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# SOME NEW CHARACTERIZATIONS OF VERONESE SURFACE AND STANDARD FLAT TORI

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*Dedicated to Professor Luis A. Santaló*

## 1. INTRODUCTION.

Let M be a (connected) surface in a Euclidean m-space $E^m$. For any point p in M and any unit vector t at p tangent to M, the vector t and and the normal space $T_p^{\perp}M$ of M at p determine an (m-1)-dimensional vector subspace E(p,t) of $E^m$ through p. The intersection of M and E(p,t) gives rise a curve $\gamma$ in a neighborhood of p which is called the *normal section* of M at p in the direction t. The surface M is said to have *planar normal sections* if normal sections of M are planar curves. In this case, for any normal section $\gamma$, we have $\gamma' \wedge \gamma'' \wedge \gamma''' = 0$ identically. A surface M is said to have *pointwise planar normal sections* if, for each point p in M, normal sections at p satisfy $\gamma' \wedge \gamma'' \wedge \gamma''' = 0$ at p (i.e., normal sections at p have "zero torsion" at p). It is clear that if a surface M lies in a linear 3-subspace $E^3$ of $E^m$, then M has planar normal sections and has pointwise planar normal sections.

We shall now define the Veronese surface. Let (x,y,z) be the natural coordinate system in $E^3$ and $(u^1,u^2,u^3,u^4,u^5)$ the natural coordinate system in $E^5$. We consider the mapping defined by

$$u^1 = \frac{1}{\sqrt{3}}\, yz \quad , \quad u^2 = \frac{1}{\sqrt{3}}\, zx \quad , \quad u^3 = \frac{1}{\sqrt{3}}\, xy \quad ,$$

$$u^4 = \frac{1}{2\sqrt{3}}\,(x^2 - y^2) \quad , \quad u^5 = \frac{1}{6}\,(x^2 + y^2 - 2z^2).$$

This defines an isometric immersion of $S^2(\sqrt{3})$ into the unit hypersphere $S^4(1)$ of $E^5$. Two points (x,y,z) and (-x,-y,-z) of $S^2(\sqrt{3})$ are mapped into the same point of $S^4(1)$, and this mapping defines an imbedding of the real projective plane into $S^4(1)$. This real projective plane imbedded in $E^5$ is called the *Veronese surface* (see, for instance, [4].)

In [2], we have proved the following.

THEOREM A. *Let* M *be a surface in* $E^m$. *If* M *has pointwise planar normal sections, then, locally,* M *lies in a linear* 5-*subspace* $E^5$ *of* $E^m$.

The classification of surfaces in $E^m$ with planar normal sections was obtained in [3].

THEOREM B. *Let* M *be a surface in* $E^m$. *If* M *has planar normal sections, then, either, locally,* M *lies in a linear* 3-*subspace* $E^3$ *or, up to similarity transformations of* $E^m$, M *is an open portion of the Veronese surface in a* $E^5$.

In view of Theorems A and B, it is an interesting problem to classify surfaces in $E^5$ with pointwise planar normal sections. As we already mentioned, every surface in $E^3$ has pointwise planar normal sections. A surface M in $E^m$ is said to *lie essentially in* $E^m$ if, locally, M does not lie in any hyperplane $E^{m-1}$ of $E^m$. According to Theorem A, the classification problem of surfaces in $E^m$ with pointwise planar normal sections remains open only for surfaces which lie essentially either in $E^5$ or in $E^4$.

In this paper, we will solve this problem completely for surfaces which lie essentially in $E^5$. Furthermore, we will obtain three classification theorems for surfaces in $E^4$. As biproducts some new geometric characterizations of the Veronese surface and standard flat tori are then obtained.

## 2. PRELIMINARIES.

Let M be a surface in $E^m$. We choose a local field of orthonormal frame $\{e_1,\dots,e_m\}$ in $E^m$ such that, restricted to M, the vectors $e_1,e_2$ are tangent to M and $e_3,\dots,e_m$ are normal to M. We denote by $\{\omega^1,\dots,\omega^m\}$ the field of dual frames. The structure equations of $E^5$ are given by

$$(2.1)\qquad d\omega^A = -\sum \omega^A_B \wedge \omega^B \quad , \quad \omega^A_B + \omega^B_A = 0 \ ,$$

$$(2.2)\qquad d\omega^A_B = -\sum \omega^A_C \wedge \omega^C_B \ ,$$

$$A,B,C,\dots = 1,2,\dots,m.$$

Restricting these forms on M, we have $\omega^r = 0$, $r,s,t,\dots = 3,\dots,m$. Since

$$(2.3)\qquad 0 = d\omega^r = -\sum \omega^r_i \wedge \omega^i \quad , \quad i,j,k\dots = 1,2 \ ,$$

Cartan's Lemma implies

$$\omega_i^r = \sum h_{ij}^r \omega^j \quad , \qquad h_{ij}^r = h_{ji}^r . \tag{2.4}$$

From these formulas we obtain

$$d\omega^i = -\sum \omega_j^i \wedge \omega^j , \tag{2.5}$$

$$\omega_j^i + \omega_i^j = 0 , \tag{2.6}$$

$$d\omega_j^i = -\sum \omega_k^i \wedge \omega_j^k + \Omega_j^i \quad , \qquad \Omega_j^i = \frac{1}{2} \sum R_{jk\ell}^i \, \omega^k \wedge \omega^\ell , \tag{2.7}$$

$$R_{jk\ell}^i = \sum (h_{ik}^r h_{j\ell}^r - h_{i\ell}^r h_{jk}^r) , \tag{2.8}$$

$$d\omega_s^r = -\sum \omega_t^r \wedge \omega_s^t + \Omega_s^r \quad , \qquad \Omega_s^\ell = \frac{1}{2} \sum R_{sij}^r \, \omega^i \wedge \omega^j , \tag{2.9}$$

$$R_{sij}^r = \sum_k (h_{ki}^r h_{kj}^s - h_{kj}^r h_{ki}^s) . \tag{2.10}$$

The Riemannian connection of M is defined by $(\omega_j^i)$. The form $(\omega_s^r)$ defines a connection D in the normal bundle of M. We call $h = \sum h_{ij}^r \omega^i \omega^j e_r$ the second fundamental form of the surface M. We call $H = \frac{1}{2} \operatorname{tr} h$ the mean curvature vector of M. We take exterior differentiation of (2.4) and define $h_{ijk}^r$ by

$$\sum h_{ijk}^r \omega^k = dh_{ij}^r - \sum h_{i\ell}^r \omega_j^\ell - \sum h_{\ell j}^r \omega_i^\ell + \sum h_{ij}^s \omega_s^r . \tag{2.11}$$

Then we have the following equation of Codazzi,

$$h_{ijk}^r = h_{ikj}^r . \tag{2.12}$$

If we denote by $\nabla$ and $\tilde{\nabla}$ the covariant derivatives of M and $E^m$, respectively, then, for any two vector fields X, Y tangent to M and any vector field $\xi$ normal to M, we have

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y) , \tag{2.13}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi , \tag{2.14}$$

where $A_\xi$ denotes the Weingarten map with respect to $\xi$. If $\langle\, , \rangle$ denotes the inner product of $E^m$, then

$$\langle A_\xi X, Y\rangle = \langle h(X,Y), \xi\rangle . \tag{2.15}$$

If we define $\overline{\nabla} h$ by

$$(\overline{\nabla}_X h)(Y,Z) = D_X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) , \tag{2.16}$$

then equation (2.12) of Codazzi becomes

$$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z) \ . \tag{2.17}$$

It is well-known that $\bar{\nabla} h$ is a normal-bundle-valued tensor of type (0,3).

