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SECTIONAL VORONOI TESSELLATIONS

R.E. Miles

Dedicated to L.A. Santaló, by way of whose delightful 'Introduction to Integral Geometry' [16] I was first exposed in 1959 to the beauties of geometry and randomness combined.

ABSTRACT. Formulae for the expected mean s-content of s-facet per polytope in the Voronoi random polytopal tessellation V of R^d, with respect to a homogeneous Poisson point process basis, are derived. s-flat sections of V yield a new class of random s-dimensional polytopal tessellations, whose properties are explored for s = 1,2,3.

1. INTRODUCTION & SUMMARY.

The area of random tessellations is an important one in stochastic geometry, and some of the earliest work is due to L.A. Santaló [12, 14,15]. An s-flat section of an ergodic homogeneous and isotropic random polytopal tessellation of R^d is a similar such tessellation in the s-flat as containing space $(1 \leqslant s \leqslant d-1)$. The main interest in this paper is in exploring properties of such sectional tessellations. In section 2, Santaló's basic formula for the expected mean projections of the isotropic uniform random section of a domain, in terms of the mean projections of the domain itself, finds useful application; in particular to sectional tessellations. The most rewarding specific random tessellations as regards sectioning are the Voronoi tessellations V considered in Section 3. An explicit formula for the mean s-content $E\{L_s\}$ of s-facet per polytope of V is derived; the case s=0 gives the mean number of vertices. Sectional Voronoi tessellations are examined in Section 4, with exact mean sectional values being obtained for s = 1,2 and asymptotic ones as $d \to \infty$ for s=3. In fact, an s-section of homogeneous V is stochastically equivalent to an s-section of a corresponding inhomogeneous (s+1)-dimensional structure. In Section 5, this aspect is explored in some detail in the line section case s=1, with an integral expression being given for the interval length distribution. Finally, in Section 6, generalized Voronoi tessellations V_n, involving the nearest n particles to a point, rather than the nearest single particle, are introduced (n = 2,3,...). An analogous formula for $E\{L_s\}$ to that obtained for V in Section 3, and an integral expression for the volume moments in s-sections, are derived.

Some of the results have been stated elsewhere [8,9], but without proofs.

PRELIMINARIES. $Q_d(x,r)$ represents the closed ball with centre x, radius r, in euclidean d-space R^d, with boundary sphere $\partial Q_d(x,r)$. $|\ldots|_m$ is used for appropriate measure, of dimension m, e.g. $|Q_d(x,r)|_d = \upsilon_d r^d$ where $\upsilon_d = \pi^{d/2}/\Gamma(\frac{d}{2}+1)$, and $|\partial Q_d(x,r)|_{d-1} = \sigma_d r^{d-1}$ where $\sigma_d = 2\pi^{d/2}/\Gamma(\frac{d}{2})$.

2. FLAT SECTIONS OF RANDOM TESSELLATIONS.

The following result is essentially due to Santaló [17; Section 5], but the form we present here is that given in [4; Relation (2.31T)]. Suppose X is a compact subset of R^d, and that $M_i\{X\}$ denotes its mean i-projection, i.e. the mean i-dimensional Lebesgue measure of its orthogonal projection onto an isotropic i-subspace in R^d ($i = 0,\ldots,d$; with $M_o \equiv 1$, $M_d \equiv |X|_d$). For smooth convex bodies, the mean projections equal, apart from constant factors, the quermassintegrals of integral geometry [4; Relation (2.27T)]. Let F_s be an isotropic uniform random (IUR) s-flat hitting X, i.e. governed by restricted and normalized invariant s-flat measure in R^d. Then $X \cap F_s$ is a random s-dimensional compact subset, which has its own set of (random) mean projections $M_j^{(s)}$ *with respect to* F_s *as containing space*, and we have the striking result

$$(2.1)\qquad E\, M_r^{(s)}\{X \cap F_s\} = M_{d-s+r}\{X\}/M_{d-s}\{X\} \qquad (0 \leqslant r \leqslant s \leqslant d)\ .$$

This extends to a corresponding result relating to a finite aggregate of compact subsets $\{{}_iX\}$ $(i = 1,\ldots,n)$ each $\subset X$, as follows. If the scalar or vector Z is some domain characteristic, then the aggregate mean value of Z is defined as

$$\bar{E}\{Z\} = n^{-1} \sum_{i=1}^{n} {}_iZ\ .$$

The (random) sectional mean $\bar{E}\{M_r^{(s)}\}$ for m independent IUR s-flat sections of X is also defined in the obvious way as the sum of the $M_r^{(s)}$ values for each flat/subset intersection, divided by the total number of such intersections; then, as $m \to \infty$, almost surely

$$(2.2) \qquad E\{M_r^{(s)}\} \longrightarrow E\{M_{d-s+r}\}/E\{M_{d-s}\} \ .$$

[7; Sections 5,6].

Although this result holds for rather general ${}_iX$, in this paper we shall only be concerned with the specific case where they form a (polytopal) tessellation, i.e. each point of X (apart from boundaries $\partial_i X$) lies in one and only one ${}_iX$ and, apart from edge effects on ∂X, the ${}_iX$ are d-dimensional convex polytopes.

