphi , \quad y'(\varphi) = R(\varphi)\sin\varphi , \quad 0 \leqslant \varphi \leqslant 2\pi ,$$

where $R(\varphi)$ is the radius of curvature of C at $P(\varphi)$. Denoting by $\psi = \psi(\varphi)$ the angle between the tangent line and the diameter, as in Fig. 1, we obtain by substitution of (2.5) into (2.4),

$$(2.6) \qquad J = R\sin(\theta-\varphi) - \lambda\theta' = R\sin\psi - \lambda\theta' .$$

Let $\bar{\rho}(\varphi)$ be the instantaneous radius of rotation of the diameter $\overline{P(\varphi)P(\varphi+\pi)}$, that is, the distance from the instantaneous center of rotation to the point $P(\varphi)$. Let ds be the element of arclength of C at $P(\varphi)$. Then we have (see Fig. 2)

$$(2.7) \qquad \bar{\rho}(\varphi)d\theta = \sin\psi\, ds = R(\varphi)\sin\psi\, d\varphi .$$

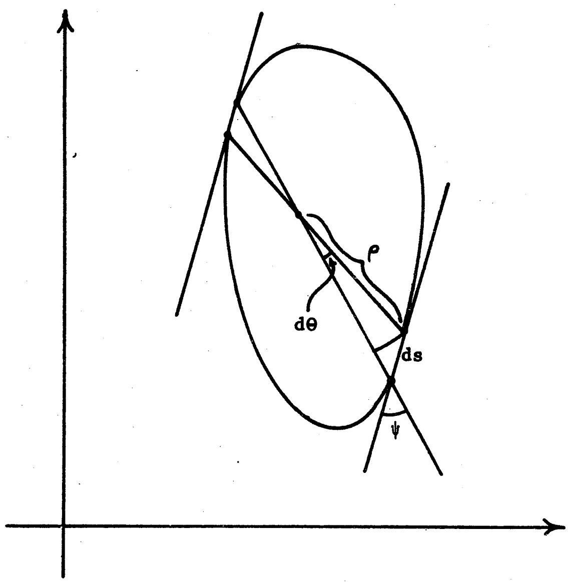

Figure 2

These relations can be derived from the results given in Hammer and Smith [4]. We have from (2.7) that $R(\varphi)\sin\psi = \bar{\rho}(\varphi)\theta'$. Substitution of this into (2.6) gives

$$(2.8) \qquad J(\varphi,\lambda) = (\bar{\rho}(\varphi)-\lambda)\theta'(\varphi) .$$

Iteration of the rightmost integral in (2.3) then gives

$$(2.9)\qquad I(K) = \frac{1}{2}\int_0^{2\pi}\{\int_0^{\overline{D}(\varphi)} |\overline{\rho}(\varphi)-\lambda|d\lambda\}\theta'(\varphi)d\varphi \ .$$

We let $\rho(\theta) = \overline{\rho}(\varphi(\theta))$ and $D(\theta) = \overline{D}(\varphi(\theta))$. Changing variables from φ to θ in (2.9) leads to

$$(2.10)\qquad I(K) = \frac{1}{2}\int_0^{2\pi}\{\int_0^{D(\theta)} |\rho(\theta)-\lambda|d\lambda\}d\theta \ .$$

Since any two diameters of an oval K intersect inside K, the centers of rotation all belong to K. Consequently $\rho(\theta) \leqslant D(\theta)$, and the inner integral in (2.10) takes the form

$$(2.11)\qquad \int_0^{D(\theta)} |\rho(\theta)-\lambda|d\lambda = \int_0^{\rho(\theta)} (\rho(\theta)-\lambda)d\lambda + \int_{\rho(\theta)}^{D(\theta)} (\lambda-\rho(\theta))d\lambda \ .$$

Evaluation of these integrals then gives, with (2.10), the required formula (1.4).

To obtain (1.5), we rewrite (1.4) in the form

$$(2.12)\qquad I(K) = \frac{1}{4}\int_0^{2\pi} [\rho^2(\theta) + (D(\theta)-\rho(\theta))^2]d\theta \ .$$

Since $\rho(\theta) + \rho(\theta+\pi) = D(\theta)$, this becomes

$$(2.13)\qquad I(K) = \frac{1}{4}\int_0^{2\pi} [\rho^2(\theta) + \rho^2(\theta+\pi)]d\theta \ ,$$

from which (1.5) follows by the periodicity of ρ.

3. THE BOUNDS ON I(K).

Write equation (1.4) in the form

$$(3.1)\qquad I(K) = \frac{1}{4}\int_0^{2\pi} D^2(\theta)d\theta - \frac{1}{2}\int_0^{2\pi} \rho(\theta)(D(\theta)-\rho(\theta))d\theta \ .$$

Applying to (3.1) the fact that $0 \leqslant \rho(D-\rho) \leqslant D^2/4$, we obtain

$$(3.2)\qquad \frac{1}{8}\int_0^{2\pi} D^2(\theta)d\theta \leqslant I(K) \leqslant \frac{1}{4}\int_0^{2\pi} D^2(\theta)d\theta \ .$$

The boundary of the difference body DK has the polar coordinate representation $r = D(\theta)$, $0 \leqslant \theta \leqslant 2\pi$, so

$$(3.3)\qquad A(DK) = \frac{1}{2}\int_0^{2\pi} D^2(\theta)d\theta \ .$$

The required bounds in (1.1) now follow from (3.2) and (3.3).