We need the following theorems for the proof of Theorem 1.

THEOREM C. (Chen [1]). *A surface* M *of* $E^m$ *has pointwise planar normal sections if and only if* $(\bar{\nabla}_t h)(t,t) \wedge h(t,t) = 0$ *for any* $t \in TM$.

THEOREM D. (Chen [2]). *Let* M *be a surface in* $E^m$ *with pointwise planar normal sections. Then* Im h *is parallel.*

## 3. CLASSIFICATION OF SURFACES IN $E^5$.

In this section we shall prove the following.

THEOREM 1. *Let* M *be a surface which lies essentially in* $E^5$. *Then, up to similarities of* $E^5$, M *is an open portion of the Veronese surface in* $E^5$ *if and only if* M *has pointwise planar normal sections.*

*Proof*. Let M be a surface in $E^5$ with pointwise planar normal sections. We choose a local field of orthonormal frame $\{e_1,e_2,e_3,e_4,e_5\}$ such that, restricted to M, $e_3$ is in the direction of the mean curvature vector H, $e_1$, $e_2$ are the principal directions of $A_3 = A_{e_3}$. Then $e_3$ is perpendicular to $h(e_1,e_2)$. We further choose $e_5$ so that $e_5$ is in the direction of $h(e_1,e_2)$. Then, with respect to $\{e_1,e_2,e_3,e_4,e_5\}$ , we have

$$A_3 = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \ , \quad A_4 = \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix} \ , \quad A_5 = \begin{pmatrix} \eta & \delta \\ \delta & -\eta \end{pmatrix}$$

Thus, we have

$$h(e_1,e_1) = \alpha e_3 + \gamma e_4 + \eta e_5 \ , \quad h(e_1,e_2) = \delta e_5 \ , \tag{3.1}$$
$$h(e_2,e_2) = \beta e_3 - \gamma e_4 - \eta e_5 \ .$$

It is easy to see that $\dim \operatorname{Im} h = 3$ if and only if $h(e_1,e_1) \wedge h(e_1,e_2) \wedge h(e_2,e_2) \neq 0$. Therefore, $\dim \operatorname{Im} h = 3$ if and only if $(\alpha+\beta)\gamma\delta \neq 0$. We put

$$M_3 = \{p \in M \mid \dim \operatorname{Im} h = 3\} \ . \tag{3.2}$$

Then $M_3$ is an open subset of M. If $M_3$ is empty, then Theorem D implies that M does not lie essentially in $E^5$. From now on, we assume that M lies essentially in $E^5$. Then $M_3$ is not empty. We denote by N a component of $M_3$. On N, we have

$$(\alpha+\beta)\gamma\delta \neq 0 \, . \tag{3.3}$$

From (2.16) and (3.1) we find

$$\begin{aligned} (\bar{\nabla}_{e_1}h)(e_1,e_1) = {} & [e_1(\alpha) + \gamma\omega_4^3(e_1) + \eta\omega_5^3(e_1)]e_3 + \\ & + [\alpha\omega_3^4(e_1) + e_1(\gamma) + \eta\omega_5^4(e_1)]e_4 + \\ & + [\alpha\omega_3^5(e_1) + \gamma\omega_4^5(e_1) + e_1(\eta) - 2\delta\omega_1^2(e_1)]e_5 \, , \end{aligned} \tag{3.4}$$

$$\begin{aligned} (\bar{\nabla}_{e_2}h)(e_1,e_1) = {} & [e_2(\alpha) + \gamma\omega_4^3(e_2) + \eta\omega_5^3(e_2)]e_3 + \\ & + [\alpha\omega_3^4(e_2) + e_2(\gamma) + \eta\omega_5^4(e_2)]e_4 + \\ & + [\alpha\omega_3^5(e_2) + \gamma\omega_4^5(e_2) + e_2(\eta) - 2\delta\omega_1^2(e_2)]e_5 \, , \end{aligned} \tag{3.5}$$

$$\begin{aligned} (\bar{\nabla}_{e_1}h)(e_1,e_2) = {} & [\delta\omega_5^3(e_1) + (\alpha-\beta)\omega_1^2(e_1)]e_3 + \\ & + [\delta\omega_5^4(e_1) + 2\gamma\omega_1^2(e_1)]e_4 + \\ & + [e_1(\delta) + 2\eta\omega_1^2(e_1)]e_5 \, , \end{aligned} \tag{3.6}$$

$$\begin{aligned} (\bar{\nabla}_{e_1}h)(e_2,e_2) = {} & [e_1(\beta) - \gamma\omega_4^3(e_1) - \eta\omega_5^3(e_1)]e_3 + \\ & + [\beta\omega_3^4(e_1) - e_1(\gamma) - \eta\omega_5^4(e_1)]e_4 + \\ & + [\beta\omega_3^5(e_1) - \gamma\omega_4^5(e_1) - e_1(\eta) - 2\delta\omega_2^1(e_1)]e_5 \, , \end{aligned} \tag{3.7}$$

$$\begin{aligned} (\bar{\nabla}_{e_2}h)(e_1,e_2) = {} & [\delta\omega_5^3(e_1) + (\alpha-\beta)\omega_1^2(e_2)]e_3 + \\ & + [\delta\omega_5^4(e_2) + 2\gamma\omega_1^2(e_2)]e_4 + \\ & + [e_2(\delta) + 2\eta\omega_1^2(e_2)]e_5 \, , \end{aligned} \tag{3.8}$$

$$\begin{aligned} (\bar{\nabla}_{e_2}h)(e_2,e_2) = {} & [e_2(\beta) - \gamma\omega_4^3(e_2) - \eta\omega_5^3(e_2)]e_3 + \\ & + [\beta\omega_3^4(e_2) - e_2(\gamma) - \eta\omega_5^4(e_2)]e_4 + \\ & + [\beta\omega_3^5(e_2) - \gamma\omega_4^5(e_1) - e_2(\eta) - 2\delta\omega_2^1(e_2)]e_5 \, . \end{aligned} \tag{3.9}$$