Since X is arbitrary, (2.2) may be extended as an almost sure identity

$$(2.3) \qquad E\{M_r^{(s)}\} = E\{M_{d-s+r}\}/E\{M_{d-s}\}$$

for an ergodic homogeneous and isotropic random polytopal tessellation in R^d [9; Section 3.4.6], where $E\{M_i\}$ are ergodic mean polytope values and $E\{M_r^{(s)}\}$ is the corresponding mean value for an arbitrary s-flat section of the tessellation.

CONSISTENCY OF (2.3). These formulae are consistent in the following sense. Write T_d for the random tessellation in R^d, T_s for the sectional random tessellation $T_d \cap F_s$ and T_t for $T_s \cap F_t$, where $t < s$ and $F_t \subset F_s$. Then the values $E\{M_i^{(t)}\}$ for T_t may be obtained either by double application of (2.3), or alternatively by a single application of (2.3) with s=t. Equating these, there results a set of consistency relations between the $E\{M_i^{(t)}\}$.

As an example, consider the random polytopal tessellation $P_d(\rho)$ determined by isotropic Poisson hyperplanes of intensity ρ in R^d, $\mathbf{P}_\rho(d-1,d)$ ([9; Section 3.4.6]; see also [3; Chapter 6]). $\mathbf{P}_\rho(d-1,d)$ is characterized by the property that the number of hyperplanes hitting any compact $X \subset R^d$ has a Poisson $(\rho M_1\{X\})$ distribution [9; Theorem 1]. For $P_d(\rho_d)$

$$(2.4) \qquad E\{M_r\} = 2^r \frac{\Gamma(\frac{d}{2}+1)}{\Gamma(\frac{d-r}{2}+1)} \left\{\frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})\rho_d}\right\}^r$$

[9; Relation (62) with t=d, s=r]. Now

$$P_d(\rho_d) \cap F_s = P_s(\rho_s)$$

for which, by (2.3), (2.4) holds with d replaced by s, and

$$\rho_s = \{\Gamma(\tfrac{s+1}{2})\Gamma(\tfrac{d}{2})/\Gamma(\tfrac{s}{2})\Gamma(\tfrac{d+1}{2})\}\rho_d \ .$$

Thus we obtain as a byproduct the intensities of sections of $P_d(\rho)$.

MEAN CROSS-SECTION OF HIT AGGREGATE. Besides the sectional tessellation $\mathcal{T}_s = \mathcal{T}_d \cap F_s$, another quantity of interest is the union U of polytopes of $\mathcal{T}_d$ hit by F_s. We now derive a formula for the mean (d-s)-content, $E\{V_{d-s}\}$, of the intersection of U with orthogonal (d-s)-flats F_{d-s}. Suppose the generic 'f' denotes ergodic densities of polytopes of $\mathcal{T}_d$, in which each polytope has equal weight. Now the 'chance' F_s hits any specific polytope T of $\mathcal{T} \propto M_{d-s}\{T\}$, so that the aggregate of cells hit by F_s has ergodic densities $\propto M_{d-s} f(M_{d-s},.)$. Hence the mean d-volume V_d of each is

$$(2.5) \qquad E'\{V_d\} = E\{M_{d-s}\, V_d\}/E\{M_{d-s}\} .$$

For $\mathcal{T}_s$, by (2.3) ,

$$(2.6) \qquad E\{V_s\} = E\{V_d\}/E\{M_{d-s}\} .$$

It follows from (2.5), (2.6) that

$$\begin{aligned} E\{V_{d-s}\} &= E\{\text{d-content of U per unit s-content of } F_s\} = \\ &= E'\{V_d\}/E\{V_s\} = E\{M_{d-s}V_d\}/E\{V_d\} , \end{aligned}$$

which is the expectation of M_{d-s} for a V_d-weighted random member of $\mathcal{T}_d$.

THE POLYTOPAL CHARACTERISTICS $Y_j^{(k)}$. Actually, (2.3) applies to random aggregates of quite general random subsets of R^d. When specializing to tessellations, the facet structure of the polytope boundaries permits (2.3) to be replaced by a larger system of such basic relations. Writing $T_{t,i}$ $(i = 1,\dots,N_t)$ for the N_t t-facets of a convex polytope T, we define

$$Y_j^{(k)}\{T\} = \sum_{i=1}^{N_k} M_j^{(k)}\{T_{k,i}\} .$$

Defining L_r to be the sum of the r-contents of the N_r r-facets of T, we have the special cases

$$Y_r^{(r)} = L_r \quad , \quad Y_r^{(d)} = M_r \quad , \quad Y_o^{(r)} = N_r \qquad (0 \leqslant r \leqslant d) .$$

(2.1) is replaced by the larger system

$$(2.7) \quad E\, Y_r^{(s+u-d)}\{T \cap F_s\} = \kappa_d(s,u) Y_{d-s+r}^{(u)}\{T\}/M_{d-s}^{(d)}\{T\} , \; (0 \leqslant r \leqslant s+u-d \leqslant s \leqslant d),$$

where

$$\kappa_d(s,u) = \Gamma(\tfrac{s+1}{2})\Gamma(\tfrac{u+1}{2})/\Gamma(\tfrac{s+u-d+1}{2})\Gamma(\tfrac{d+1}{2})$$

[7; Section 10]. As (2.1) becomes (2.3) for a random tessellation, so

(2.7) must be replaced by the same relation having two expectations on the right side. An important formula for convex polytopes is

$$(2.8)\qquad M_s\{T\} = \{\Gamma(\tfrac{s+1}{2})\Gamma(\tfrac{d-s+1}{2})/\pi^{\frac{1}{2}}\Gamma(\tfrac{d+1}{2})\}\sum_{i=1}^{N_s} L_{s,i}\psi_{s,i}\quad ,$$

where $\psi_{s,i}$ is the normalized (so that the total angle at an s-facet, in the orthogonal (d-s)-subspace, is 1) exterior angle at the s-facet $T_{s,i}$ [4; Relation (2.18T)].