Equality holds on the lefthand side of (3.2), and so of (1.1), iff

$\rho(\theta)(D(\theta)-\rho(\theta)) \equiv D^2(\theta)/4$, which happens precisely when $\rho(\theta) \equiv D(\theta)/2$. In this case each diameter of K is an area bisector, and it follows that K is centrally symmetric (see Hammer and Smith [4]). As a further consequence, since $A(DK) = 4A(K)$ iff K is centrally symmetric, we see that $n(K) = 1$ iff K is centrally symmetric.

The theorems of Hammer and Sobczyk [5] imply that when K is not centrally symmetric there exist three diameters surrounding a triangle Δ such that $n(x,y) \geqslant 3$ for $(x,y) \in \Delta$. In this case, since $n(x,y) \geqslant 1$ for all $(x,y) \in K$, one must have that $n(K) > 1$. This shows in another way that $n(K) = 1$ only if K is centrally symmetric.

Equality can hold on the righthand side of (3.2) and (1.1) iff $\rho(\theta)(D(\theta)-\rho(\theta)) \equiv 0$. This is not possible for our class of ovals; however we can find ovals K for which I(K) is arbitrary close to A(DK)/2. For example, appropriate approximations of triangles will have this property, and we can find such K with n(K) as close to 3 as we please. In that sense the bounds in (1.3) are sharp.

4. PROOF OF (1.6).

If $w(\varphi)$ is the width of K, then we have $w(\varphi) = D(\theta)\sin\psi$ (see Fig.1). Thus from (2.7) we obtain

$$\rho(\theta)d\theta = \sin\psi\, ds = \frac{w(\varphi)}{D(\theta)} R(\varphi)d\varphi . \tag{4.1}$$

Since $w(\varphi+\pi) = w(\varphi)$ and $D(\theta+\pi) = D(\theta)$, we also have

$$\rho(\theta+\pi)d\theta = \frac{w(\varphi)}{D(\theta)} R(\varphi+\pi)d\varphi . \tag{4.2}$$

Comparison of (4.1) and (4.2) yields

$$\frac{\rho(\theta+\pi)}{\rho(\theta)} = \frac{R(\varphi+\pi)}{R(\varphi)} ,$$

from which it follows that

$$\frac{D(\theta)}{\rho(\theta)} = \frac{\rho(\theta)+\rho(\theta+\pi)}{\rho(\theta)} = \frac{R(\varphi)+R(\varphi+\pi)}{R(\varphi)} .$$

Thus we have

$$\rho(\theta) = \frac{D(\theta)R(\varphi)}{R(\varphi)+R(\varphi+\pi)} . \tag{4.3}$$

Then (4.1) and (4.3) yield

$$\rho^2(\theta)d\theta = \rho(\theta)\rho(\theta)d\theta = \frac{R^2(\varphi)w(\varphi)}{R(\varphi)+R(\varphi+\pi)} d\varphi , \tag{4.4}$$

which on integration gives the required formula (1.6).

5. INTERPRETATION OF (1.6) IN RELATIVE GEOMETRY.

In relative differential geometry in the plane (see, for example, Bonnesen and Fenchel [1]), one replaces the ordinary Euclidean unit disk by an arbitrary centrally symmetric convex body E centered at the origin. The *relative width* of a convex set K is the Euclidean width divided by half the width of E in the same direction. Then K has *constant relative width* b iff $DK = K + (-K) = bE$.

Given an oval K, we take $E = DK$ as our unit disk for a relative geometry. Then K has contant relative width 1, relative to E. Let $ds(\varphi)$ be the Euclidean element of arclength of K at $P(\varphi)$, and $dS(\varphi)$ the Euclidean element of arclength of E at the boundary point with outward normal parallel to the outward normal of K at $P(\varphi)$. The *relative radius of curvature* of K at $P(\varphi)$, denoted by $\tilde{R}(\varphi)$, is

$$\tilde{R}(\varphi) = \frac{ds(\varphi)}{dS(\varphi)} .$$

But we have $ds(\varphi) = R(\varphi)d\varphi$ and, since $E = DK$, $dS(\varphi) = (R(\varphi)+R(\varphi+\pi))d\varphi$. Hence

$$\tilde{R}(\varphi) = \frac{R(\varphi)}{R(\varphi)+R(\varphi+\pi)} . \tag{5.1}$$

The *relative arclength element* of E, at a boundary point where the supporting line makes angle φ with the horizontal, is $d\tilde{S}(\varphi) = h(E,\varphi)dS(\varphi)$, where $h(E,\varphi)$ is the supporting function of E. Since $E = DK$ we have $h(E,\varphi) = w(\varphi)$ = the width of K. This gives

$$d\tilde{S}(\varphi) = w(\varphi)dS(\varphi) = w(\varphi)(R(\varphi)+R(\varphi+\pi))d\varphi . \tag{5.2}$$

From (5.1) and (5.2) we obtain then for (1.6) the form,

$$I(K) = \frac{1}{2}\int \tilde{R}^2 \, d\tilde{S} , \tag{5.3}$$

where the integration is over the boundary of $E = DK$ with respect to the relative arclength induced by E. Thus (1.6) may be viewed as the generalization of (1.7) to sets of constant relative width.

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