Because M has pointwise planar normal sections, Theorem C implies

$$(\bar{\nabla}_{e_1}h)(e_1,e_1) = \lambda_1 h(e_1,e_1) \quad , \quad (\bar{\nabla}_{e_2}h)(e_2,e_2) = \lambda_2 h(e_2,e_2) \, , \tag{3.10}$$

for some local functions $\lambda_1,\lambda_2$. Combining (3.1), (3.4), (3.9) with (3.10) we obtain

$$(3.11)\qquad e_1(\alpha) = \alpha\lambda_1 + \gamma\omega_3^4(e_1) + \eta\omega_3^5(e_1) \quad ,$$

$$(3.12)\qquad e_1(\gamma) = \gamma\lambda_1 - \alpha\omega_3^4(e_1) + \eta\omega_4^5(e_1) \quad ,$$

$$(3.13)\qquad e_1(\eta) = \eta\lambda_1 - \alpha\omega_3^5(e_1) - \gamma\omega_4^5(e_1) + 2\delta\omega_1^2(e_1) \quad ,$$

$$(3.14)\qquad e_2(\beta) = \beta\lambda_2 - \gamma\omega_3^4(e_2) - \eta\omega_3^5(e_2) \quad ,$$

$$(3.15)\qquad e_2(\gamma) = \gamma\lambda_2 + \beta\omega_3^4(e_2) + \eta\omega_4^5(e_2) \quad ,$$

$$(3.16)\qquad e_2(\eta) = \eta\lambda_2 + \beta\omega_3^5(e_2) - \gamma\omega_4^5(e_2) + 2\delta\omega_1^2(e_2) \ .$$

Moreover, from (3.5), (3.6), (3.7), (3.8) and equation (2.17) of Codazzi, we also have

$$(3.17)\qquad e_2(\alpha) = \gamma\omega_3^4(e_2) - \delta\omega_3^5(e_1) + \eta\omega_3^5(e_2) + (\alpha-\beta)\omega_1^2(e_1) \quad ,$$

$$(3.18)\qquad e_1(\beta) = -\gamma\omega_3^4(e_1) - \delta\omega_3^5(e_2) - \eta\omega_3^5(e_1) + (\alpha-\beta)\omega_1^2(e_2) \quad ,$$

$$(3.19)\qquad e_1(\delta) = \eta\lambda_2 + (\alpha+\beta)\omega_3^5(e_2) - 2\eta\omega_1^2(e_1) \quad ,$$

$$(3.20)\qquad e_2(\delta) = -\eta\lambda_1 + (\alpha+\beta)\omega_3^5(e_1) - 2\eta\omega_1^2(e_2) \quad ,$$

$$(3.21)\qquad \lambda_1\gamma - (\alpha+\beta)\omega_3^4(e_1) - \delta\omega_4^5(e_2) + 2\gamma\omega_1^2(e_2) = 0 \quad ,$$

$$(3.22)\qquad \lambda_2\gamma + (\alpha+\beta)\omega_3^4(e_2) + \delta\omega_4^5(e_1) - 2\gamma\omega_1^2(e_1) = 0 \ .$$

Let $t = e_1 + ke_2$. Then, from Theorem C, we have

$$(3.23)\qquad (\overline{\nabla}_{e_1+ke_2}h)(e_1+ke_2, e_1+ke_2) \wedge h(e_1+ke_2, e_1+ke_2) = 0$$

for any k. Because $e_3 \wedge e_4$ , $e_3 \wedge e_5$ and $e_4 \wedge e_5$ are linearly independent, (3.1), (3.3), (3.4) - (3.10), and (3.23) imply

$$(3.24)\qquad -\gamma\delta\omega_3^5(e_1) + \alpha\delta\omega_4^5(e_1) - (\alpha+\beta)\gamma\omega_1^2(e_1) = 0 \quad ,$$

$$(3.25)\qquad (\alpha+\beta)\gamma\lambda_1 + 3\gamma\delta\omega_3^5(e_2) - 3\alpha\delta\omega_4^5(e_2) + 3(\alpha+\beta)\gamma\omega_1^2(e_2) = 0 \quad ,$$

$$(3.26)\qquad (\alpha+\beta)\gamma\lambda_2 + 3\gamma\delta\omega_3^5(e_1) + 3\beta\delta\omega_4^5(e_1) - 3(\alpha+\beta)\gamma\omega_1^2(e_1) = 0 \quad ,$$

$$(3.27)\qquad \gamma\delta\omega_3^5(e_2) + \beta\delta\omega_4^5(e_2) - (\alpha+\beta)\gamma\omega_1^2(e_2) = 0 \quad ,$$

$$(3.28)\qquad 2\gamma\delta\lambda_1 - 3\gamma\eta\lambda_2 - 3(\alpha+\beta)\gamma\omega_3^5(e_2) - 3\delta\eta\omega_4^5(e_1) + 6\gamma\eta\omega_1^2(e_1) = 0 \ ,$$

$$(3.29)\qquad -3\gamma\eta\lambda_1 - 2\gamma\delta\lambda_2 + 3(\alpha+\beta)\gamma\omega_3^5(e_1) + 3\delta\eta\omega_4^5(e_2) - 6\gamma\eta\omega_1^2(e_2) = 0 \ .$$

From (3.25) and (3.27) we find

$$\gamma\lambda_1 - 3\delta\omega_4^5(e_2) + 6\gamma\omega_1^2(e_2) = 0 . \tag{3.30}$$

From (3.24) and (3.26) we find

$$\gamma\lambda_2 + 3\delta\omega_4^5(e_1) - 6\gamma\omega_1^2(e_1) = 0 . \tag{3.31}$$

Similarly, from (3.21), (3.22), (3.28) and (3.29), we also have

$$2\gamma\delta\lambda_1 + 3(\alpha+\beta)\eta\omega_3^4(e_2) - 3(\alpha+\beta)\gamma\omega_3^5(e_2) = 0 , \tag{3.32}$$

$$-2\gamma\delta\lambda_2 - 3(\alpha+\beta)\eta\omega_3^4(e_1) + 3(\alpha+\beta)\gamma\omega_3^5(e_1) = 0 . \tag{3.33}$$

From (3.22) and (3.24) we find

$$-\alpha\gamma\lambda_2 - \alpha(\alpha+\beta)\omega_3^4(e_2) - \gamma\delta\omega_3^5(e_1) + (\alpha-\beta)\gamma\omega_1^2(e_1) = 0 . \tag{3.34}$$

Similarly, from (3.21) and (3.27) we get

$$\beta\gamma\lambda_1 - \beta(\alpha+\beta)\omega_3^4(e_1) + \gamma\delta\omega_3^5(e_2) - (\alpha-\beta)\gamma\omega_1^2(e_2) = 0 . \tag{3.35}$$

From (3.21), (3.30) and (3.22) and (3.31), we obtain, respectively,

$$(\alpha+\beta)\omega_3^4(e_1) - 2\delta\omega_4^5(e_2) + 4\gamma\omega_1^2(e_2) = 0 , \tag{3.36}$$

$$(\alpha+\beta)\omega_3^4(e_2) - 2\delta\omega_4^5(e_1) + 4\gamma\omega_1^2(e_1) = 0 . \tag{3.37}$$