3. VORONOI TESSELLATIONS.

In geometrical statistical applications, it is desirable to have a variety of specific random tessellations, for modelling purposes. A natural source of such models are three dimensional flat sections of higher dimensional tessellations. As we have just seen, sectioning P tessellations leads to nothing new. However, this is not the case for the other basic class of specific tessellations, the Voronoi (sometimes Thiessen, or Dirichlet) tessellations. We now determine basic properties of Voronoi tessellations, before considering their flat sections in Section 4.

The basic building block for a Voronoi tessellation is an underlying stochastic point process. For simplicity, we shall simply take the latter as the homogeneous Poisson point process $\mathbf{P}_\rho(0,d)$ of intensity ρ in R^d, for which the number of point *particles* falling in any measurable set X has a Poisson $(\rho|X|_d)$ distribution, and realizations in disjoint sets are mutually independent. Each point $x \in R^d$ has an (almost surely well-defined) nearest particle of $\mathbf{P}_\rho(0,d)$. The set of all x with the same nearest particle is (almost surely) the intersection of a finite number of (open) halfspaces in mutual general position, and so is a (simple convex) polytope T_x [1;p.58]; $x \in T_x$ and may be regarded as its *nucleus* (particle). Being simple, every s-facet of T_x lies in the boundaries of $\binom{d-s}{d-t}$ t-facets of T_x $(0 \leqslant s \leqslant t \leqslant d)$.
The aggregate of such polytopal *cells* constitutes a random tessellation $V = V(d)$ of R^d, which is ergodic, homogeneous and isotropic. $V(1)$ is a sequence of random intervals in R^1. It is easily analysed, with the interval distribution being $\Gamma(2,2\rho)$, i.e. the distribution of the sum of two independent exponential (2ρ) random variables. For discussions of $V(2)$ and $V(3)$, the reader may consult [2] and [8], respectively.

BASIC (ALMOST SURE) PROPERTIES OF V. As with all polytopal tessellations, each (d-1)-facet bounds two cells, but in this case it is a portion of the perpendicular hyperplane bisector of the segment joining the two associated nuclei. More generally, each s-facet lies in the boundaries of d-s+1 cells ($s = 0,\dots,d-1$): tessellations having this property we call *normal*, because real-life tessellations for $d = 1,2,3$ commonly possess this property. Moreover, for V, each s-facet is a portion of the s-flat all of whose points are equidistant from the associated d-s+1 nuclei. In particular, each vertex (0-facet) is a vertex of d+1 cells and is the circumcentre of the circumsphere through the associated d+1 nuclei.

Now for some notation. For particles $x_o,\dots,x_{d-s}$ in general position in R^d,

$$F_s = \{y: |yx_o|_1 = |yx_1|_1 = \dots = |yx_{d-s}|_1\}$$

is the equidistant s-flat ($s = 0,\dots d-1$). $y \in F_s$ lies in the common s-facet of the cells with nuclei $x_o,\dots,x_{d-s}$ iff the unique d-sphere centre y through $x_o,\dots,x_{d-s}$ contains no other particles of $\mathbf{P}_\rho(0,d)$.

THE VALUE OF $E\{L_s\}$ FOR $V(d)$. If obvious interest are the ergodic distributions and moments of characteristics of the members of $V(d)$. Writing $V_d \equiv L_d^{(d)}$, one obvious one is

$$E\{V_d\} = \rho^{-1} , \tag{3.1}$$

true whatever the underlying (ergodic) stochastic point process. We shall now derive the values of the other $E\{L_s\}$, and apply them in investigating sectional Voronoi tessellations (Section 4).

The method relies heavily on a re-parametrization of $x_o,\dots,x_{d-s}$ - supposed to have general position in R^d - which lie in a unique (d-s)-flat F_{d-s}. Write ∇_{d-s} for (d-s)! times the (d-s)-content of the (d-s)-simplex with vertices $x_o,\dots,x_{d-s}$, and suppose $Q_{d-s}(z,R)$ is the unique (d-s)-sphere through $x_o,\dots,x_{d-s}$. Then we have, in polar coordinates within F_{d-s},

$$\vec{zx_i} = R\, u_i^{(d-s)}$$

for unit vectors $u_i^{(d-s)}$; write $dO_i^{(d-s)}$ for the volume element of a unit sphere in F_{d-s} corresponding to $u_i^{(d-s)}$ ($i = 0,\dots,d-s$). Finally, write $F_{s(0)}$ for the s-subspace orthogonal to F_{d-s}, and parallel to $F_s = \{z\} + F_{s(0)}$.