From (3.21) and (3.36), we obtain

$$-2\gamma\lambda_1 + 3(\alpha+\beta)\omega_3^4(e_1) = 0 . \tag{3.38}$$

Similarly, from (3.22) and (3.37), we obtain

$$2\gamma\lambda_2 + 3(\alpha+\beta)\omega_3^4(e_2) = 0 . \tag{3.39}$$

Combining (3.21) and (3.38) we have

$$\gamma\lambda_1 - 3\delta\omega_4^5(e_2) + 6\gamma\omega_1^2(e_2) = 0 . \tag{3.40}$$

Equations (3.22) and (3.39) imply

$$\gamma\lambda_2 + 3\delta\omega_4^5(e_1) - 6\gamma\omega_1^2(e_1) = 0 . \tag{3.41}$$

From (3.34) and (3.39) we find

$$\alpha\lambda_2 + 3\delta\omega_3^5(e_1) - 3(\alpha-\beta)\omega_1^2(e_1) = 0 . \tag{3.42}$$

Similarly, we have

$$\beta\lambda_1 + 3\delta\omega_3^5(e_2) - 3(\alpha-\beta)\omega_1^2(e_2) = 0 . \tag{3.43}$$

From (3.32) and (3.39) we find

$$2\delta\lambda_1 - 2\eta\lambda_2 - 3(\alpha+\beta)\omega_3^5(e_2) = 0 . \tag{3.44}$$

Similarly, we also have

$$2\eta\lambda_1 + 2\delta\lambda_2 - 3(\alpha+\beta)\omega_3^5(e_1) = 0 . \tag{3.45}$$

Now, we want to claim that N is pseudo-umbilical in $E^5$, i.e., $\alpha\equiv\beta$ on N. Assume that $\alpha \neq \beta$ at a point $p \in N$. Then there is an open neighborhood U of p in N such that $\alpha \neq \beta$ everywhere on U. From (3.38) - (3.45), we obtain the following expression of $\omega_1^2$ and $\omega_r^s$ on U,

$$\omega_1^2 = \left\{\frac{2\delta\eta\lambda_1 + [\alpha(\alpha+\beta) + 2\delta^2]\lambda_2}{3(\alpha^2-\beta^2)}\right\}\omega^1 + \tag{3.46}$$

$$+ \left\{\frac{[\beta(\alpha+\beta) + 2\delta^2]\lambda_1 - 2\delta\eta\lambda_2}{3(\alpha^2-\beta^2)}\right\}\omega^2 ,$$

$$\omega_3^4 = \left\{\frac{2\gamma\lambda_1}{3(\alpha+\beta)}\right\}\omega^1 - \left\{\frac{2\gamma\lambda_2}{3(\alpha+\beta)}\right\}\omega^2 , \tag{3.47}$$

$$\omega_3^5 = \left\{\frac{2\eta\lambda_1 + 2\delta\lambda_2}{3(\alpha+\beta)}\right\}\omega^1 + \left\{\frac{2\delta\lambda_1 - 2\eta\lambda_2}{3(\alpha+\beta)}\right\}\omega^2 , \tag{3.48}$$

$$\omega_4^5 = \left\{\frac{4\gamma\delta\eta\lambda_1 + \gamma[(\alpha+\beta)^2 + 4\delta^2]\lambda_2}{3\delta(\alpha^2-\beta^2)}\right\}\omega^1 + \tag{3.49}$$

$$+ \left\{\frac{\gamma[(\alpha+\beta)^2 + 4\delta^2]\lambda_1 - 4\gamma\delta\eta\lambda_2}{3\delta(\alpha^2-\beta^2)}\right\}\omega^2 .$$

Now, we shall make a careful study of the integrability condition to obtain a contradiction. In order to do so, we need to compute the exterior derivatives of $(\omega_r^s)$.

From (3.47) we have

$$d\omega_3^4 = d\left(\frac{2\gamma}{3(\alpha+\beta)}\right) \wedge (\lambda_1\omega^1 - \lambda_2\omega^2) + \left(\frac{2\gamma}{3(\alpha+\beta)}\right) d(\lambda_1\omega^1 - \lambda_2\omega^2) . \tag{3.50}$$

Thus, by applying (3.11) - (3.18), (3.46) and a direct long computation, we may find

$$d\omega_3^4 = -\frac{2\gamma}{3(\alpha+\beta)}\left\{e_2(\lambda_1) + e_1(\lambda_2) - \frac{\lambda_1\lambda_2}{3} + \right. \tag{3.51}$$

$$+ \frac{\eta[(\alpha+\beta)^2+2\delta^2]}{3\delta(\alpha^2-\beta^2)}(\lambda_1^2+\lambda_2^2)\Big\}\ \omega^1\wedge\omega^2 .$$

Similarly, we may also obtain

$$(3.52)\qquad d\omega_3^5 = \frac{1}{9\delta(\alpha+\beta)^2(\alpha-\beta)}\{6(\alpha^2-\beta^2)\delta^2[e_1(\lambda_1)-e_2(\lambda_2)] -$$
$$- 6(\alpha^2-\beta^2)\delta\eta[e_2(\lambda_1)+e_1(\lambda_2)] -$$
$$- 2\{(\delta^2-\gamma^2)[(\alpha+\beta)^2+4\delta^2]+2\delta^2\eta^2\}(\lambda_1^2+\lambda_2^2) +$$
$$+ 2\delta^2[\beta(\alpha+\beta)+2\delta^2]\lambda_1^2+2\delta^2[\alpha(\alpha+\beta)+2\delta^2]\lambda_2 +$$
$$+ 2\delta(\alpha^2-\beta^2)\eta\lambda_1\lambda_2\}\ \omega^1\wedge\omega^2 ,$$

$$(3.53)\qquad d\omega_4^5 = \frac{1}{9\delta(\alpha^2-\beta^2)^2}\cdot\{3(\alpha^2-\beta^2)\gamma[(\alpha+\beta)^2+4\delta^2][e_1(\lambda_1)-e_2(\lambda_2)] -$$
$$- 12\gamma\delta\eta(\alpha^2-\beta^2)[e_2(\lambda_1)+e_1(\lambda_2)] -$$
$$- [(\alpha+\beta)^2(\alpha^2+\alpha\beta+\beta^2)\gamma +$$
$$+ 2\gamma\delta^2(5\alpha^2+5\beta^2+4\eta^2+4\delta^2)](\lambda_1^2+\lambda_2^2) +$$
$$+ \gamma[(\alpha+\beta)^2+4\delta^2](\beta^2\lambda_1^2+\alpha^2\lambda_2^2) +$$
$$+ 4\gamma\delta\eta(\alpha^2-\beta^2)\lambda_1\lambda_2\}\ \omega^1\wedge\omega^2 .$$

On the other hand, by using (2.10) and (3.1), we have

$$(3.54)\qquad R^4_{312} = 0 ,$$

$$(3.55)\qquad R^5_{312} = (\beta-\alpha)\delta ,$$

$$(3.56)\qquad R^5_{412} = -2\gamma\delta .$$

Therefore, by equation (2.9) of Ricci, equations (3.47) - (3.49) and (3.54) - (3.56), we also have

$$(3.57)\qquad d\omega_3^4 = -\left(\frac{2\gamma\eta}{9\delta(\alpha-\beta)}\right)(\lambda_1^2+\lambda_2^2)\ \omega^1\wedge\omega^2$$

$$(3.58)\qquad d\omega_3^5 = \frac{1}{9\delta(\alpha^2-\beta^2)(\alpha+\beta)}\{2\gamma^2[(\alpha+\beta)^2+4\delta^2](\lambda_1^2+\lambda_2^2) -$$
$$- 9\delta^2(\alpha^2-\beta^2)^2\}\ \omega^1\wedge\omega^2 ,$$

$$(3.59)\qquad d\omega_4^5 = \frac{-2\gamma\delta}{9(\alpha+\beta)^2}\{2(\lambda_1^2+\lambda_2^2)+9(\alpha+\beta)^2\}\ \omega^1\wedge\omega^2 .$$

Comparing (3.51) with (3.57), we find

(3.60) $$e_2(\lambda_1)+e_1(\lambda_2) = \frac{1}{3}\lambda_1\lambda_2 - \frac{2\eta\delta}{3(\alpha^2-\beta^2)}(\lambda_1^2+\lambda_2^2) .$$

Comparing (3.52) with (3.58), we find

(3.61) $$\delta[e_1(\lambda_1)-e_2(\lambda_2)]-\eta[e_2(\lambda_1)+e_1(\lambda_2)] =$$
$$= \frac{\delta}{3(\alpha^2-\beta^2)}\{[\alpha(\alpha+\beta)+2\delta^2+2\eta^2]\lambda_1^2+[\beta(\alpha+\beta)+2\delta^2+2\eta^2]\lambda_2^2\} -$$
$$- \frac{1}{3}\eta\lambda_1\lambda_2 - \frac{3}{2}(\alpha^2-\beta^2)\delta .$$

Combining (3.53) with (3.59), we get

(3.62) $$[(\alpha+\beta)^2+4\delta^2][e_1(\lambda_1)-e_2(\lambda_2)]-4\delta\eta[e_2(\lambda_1)+e_1(\lambda_2)] =$$
$$= \frac{1}{3(\alpha^2-\beta^2)}\{(\alpha+\beta)^2(\alpha^2+\alpha\beta+\beta^2) +$$
$$+ 2\delta^2(3\alpha^2+4\alpha\beta+3\beta^2+4\eta^2+4\delta^2)](\lambda_1^2+\lambda_2^2) -$$
$$- \frac{1}{3(\alpha^2-\beta^2)}\{[(\alpha+\beta)^2+4\delta^2](\beta^2\lambda_1^2+\alpha^2\lambda_2^2)\} -$$
$$- \frac{4}{3}\delta\eta\lambda_1\lambda_2-6\delta^2(\alpha^2-\beta^2) .$$

Substituting (3.60) into (3.61), we obtain

(3.63) $$e_1(\lambda_1)-e_2(\lambda_2) = \frac{1}{3(\alpha^2-\beta^2)}\{[\alpha(\alpha+\beta)+2\delta^2]\lambda_1^2 +$$
$$+ [\beta(\alpha+\beta)+2\delta^2]\lambda_2^2\}-\frac{3}{2}(\alpha^2-\beta^2) .$$

Substituting (3.60) and (3.63) into (3.62), we may obtain

(3.64) $$\alpha^2 - \beta^2 = 0 .$$

This contradicts to (3.3) because we assume that $\alpha \not\equiv \beta$. Therefore, we have proved that $\alpha = \beta$ identically on N, i.e., N is pseudo-umbilical in $E^5$. Because $\alpha \equiv \beta$, (3.42), (3.43), (3.44) and (3.45) reduce to

(3.65) $$\alpha\lambda_2 + 3\delta\omega_3^5(e_1) = 0 ,$$

(3.66) $$\beta\lambda_1 + 3\delta\omega_3^5(e_2) = 0 ,$$

(3.67) $$(\alpha+\beta)\beta\lambda_1 = -2\delta^2\lambda_1 + 2\delta\eta\lambda_2 ,$$

(3.68) $$(\alpha+\beta)\alpha\lambda_2 = -2\delta\eta\lambda_1 - 2\delta^2\lambda_2 .$$

From (3.67) and (3.68) we obtain

$$\lambda_1 = \lambda_2 = 0 . \tag{3.69}$$

Thus, from (3.30) and (3.31), we have

$$\delta\omega_4^5 = 2\gamma\omega_1^2 . \tag{3.70}$$

From (3.38), (3.39), (3.42) and (3.43), we find

$$\omega_3^4 = \omega_3^5 = 0 . \tag{3.71}$$

Substituting (3.69) and (3.71) into (3.11), (3.14), (3.17) and (3.18), we find

$$\alpha = \beta = \text{constant on } N . \tag{3.72}$$

From (3.12), (3.15), (3.69) and (3.71), we obtain

$$d\gamma = \eta\omega_4^5 . \tag{3.73}$$

From (2.9), (2.10), (3.1) and (3.71), we find

$$d\omega_4^5 = -2\gamma\delta\ \omega^1 \wedge \omega^2 . \tag{3.74}$$

Using (3.13), (3.16), (3.69), (3.70) and (3.71), we have

$$d\eta = \left(\frac{\delta^2 - \gamma^2}{\gamma}\right) \omega_4^5 . \tag{3.75}$$

Taking exterior differentiation of (3.73) and applying (2.9), (2.10), and (3.74), we obtain

$$0 = d^2\gamma = -2\gamma\delta\eta\ \omega^1 \wedge \omega^2 . \tag{3.76}$$

From (3.76) we get

$$\eta = 0 . \tag{3.77}$$

Since (3.74) shows that $\omega_4^5 \neq 0$, (3.75) and (3.77) give $\delta^2 = \gamma^2$. Without loss of generality, we may assume that

$$\delta = -\gamma . \tag{3.78}$$

From (3.70) and (3.78), we find

$$\omega_4^5 = -2\omega_1^2 . \tag{3.79}$$

From (3.73) and (3.77), we see that $\delta = -\gamma$ is a nonzero constant on N. Thus, by the definition of N and continuity, we conclude that N is the whole surface M.

From (2.7), (2.9), (3.1), (3.74), (3.78) and (3.79) we find

$$\alpha^2 = 3\gamma^2 . \tag{3.80}$$

Consequently, we may assume that $\alpha = -\sqrt{3}\,\gamma$. Therefore, by combining (3.71), (3.77), (3.79) and (3.80), we conclude that the connection form $(\omega_A^B)$, restricted to N, is given by

$$\begin{pmatrix} 0 & \omega_1^2 & \sqrt{3}\,\gamma\omega^1 & -\gamma\omega^1 & \gamma\omega^2 \\ \omega_2^1 & 0 & \sqrt{3}\,\gamma\omega^2 & \gamma\omega^2 & \gamma\omega^1 \\ -\sqrt{3}\,\gamma\omega^1 & -\sqrt{3}\,\gamma\omega^2 & 0 & 0 & 0 \\ \gamma\omega^1 & -\gamma\omega^2 & 0 & 0 & 2\omega_1^2 \\ -\gamma\omega^2 & -\gamma\omega^1 & 0 & 2\omega_2^1 & 0 \end{pmatrix}$$

This shows that, up to similarity transformations of $E^5$, M coincides locally with the Veronese surface [4].