Now the (d-s+1)-set $x_o,\dots,x_{d-s}$ is alternatively parametrized by

$$z,\ R,\ F_{s(0)},\ u_o^{(d-s)},\dots,u_{d-s}^{(d-s)}$$

and we have the corresponding integral geometric density relationship

(3.2) $$dx_o \dots dx_{d-s} = \nabla_{d-s}^{s+1} R^{d(d-s)-1}\, dz\, dR\, dF_{s(0)}\, dO_o^{(d-s)} \dots dO_{d-s}^{(d-s)} ,$$

due to Blaschke & Petkantschin [9; Relation [74)]. Next, we express points y in F_s in terms of polar coordinates $(S,v^{(s)})$ within F_s with respect to z as origin, so that

$$|y\ x_i|_1 = (R^2 + S^2)^{1/2} \equiv T \qquad (i = 0,\dots d-s)$$

say, and

(3.3) $$\begin{aligned}&\Pr\{y \in \text{associated s-facet of } V|\text{particles at } x_o,\dots,x_{d-s}\}\\ &= \Pr\{\text{int } Q_d(y,T) \text{ contains no particles}\}\\ &= \exp(-\rho\, \upsilon_d T^d) .\end{aligned}$$

The probability element for particles of $\mathbf{P}_\rho(0,d)$ in $dx_o,\dots,dx_{d-s}$ is $\rho^{d-s+1}\, \Pi_{i=1}^{d-s}\, dx_i$, so that the probability element for a $(d-s+1)$-set of particles within limitations dz, dR, $dF_{s(0)}, dO_o^{(d-s)},\dots,dO_{d-s}^{(d-s)}$ is

$$\rho^{d-s+1}\, \nabla_{d-s}^{s+1}\, R^{d(d-s)-1}\, dz\, dR\, dF_{s(0)}\, dO_o^{(d-s)} \dots dO_{d-s}^{(d-s)} .$$

Now consider the contribution ℓ_s from given particles at $x_o,\dots x_{d-s}$ to the total s-facet content. We may write

$$\ell_s = \iint I(S,v^{(s)}) S^{s-1}\, dS\, dO^{(s)}$$

where $I(S,v^{(s)})$ indicates that $(S,v^{(s)})$ lies in an s-facet of V. Hence, by the complete independence of Poisson point processes,

(3.4) $$\begin{aligned}&E\{\ell_s|\text{particles at } x_o,\dots,x_{d-s}\}\\ &= \iint E\{I(S,v^{(s)})\}S^{s-1}\, dS\, dO^{(s)}\\ &= \sigma_s \int \exp(-\rho\upsilon_j T^d)\ S^{s-1}\, dS .\end{aligned}$$

It follows from (3.3) and (3.4) that

(3.5) $$E\{\ell_s \text{ from particle } (d-s+1)\text{-sets with circumcentre in } dz\} =$$

$$= \frac{\rho^{d-s+1} dz}{(d-s+1)!} \underbrace{\int dF_{s(0)}}_{J_1} \underbrace{\int_0^\infty\!\!\int_0^\infty R^{d(d-s)-1}\ S^{s-1} \exp(-\rho\upsilon_d T^d) dR\, dS}_{J_2}$$

$$\underbrace{\int\dots\int \nabla_{d-s}^{s+1}\, dO_o^{(d-s)} \dots\, dO_{d-s}^{(d-s)}}_{J_3}$$

$$\equiv X_s \, dz ,$$

say. The (d-s+1)! factor arises because with total integration every particle (d-s+1)-set is counted this many times. By [6; Relation (12)]

$$J_1 = \sigma_{d-s+1} \cdots \sigma_d / \sigma_1 \cdots \sigma_s , \tag{3.6}$$

while

$$J_2 = \frac{\Gamma(d-s+\frac{s}{d})}{d(\rho v_d)^{d-s+(s/d)}} \; \frac{\Gamma(\frac{d(d-s)}{2})\Gamma(\frac{s}{2})}{2\,\Gamma(\frac{d(d-s)+s}{2})} . \tag{3.7}$$

As for J_2, this is σ_{d-s}^{d-s+1} times the mean value of ∇_{d-s}^{s+1} for d-s+1 particles chosen independently and uniformly on the unit sphere in R^{d-s}. Its value,

$$J_3 = \frac{\Gamma(\frac{d^2-sd+s+1}{2})}{\Gamma(\frac{d^2-sd}{2})} \left\{ \frac{\Gamma(\frac{d-s}{2})}{\Gamma(\frac{d+1}{2})} \right\}^{d-s} \frac{\Gamma(\frac{s+2}{2}) \cdots \Gamma(\frac{d}{2})}{\Gamma(\frac{1}{2}) \cdots \Gamma(\frac{d-s-1}{2})} \, \sigma_{d-s}^{d-s+1} , \tag{3.8}$$

is derived in [6; Theorem 2], essentially by manipulation of the basic Blaschke-Petkantschin formula (3.2). Now X_s in (3.5) is the average s-content of s-facet per unit d-content of R^d. Hence, since each s-facet is an s-facet of d-s+1 distinct cells of V, we have

$$E\{L_s\} = (d-s+1) \, X_s \, E(V_d) \tag{3.9}$$

which, by (3.1) and (3.5) - (3.8) ,

$$= \frac{2^{d-s+1}\pi^{(d-s)/2}\Gamma(\frac{d^2-sd+s+1}{2})\Gamma(\frac{d}{2}+1)^{d-s+(s/d)}\Gamma(d-s+\frac{s}{d})}{(d-s)!\; d\Gamma(\frac{d^2-sd+s}{2})\;\Gamma(\frac{d+1}{2})^{d-s}\;\Gamma(\frac{s+1}{2})\;\rho^{s/d}} \qquad (0 \leqslant s \leqslant d).$$