Conversely, if, up to similarity transformations of $E^5$, M is an open portion of the Veronese surface, then M has parallel second fundamental form, i.e., $\bar{\nabla}h \equiv 0$. Thus, by Theorem C of Chen [1], we conclude that M has pointwise planar normal sections. This completes the proof of Theorem 1.

## 4. SURFACES IN $E^4$ WITH CONSTANT MEAN CURVATURE.

In this and the next two sections, we will study surfaces in $E^4$. Assume that M is a surface in $E^4$ with pointwise planar normal sections. We choose a local field of orthonormal frame $\{e_1,e_2,e_3,e_4\}$ so that, restricted to M, $e_3$ is in the direction of H, $e_1$, $e_2$ are the principal directions of $A_3$. Then $e_3$ is perpendicular to $h(e_1,e_2)$. With respect to $\{e_1,e_2,e_3,e_4\}$, we have

$$A_3 = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} , \quad A_4 = \begin{pmatrix} \eta & \delta \\ \delta & -\eta \end{pmatrix}$$

Thus we have

$$h(e_1,e_1) = \alpha e_3+\eta e_4 , \quad h(e_1,e_2) = \delta e_4 , \quad h(e_2,e_2) = \beta e_3-\eta e_4 . \tag{4.1}$$

It is easy to find that the mean curvature, the normal curvature and the Gauss curvature of M in $E^4$ are given respectively by
$|H| = \frac{1}{2}|\alpha+\beta|$ , $K^N = 2(\alpha-\beta)^2\delta^2$ and $K = \alpha\beta - \eta^2 - \delta^2$.
Since M has pointwise planar normal sections, Theorem C implies

$$(4.2)\qquad (\overline{\nabla}_{e_1}h)(e_1,e_1) = \lambda_1 h(e_1,e_2) ,$$
$$(\overline{\nabla}_{e_2}h)(e_2,e_2) = \lambda_2 h(e_2,e_2)$$

for some local functions $\lambda_1$, $\lambda_2$. Using the same method as before, we have the following

$$(4.3)\qquad e_1(\alpha) = \alpha\lambda_1 + \eta\omega_3^4(e_1) ,$$

$$(4.4)\qquad e_1(\beta) = -\eta\omega_3^4(e_1) - \delta\omega_3^4(e_2) + (\alpha-\beta)\omega_1^2(e_2) ,$$

$$(4.5)\qquad e_1(\eta) = \eta\lambda_1 - \alpha\omega_3^4(e_1) + 2\delta\omega_1^2(e_1) ,$$

$$(4.6)\qquad e_1(\delta) = \eta\lambda_2 + (\alpha+\beta)\omega_3^4(e_2) - 2\eta\omega_1^2(e_1) ,$$

$$(4.7)\qquad e_2(\alpha) = -\delta\omega_3^4(e_1) + \eta\omega_3^4(e_2) + (\alpha-\beta)\omega_1^2(e_1) ,$$

$$(4.8)\qquad e_2(\beta) = \beta\lambda_2 - \eta\omega_3^4(e_2) ,$$

$$(4.9)\qquad e_2(\eta) = \eta\lambda_2 + \beta\omega_3^4(e_2) + 2\delta\omega_1^2(e_2) ,$$

$$(4.10)\qquad e_2(\delta) = -\eta\lambda_1 + (\alpha+\beta)\omega_3^4(e_1) - 2\eta\omega_1^2(e_2) ,$$

$$(4.11)\qquad 2\alpha\delta\lambda_1 - 3\alpha\eta\lambda_2 - 3\eta\delta\omega_3^4(e_1) - 3\alpha(\alpha+\beta)\omega_3^4(e_2) + $$
$$+ 3(\alpha-\beta)\eta\omega_1^2(e_1) = 0 ,$$

$$(4.12)\qquad (2\alpha-\beta)\eta\lambda_1 - 3(\alpha^2+\alpha\beta+2\delta^2)\omega_3^4(e_1) - 3\eta\delta\omega_3^4(e_2) +$$
$$+ 6(\alpha-\beta)\delta\omega_1^2(e_1) + 3(\alpha-\beta)\eta\omega_1^2(e_2) = 0 ,$$

$$(4.13)\qquad (\alpha-2\beta)\eta\lambda_2 + 3\eta\delta\omega_3^4(e_1) - 3(\alpha\beta+\beta^2+2\delta^2)\omega_3^4(e_2) -$$
$$- 3(\alpha-\beta)\eta\omega_1^2(e_1) + 6(\alpha-\beta)\delta\omega_1^2(e_2) = 0 ,$$

$$(4.14)\qquad 3\beta\eta\lambda_1 + 2\beta\delta\lambda_2 - 3(\alpha+\beta)\beta\omega_3^4(e_1) + 3\eta\delta\omega_3^4(e_2) -$$
$$- 3(\alpha-\beta)\eta\omega_1^2(e_2) = 0 .$$

THEOREM 2. *Let* M *be a surface which lies essentially in* $E^4$. *Then* M *is an open portion of the product surface of two planar circles if and only if* M *has pointwise planar normal sections and constant mean curvature.*

*Proof*. If M is an open portion of the product surface of two planar circles, then it is easy to check that M has constant mean curvature and pointwise planar normal sections.

Now, let M be a surface which lies essentially in $E^4$. Assume that M has constant mean curvature and pointwise planar normal sections. Then, by using Theorem 4 of [2], we see that $\alpha+\beta \neq 0$. We want to claim that $(\alpha-\beta)\delta = 0$. Assume that $(\alpha-\beta)\delta \neq 0$. If $\eta \neq 0$, then by eliminating $\omega_1^2(e_1)$, $\omega_1^2(e_2)$ from (4.12) and (4.13) with the help of (4.11), (4.14), we have

$$(4.15) \qquad 2[(\alpha+\beta)\eta^2 - 2\alpha\delta^2]\lambda_1 + 2(3\alpha+\beta)\eta\delta\lambda_2 - 3(\alpha+\beta)^2\eta\omega_3^4(e_1) + 6\alpha(\alpha+\beta)\delta\omega_3^4(e_2) = 0 \quad ,$$

$$(4.16) \qquad -2(\alpha+3\beta)\eta\delta\lambda_1 + 2[(\alpha+\beta)\eta^2 - 2\beta\delta^2]\lambda_2 + 6(\alpha+\beta)\beta\delta\omega_3^4(e_1) + 3(\alpha+\beta)^2\eta\omega_3^4(e_2) = 0 \ .$$