Special cases are, for s=d, (3.1) and, for s=0, the mean number of vertices

$$E\{N_o\} = \frac{2^{d+1}\,\pi^{\frac{d-1}{2}}}{d^2} \; \frac{\Gamma(\frac{d^2+1}{2})}{\Gamma(\frac{d^2}{2})} \left\{ \frac{\Gamma(\frac{d}{2}+1)}{\Gamma(\frac{d+1}{2})} \right\}^d \qquad (d = 1,2,\ldots) .$$

No other ergodic distributions or moments of V are known. Obvious targets are formulae for $E\{M_s\}$ and $E\{N_s'\}$.

Each vertex of V is the circumcentre of a set of d+1 particles of $\mathbf{P}_\rho(0,d)$, the convex hull of which is a simplex. It turns out that the aggregate of such simplices is a random tessellation - the Delaunay tessellation [11]. Its ergodic distribution and the values of $E\{V_d^k\}$

are derived in [9; Relations (76),(77)].

4. SECTIONAL VORONOI TESSELLATIONS.

Our main concern in this paper is with the sectional Voronoi tessellations

$$V(s,d) = V(d) \cap F_s$$

for an arbitrary s-flat F_s. Note that, in this notation, $V = V(d) = V(d,d)$. Providing the intersections are nonvoid, F_s intersects cells of V in simple s-polytopes and t-facets in (s+t-d)-facets. As expected, each such (s+t-d)-facet lies in the boundaries of s-(s+t-d)+1 = d-t+1 cells of $V(s,d)$. Thus, topologically, $V(s,d)$ has the same properties relative to F_s as V has relative to R^d, and is a normal tessellation.

We now investigate the application of (2.3) to $V(s,d)$.

s = 1: Hence F_1 intersects the polytope boundaries of V in an ergodic stationary (= homogeneous) stochastic point process. $V(1,d)$ comprises the intervals so formed, and the obvious goal here is to determine the (ergodic) interval length (L) distribution. (2.3) reduces to one relation, viz.

$$E\{L\} = E\{V_d\}/E\{M_{d-1}\} \tag{4.1}$$

which, by [4; Relation (2.21)] ,

$$= \{2\pi^{\frac{1}{2}}\Gamma(\tfrac{d+1}{2})/\Gamma(\tfrac{d}{2})\}\ E(V_d)/E(L_{d-1})$$

which, by (3.9) ,

$$= \frac{\Gamma(d-\frac{1}{2})\cdot\Gamma(\frac{d+1}{2})^2}{(d-1)!\,\Gamma(\frac{d}{2})\Gamma(\frac{d}{2}+1)^{1-(1/d)}\Gamma(2-\frac{1}{d})\rho^{1/d}} .$$

Note that, as $d \to \infty$,

$$E\{L\} \to (2e)^{-\frac{1}{2}} = 0.4289 .$$

This limiting process is examined in closer detail, and an integral expression for the distribution of L is given, in the next section.

s = 2: Here $V(2,d)$ is a planar tessellation, and (2.3) yields the two relations

$$E\{A\} = E\{V_d\}/E\{M_{d-2}\} , \tag{4.2}$$

$$\pi^{-1}E\{B\} = E\{M_{d-1}\}/E\{M_{d-2}\} \tag{4.3}$$

(A = area, B = perimeter). Simple geometric considerations in a 2-flat orthogonal to any (d-2)-facet show that the sum of the three exterior angles there is 1/2. It follows from (2.8) that, for V,

$$E\{M_{d-2}\} = E\{L_{d-2}\}/6(d-1)$$

and so (4.2), (4.3) become

$$(4.4) \qquad E\{A\} = \frac{3\ d\ \Gamma(\frac{3d}{2}-1)\ \Gamma(\frac{d+1}{2})^3}{\pi\Gamma(\frac{3d-1}{2})\ \Gamma(\frac{d}{2}+1)^{3-(2/d)}\ \Gamma(3-\frac{2}{d})\ \rho^{2/d}},$$

$$(4.5) \qquad E\{B\} = \frac{6(d-1)!\ \Gamma(\frac{d+1}{2})\Gamma(\frac{3d}{2}-1)\ \Gamma(2-\frac{1}{d})}{\Gamma(d-\frac{1}{2})\ \Gamma(\frac{d}{2}+1)^{1-(1/d)}\ \Gamma(\frac{3d-1}{2})\ \Gamma(3-\frac{2}{d})\ \rho^{1/d}}.$$

Of course, because each vertex in $V(2,d)$ is vertex of three polygons, the mean number of vertices

$$E\{N\} = 6 .$$

As $d \to \infty$,

$$E\{A\} \to 3^{\frac{1}{2}}/\pi e = 0.2028$$

and

$$E\{B\} \to (6/e)^{\frac{1}{2}} = 1.486 .$$

The reader may check the consistency of (4.1) and (4.4), (4.5), by considering a line section of $V(2,d)$.

The dimensionless aggregate polygon rotundness measure

$$\theta = 4\pi\ E\{A\}/E\{B\}^2 ,$$

as a function of d, is of interest. As d increases from 2 to ∞, it increases from 0.785 to 1.155, suggesting that the polygons become more rotund on average as d increases (cf. [5; p.119]).