Combining (4.15) and (4.16), we have

$$(4.17) \qquad [(\alpha+\beta)^2\eta^2 + 4\alpha\beta\delta^2][2\eta\lambda_1 + 2\delta\lambda_2 - 3(\alpha+\beta)\omega_3^4(e_1)] = 0 \ .$$

If $(\alpha+\beta)^2\eta^2 + 4\alpha\beta\delta^2 \neq 0$. We have from (4.11) - (4.17)

$$(4.18) \qquad \omega_1^2 = \frac{2\eta\delta\lambda_1 + (\alpha^2+\alpha\beta+2\delta^2)\lambda_2}{3(\alpha^2-\beta^2)}\,\omega^1 + \frac{(\alpha\beta+\beta^2+2\delta^2)\lambda_1 - 2\eta\delta\lambda_2}{3(\alpha^2-\beta^2)}\,\omega^2 \ ,$$

$$(4.19) \qquad \omega_3^4 = \frac{2(\eta\lambda_1+\delta\lambda_2)}{3(\alpha+\beta)}\,\omega^1 + \frac{2(\delta\lambda_1-\eta\lambda_2)}{3(\alpha+\beta)}\,\omega^2 \ .$$

If $(\alpha+\beta)^2\eta^2 + 4\alpha\beta\delta^2 = 0$, differentiating this relation, we have, with the help of (4.3) - (4.10),

$$(4.20) \qquad [\alpha(\alpha+\beta)\eta^2 - 2\alpha\beta\delta^2]\lambda_1 + 4\alpha\beta\eta\delta\lambda_2 - [\alpha(\alpha+\beta)^2 + 2(\alpha-\beta)\delta^2]\eta\omega_3^4(e_1) + [4\alpha\beta(\alpha+\beta) - (\alpha+\beta)\eta^2 - 2\alpha\delta^2]\delta\omega_3^4(e_2) + 2(\alpha-\beta)^2\eta\delta\omega_1^2(e_1) + (\alpha-\beta)[(\alpha+\beta)\eta^2 + 2\alpha\delta^2]\omega_1^2(e_2) = 0 \ .$$

$$(4.21) \qquad -4\alpha\beta\eta\delta\lambda_1 + [\beta(\alpha+\beta)\eta^2 - 2\alpha\beta\delta^2]\lambda_2 + [4\alpha\beta(\alpha+\beta) - (\alpha+\beta)\eta^2 - 2\beta\delta^2]\delta\omega_3^4(e_1) +$$

$$+ [\beta(\alpha+\beta)^2 - 2(\alpha-\beta)\delta^2]\eta\omega_3^4(e_2) \quad +$$
$$+ (\alpha-\beta)[(\alpha+\beta)\eta^2 + 2\beta\delta^2]\omega_1^2(e_1) \quad +$$
$$+ 2(\alpha-\beta)^2\eta\delta\omega_1^2(e_2) = 0 \ .$$

From (4.11) - (4.14) and (4.20), (4.21), we still have (4.18), (4.19). Because $|H|$ is constant, differentiating the relation $\alpha+\beta$ = constant, we have

$$\alpha\lambda_1 - \delta\omega_3^4(e_2) + (\alpha-\beta)\omega_1^2(e_2) = 0 \tag{4.22}$$

$$\beta\lambda_2 - \delta\omega_3^4(e_1) + (\alpha-\beta)\omega_1^2(e_1) = 0 \ . \tag{4.23}$$

Substituting (4.18), (4.19) into (4.22), (4.23), we get

$$(3\alpha+\beta)\lambda_1 = 0 \ . \tag{4.24}$$

$$(3\beta+\alpha)\lambda_2 = 0 \ . \tag{4.25}$$

Thus we have (i) $\lambda_1 = \lambda_2 = 0$, or (ii) $3\alpha+\beta = 0$, $3\beta+\alpha = 0$, or (iii) $3\alpha+\beta = 0$, $\lambda_2 = 0$, or (iv) $3\beta+\alpha = 0$, $\lambda_1 = 0$. If case (i) occurs, (4.18) and (4.19) imply $\omega_1^2 = \omega_3^4 = 0$. In particular, we have $K^N = 0$. Thus, by applying Theorem 5 of Chen [2], we see that M is an open portion of the product surface of two planar circles. In particular, we have $\delta = 0$. This is a contradiction. If case (ii) occurs, we have $\alpha = \beta = 0$. This contradicts to $\alpha+\beta \neq 0$. For case (iii), differentiating $3\alpha+\beta = 0$, we have

$$3e_2(\alpha) + e_2(\beta) = 0 \ . \tag{4.26}$$

Since $\lambda_2 = 0$, (4.7), (4.8), (4.18), (4.19), and (4.26) imply

$$\eta\delta\lambda_1 = 0 \ . \tag{4.27}$$

From this we may again obtain a contradiction. The last case is similar to case (iii). Consequently, we have $\eta = 0$.

If $(\alpha-\beta)\delta \neq 0$ and $\alpha\beta \neq 0$, then from (4.3) - (4.14) we have $\alpha\beta+\delta^2 = 0$ and

$$\omega_1^2 = \frac{\alpha\lambda_2}{3(\alpha+\beta)}\,\omega^1 - \frac{\beta\lambda_1}{3(\alpha+\beta)}\,\omega^2 \ , \tag{4.28}$$

$$\omega_3^4 = \frac{2\delta\lambda_2}{3(\alpha+\beta)}\,\omega^1 + \frac{2\delta\lambda_1}{3(\alpha+\beta)}\,\omega^2 \ . \tag{4.29}$$

Differentiating $\alpha+\beta$ = constant, we have (4.22) and (4.23). By substituting (4.28) and (4.29) into (4.22) and (4.23), we obtain

$$(4.30)\qquad (3\alpha^2 + 2\alpha\beta + \beta^2 - 2\delta^2)\lambda_1 = 0\ ,$$

$$(4.31)\qquad (\alpha^2 + 2\alpha\beta + 3\beta^2 - 2\delta^2)\lambda_2 = 0\ .$$

Thus, (i) $\lambda_1 = \lambda_2 = 0$, or (ii) $3\alpha^2 + 2\alpha\beta + \beta^2 - 2\delta^2 = 0$ and $\alpha^2 + 2\alpha\beta + 3\beta^2 - 2\delta^2 = 0$, or (iii) $3\alpha^2 + 2\alpha\beta + \beta^2 - 2\delta^2 = 0$ and $\lambda_2 = 0$, or (iv) $\lambda_1 = 0$ and $\alpha^2 + 2\alpha\beta + 3\beta^2 - 2\delta^2 = 0$.

Case (i) contradicts the assumption. Case (ii) implies $\alpha^2 = \beta^2$ which contradicts the assumption too. For case (iii), since $\alpha\beta + \delta^2 = 0$ , we obtain

$$(4.32)\qquad 3\alpha^2 + 4\alpha\beta + \beta^2 = 0\ .$$

This implies $3\alpha+\beta = 0$. We know that this is impossible. The last case is similar to case (iii).

If $(\alpha-\beta)\delta \neq 0$ and $\alpha\beta = 0$, then without loss of generality, we may assume $\beta = 0$. From (4.3) - (4.14), we have

$$(4.33)\qquad e_1(\beta) = -\delta\omega_3^4(e_2) + \alpha\omega_1^2(e_2) = 0\ ,$$

$$(4.34)\qquad e_2(\eta) = 2\delta\omega_1^2(e_2) = 0\ ,$$

$$(4.35)\qquad 2\delta\lambda_1 = 3\alpha\omega_3^4(e_2) = 0\ .$$

These imply $\lambda_1 = 0$ and since $\beta = \eta = 0$, we have $h(e_2,e_2) = 0$. Thus, by (4.2), we may choose $\lambda_2 = 0$. From these we obtain a contradiction. Consequently, we obtain $(\alpha-\beta)\delta = 0$. Thus, $K^N = 0$, from which we obtain Theorem 2 by applying Theorem 5 of Chen [2]. (Q.E.D.)

## 5. SURFACES IN $E^4$ WITH CONSTANT NORMAL CURVATURE.

In this section, we give the following classification result.

THEOREM 3. *Let* M *be a surface which lies essentially in* $E^4$. *Then* M *is an open portion of the product surface of two planar circles if and only if* M *has pointwise planar normal sections and constant normal curvature.*

*Proof*. Let M be a surface which lies essentially in $E^4$. Assume M has constant normal curvature and pointwise planar normal sections. As mentioned in the proof of Theorem 2 we may assume that $\alpha+\beta \neq 0$. We want to claim that $(\alpha-\beta)\delta = 0$. Assume that $(\alpha-\beta)\delta \neq 0$. Because, $(\alpha-\beta)\delta$ = constant, we have

$$(5.1)\qquad \delta[e_i(\alpha) - e_i(\beta)] + (\alpha-\beta)e_i(\delta) = 0\ ,\quad i = 1,2\ .$$

Assume that $\eta \neq 0$. Using (4.3) - (4.10) and (4.18), (4.19), we obtain from (5.1),

$$(5.2)\qquad \delta(5\alpha-3\beta)\lambda_1 - \eta(\alpha+\beta)\lambda_2 = 0\ ,$$

$$(5.3)\qquad -\eta(\alpha+\beta)\lambda_1 + \delta(3\alpha-5\beta)\lambda_2 = 0\ .$$

From these, we know that either $\lambda_1 = \lambda_2 = 0$ or $\lambda_1^2 + \lambda_2^2 = 0$ and

$$(5.4)\qquad \delta^2(15\alpha^2 + 15\beta^2 - 34\alpha\beta) = \eta^2(\alpha+\beta)^2\ .$$

The first case implies that $\omega_3^4 = 0$ which gives $(\alpha-\beta)\delta = 0$. In the second case, we differentiate (5.4) to obtain

$$(5.5)\qquad \delta\lambda_1 = \eta\lambda_2\quad ,\quad \eta\lambda_1 = -\delta\lambda_2\ ,$$

where we have used (4.3) - (4.10) and (4.18), (4.19). From (5.5) we find $\eta^2+\delta^2 = 0$ which contradicts to the assumption. Consequently, we have $\eta = 0$.

If $\alpha\beta \neq 0$ and $(\alpha-\beta)\delta \neq 0$, then, from (4.3)-(4.14), we have (4.28) and (4.29) and $\alpha\beta+\delta^2 = 0$. Differentiating $K^N$, we find

$$(5.6)\qquad (3\alpha-\beta)\beta e_i(\alpha) + (\alpha-3\beta)\alpha e_i(\beta) = 0\quad ,\quad i = 1,2\ .$$

Using (4.3), (4.4), (4.7), (4.8), (4.28) and (4.29), we have from (5.6),

$$(5.7)\qquad (5\alpha-3\beta)\lambda_1 = (3\alpha-5\beta)\lambda_2 = 0\ .$$

Since $\alpha\beta+\delta^2 = 0$, $5\alpha-3\beta$ and $3\alpha-5\beta$ are nonzero. Thus, $\lambda_1 = \lambda_2 = 0$. This will give a contradiction. If $(\alpha-\beta)\delta \neq 0$ and $\alpha\beta = 0$, then, by the same argument as given in section 4, we also have a contradiction. Thus, we have $(\alpha-\beta)\delta = 0$, i.e., $K^N = 0$. Therefore, by Theorem 5 of Chen [2], M is an open portion of the product surface of two planar circles. The converse of this is clear. (Q.E.D.)

## 6. SURFACES IN $E^4$ WITH CONSTANT GAUSS CURVATURE.

THEOREM 4. *Let* M *be a surface which lies essentially in* $E^4$. *If* M *has pointwise planar normal sections and constant Gauss curvature, then* M *has vanishing Gauss curvature.*

*Proof.* Let M be a surface which lies essentially in $E^4$. Assume that M has constant Gauss curvature K and pointwise planar normal sections. We may assume that $\alpha+\beta \neq 0$ by Theorem 4 of [2]. If $(\alpha-\beta)\eta\delta \neq 0$, then, by differentiating K, we have

$$\beta e_i(\alpha) + \alpha e_i(\beta) - 2\eta e_i(\eta) - 2\delta e_i(\delta) = 0 , \quad i = 1,2 . \tag{6.1}$$

Using (4.3) - (4.10), (4.18), (4.19) and (6.1) we find

$$(\alpha\beta - \eta^2 - \delta^2)\lambda_1 = (\alpha\beta - \eta^2 - \delta^2)\lambda_2 = 0 . \tag{6.2}$$

From this, we may conclude that $K = \alpha\beta - \eta^2 - \delta^2 = 0$.

If $(\alpha-\beta)\delta \neq 0$, $\alpha\beta \neq 0$, but $\eta = 0$, then we have (4.28), (4.29) and $\alpha\beta+\delta^2 = 0$. Differentiating $K = \alpha\beta-\delta^2 =$ constant, we have

$$\beta e_i(\alpha) + \alpha e_i(\beta) - 2\delta e_i(\delta) = 0 , \quad i = 1,2 . \tag{6.3}$$

From (4.3), (4.4), (4.6), (4.7), (4.8), (4.10), (4.28) and (4.29), we have

$$(\alpha\beta-\delta^2)\lambda_1 = (\alpha\beta-\delta^2)\lambda_2 = 0 . \tag{6.4}$$

Thus, we have $\alpha\beta-\delta^2 = 0$ which contradicts $\alpha\beta+\delta^2 = 0$. If $(\alpha-\beta)\delta \neq 0$ but $\eta = \alpha\beta = 0$, then by a similar argument as given in section 4, we have a contradiction too.

When $(\alpha-\beta)\delta = 0$, $K^N = 0$. In this case, Theorem 5 of [2] implies that M is an open portion of a flat torus. Thus, $K = 0$. (Q.E.D.)

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