<u>s = 3</u>: Here application of (2.3) (and (2.7)) yields

$$(4.6) \qquad E\{V\} = E\{V_d\}/E\{M_{d-3}^{(d)}\}$$

$$(4.7) \qquad E\{S\} = \{2\Gamma(\tfrac{d}{2})/\pi^{\frac{1}{2}}\Gamma(\tfrac{d+1}{2})\}E\{L_{d-1}^{(d)}\}/E\{M_{d-3}^{(d)}\}$$

$$(4.8) \qquad E\{M_1^{(3)}\} = E\{M_{d-2}^{(d)}\}/E\{M_{d-3}^{(d)}\}$$

$$(4.9) \qquad E\{L_1^{(3)}\} = \{2/(d-1)\}\ E\{L_{d-2}^{(d)}\}/L\{M_{d-3}^{(d)}\}$$

$$(4.10) \qquad E\{N_o^{(3)}\} = \{\Gamma(\tfrac{d-2}{2})/\pi^{\frac{1}{2}}\Gamma(\tfrac{d+1}{2})\}E\{L_{d-3}^{(d)}\}/E\{M_{d-3}^{(d)}\} ,$$

where S = surface area. Note that, by Euler's formula and the fact that each vertex lies in three faces, $N_1^{(3)} = (3/2)N_o^{(3)}$ and $N_2^{(3)} = (N_o^{(3)}) + 2$. Unfortunately, there appears to be no geometrical identity governing the exterior angles at (d-3)-facets, allowing $E\{M_{d-3}^{(d)}\}$ to be determined by means of (2.8). However, we know from (3.3) with $s = d-3$ that the conditional orientation density of the four particles generating a (d-3)-facet of V is proportional to V_3^{d-2} on the orthogonal 3-sphere. As $d \to \infty$, this distribution tends to degeneracy, in which the four particles form an (isotropically oriented) equilateral tetrahedron. Consider now the consequences for the interior and exterior angles at (d-3)-facets of V. The interior angles actually correspond to the spherical Voronoi division of the 3-sphere generated by the particle orientations, and so each tends to 1/4. The exterior angles are those of the dual regions on the 3-sphere [13; p.708]; by this duality

$$A + B^* = A^* + B = \frac{1}{2} \cdot$$

From this it follows that each exterior angle tends to

$$\psi = \frac{1}{8} - \frac{3}{4\pi} \sin^{-1}(\tfrac{1}{3}) ,$$

so that, by (2.8), as $d \to \infty$,

$$E\{M_{d-3}^{(d)}\}/E\{L_{d-3}^{(d)}\} \to \{\Gamma(\tfrac{d}{2} - 1)/\pi^{\frac{1}{2}}\Gamma(\tfrac{d+1}{2})\}\ \psi \quad .$$

Application of this and other formulae to (4.6) - (4.10) yields

$$E\{V\} \to 1/16\pi e^{3/2}\ \psi \qquad = 0.1012$$

$$E\{S\} \to 1/2^{3/2}\pi e\ \psi \qquad = 0.9437$$

$$E\{M_1^{(3)}\} = E\{L_1^{(3)}\}/12 \to 1/16(3e)^{1/2}\ \psi \qquad = 0.4989$$

$$E\{N_2^{(3)}\} \to 2 + (1/2\psi) \qquad = 13.39$$

$$E\{N_1^{(3)}\} \to 3/2\ \psi \qquad = 34.19$$

$$E\{N_o^{(3)}\} \to 1/\psi \qquad = 22.79$$

as $d \to \infty$. Real-life observational and experimental models have indicated the common ocurrence of random normal tessellations with values of $E\{N_2^{(3)}\}$ between 13 and 15. Thus, assuming that $E\{N_2^{(3)}\}$ for $V(3,d)$ decreases monotonically from 15.54 to 13.39 as d increases from 3 to ∞, the random tessellations $\{V(3,d)\}$ ($d = 3,4,\ldots$) may be advanced as natural stochastic models for these phenomena. For further details, see [8; Section 6].

$s \geq 4$: The above cases $s = 1,2,3$ are those of obvious practical significance from a modelling viewpoint. In principle, such theory may be carried out for general s, with its asymptotics involving an equilateral $(s+1)$-simplex inscribed in an s-sphere.

5. REDUCED DIMENSION STOCHASTIC EQUIVALENCE.

The effect of any particle x in $\mathbf{P}_\rho(0,d)$ on $V(s,d)$ is only by way of its nearest point y of F_s and the distance $|xy|_1$. Hence $V(s,d)$ is stochastically equivalent to $V(s,s+1)$ with respect to a new $\mathbf{P}(0,s+1)$ density which is that of $\mathbf{P}_\rho(0,d)$ collapsed by rotation onto a half-F_{s+1} with bounding s-flat F_s. This density is $\rho\, \sigma_{d-s} r^{d-s-1}$, where r denotes orthogonal distance from F_s; note that it is inhomogeneous. We illustrate this stochastic equivalence in the case s=1 by considering, in the first instance, the case in which the particles form an inhomogeneous Poisson process in R^2 of intensity $\rho(y)$ $(y \geqslant 0)$, with $F_1 =$ = the x-axis.

We begin by exploring the joint distribution of two particles in R^2, given that they give rise, as the intersection of their perpendicular bisector with Ox, to an endpoint H of the interval process $V(1,2)$ on Ox. The necessary and sufficient condition for this to occur is shown in Fig.1.

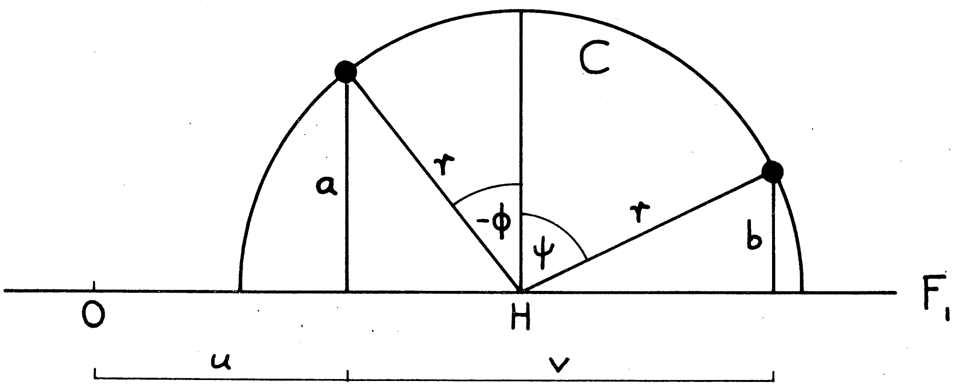

Fig.1. Geometry of an interval end point H of $V(1,d)$.

That is, there are two particles on the semicircle C with centre H, radius r, and none within C. We suppose those particles have angular coordinates ϕ,ψ $(-\pi/2 \leqslant \phi \leqslant \psi \leqslant \pi/2)$ with respect to the orthogonal to Ox at H, and are interested in the joint distribution of $(r;\phi,\psi)$ given that the two particles give rise to H. The method is elementary, and uses the alternative coordinates shown in Fig.1, i.e. particles at (u,a) and (u+v,b) $(a,v,b > 0)$. We have

(5.1) $\quad$ Pr{particles in (du,da) and $(d\overline{u+v},db)$, and none in int C} =

$$= \rho(a)\,\rho(b)\,du\,da\,dv\,db\,\exp\{-2\int_0^r \rho(y)(r^2-y^2)^{1/2}dy\}\ .$$

Integrating this respect to u over an interval of unit length, we obtain $E\{L\}^{-1} f(a,v,b)\,da\,dv\,db$, where the joint density f(a,v,b) relates to a random such configuration on F_1. (Here, and below, f(*) means 'density of *'). Thus, transforming to the polar coordinates (r,ϕ,ψ) (Fig.1), we have the ergodic density

(5.2)
$$f(r,\phi,\psi) \propto \rho(r\cos\phi)\rho(r\cos\psi)\, r^2(\sin\psi - \sin\phi)\,\exp\{-2\int_0^r \rho(y)(r^2-y^2)^{1/2}\,dy\}\ .$$

We now specialise to $V(1,d)$, for which $\rho(y) = \rho\sigma_{d-1}y^{d-2}$. Substitution of this in (5.2) shows that r and (ϕ,ψ) are independent, with normalized marginal probability densities

$$f(r) = \frac{d}{\Gamma(2-\frac{1}{d})}\left\{\frac{\rho\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}\right\}^{2-\frac{1}{d}} r^{2d-2}\exp\left\{-\frac{\rho\pi^{d/2}r^d}{\Gamma(\frac{d}{2}+1)}\right\} \qquad (r \geqslant 0)\ ,$$

$$f(\phi,\psi) = \frac{(2d-2)!}{2^{2d-1}\,(d-2)!^2}(\cos\phi\cos\psi)^{d-2}(\sin\psi - \sin\phi) \quad (-\tfrac{\pi}{2} \leq \phi \leq \psi \leq \tfrac{\pi}{2}).$$

A swift integration gives

$$E\{r\} = \frac{1}{\Gamma(2-\frac{1}{d})}\left\{\frac{\Gamma(\frac{d}{2}+1)}{\rho\pi^{d/2}}\right\}^{\frac{1}{d}} \sim (d/2\pi e)^{1/2} \quad \text{as} \quad d \to \infty\ ,$$

while the mean projected particle separation onto Ox is

$$E\{r(\sin\psi - \sin\phi)\} = E\{r\}.E\{\sin\psi - \sin\phi\} = E\{L\}$$

given in (4.1).

To investigate the limiting behaviour of the distributions of r, (ϕ,ψ), we consider the new variables

$$R = \{(2e)^{\frac{d}{2}}\,\pi^{\frac{d-1}{2}} / d^{\frac{d+1}{2}}\}\, r^d$$

$$\alpha = d^{1/2}\phi \quad , \quad \beta = d^{1/2}\psi\ .$$

Then it is easily shown that, as $d \to \infty$,

(i) the distribution of $R \to \Gamma(2,\rho)$, so that $r/E\{r\} \to 1$ in probability;

(ii) the joint probability density

$$f(\alpha,\beta) \to (2\pi^{\frac{1}{2}})^{-1}(\beta-\alpha)\exp\{-\tfrac{1}{2}(\alpha^2+\beta^2)\} \qquad (-\infty < \alpha \leqslant \beta < \infty) \quad ,$$

with limiting marginal density

$$f(\alpha) = (2\pi^{\frac{1}{2}})^{-1}[e^{-\alpha^2} - (2\pi)^{\frac{1}{2}}\alpha e^{-\alpha^2/2}\{1-\Phi(\alpha)\}] \qquad (-\infty < \alpha < \infty).$$

For the marginal of β, note that β and $-\alpha$ have the same distribution. Note also that, as $d \to \infty$, the projection of the segment joining the two particles onto $F_1 \sim r(\psi-\phi) \sim (2\pi e)^{-\frac{1}{2}}\,(\beta-\alpha)$.

This approach may be extended to two adjacent semicircles, resulting in an integral expression for the distribution of interval length L in $V(1,d)$.

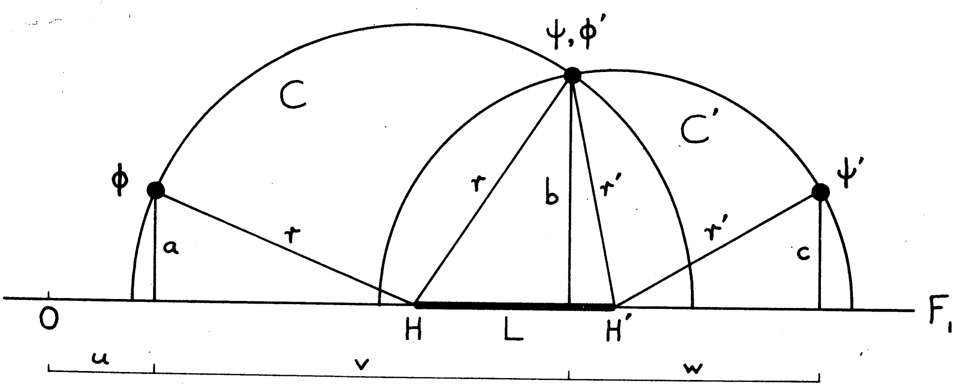

Fig.2. Geometry of an interval HH' of $V(1,d)$.

Fig.2 shows the geometry. We have particles P_1,P_2,P_3 at the points (u,a), $(u+v,b)$ and $(u+v+w,c)$, respectively $(a,b,c,v,w > 0)$. P_1 and P_2 determine the semicircle C with centre $H \in F_1$, as above, and likewise P_2 and P_3 determine the semicircle C' with centre H' also $\in F_1$. The Voronoi geometry requires that there are no other particles within $U = C \cup C'$, so that HH', of length L, is a typical interval of $V(1,d)$. Then, with Poisson intensity $\rho(y)$, the analogue of (5.1) is

$$\Pr\{\text{particles in } (du,da),\ (d\overline{u+v},db) \text{ and } (d\overline{u+v+w},dc)\ ,$$
$$\text{and none in int } U\}$$
$$= \rho(a)\rho(b)\rho(c)\ du\ da\ dv\ db\ dw\ dc\quad \exp\{-\int_0^{\max(r,r')} \rho(y)\ell(y)\ dy\ ,$$

where $\ell(y)$ is the length of the intersection of a line parallel to, and distant y from, F_1 with U, and r,r' are the radii of C,C'. Again integration with respect to u over a unit interval gives

$E\{L\}^{-1}$ $f(a,b,c,v,w)$ da db dc dv dw. Next we switch from (a,b,c,v,w) to $(L,\phi,\psi,\phi',\psi')$, where

$$P_1 \text{ has polar coordinates } (r,\phi) \text{ with respect to } H$$

$$P_2 \text{ has polar coordinates} \begin{cases} (r,\psi) \text{ with respect to } H \\ (r',\phi') \text{ with respect to } H' \end{cases}$$

$$P_3 \text{ has polar coordinates } (r',\psi') \text{ with respect to } H'$$

(fig.2, cf. also Fig.1). The transformation relations are

$$\begin{aligned} a &= L \cos\phi \cos\phi'/\sin(\psi-\phi') \\ b &= L \cos\psi \cos\phi'/\sin(\psi-\phi') \\ c &= L \cos\psi \cos\psi'/\sin(\psi-\phi') \\ v &= L \cos\phi'(\sin\psi - \sin\phi)/\sin(\psi-\phi') \\ w &= L \cos\psi\,(\sin\psi' - \sin\phi')/\sin(\psi-\phi') \end{aligned}$$

with

$$\frac{\partial(a,b,c,v,w)}{\partial(L,\phi,\psi,\phi',\psi')} = \frac{L^4(\cos\psi\cos\phi')^2}{\sin\phi\sin\psi'\sin^7(\psi-\phi')} \cdot$$

$$\cdot\ \{\cos\psi\cos(\psi-\phi') + \sin\phi\sin(\psi-\phi') - \cos\phi'\}\{\cos\phi'\cos(\psi-\phi') - \\ - \sin\psi'\sin(\psi-\phi') - \cos\psi\}.$$

Thus in principle we have the joint density

$$f(L,\phi,\psi,\phi',\psi') = E\{L\}\left|\frac{\partial(a,b,c,v,w)}{\partial(L,\phi,\psi,\phi',\psi')}\right| \rho(a)\rho(b)\rho(c) \\ \exp\{-\int_0^{\max(r,r')} \rho(y)\ell(y)\,dy\}$$

where

$$r = L\cos\phi'/\sin(\psi-\phi') \quad , \quad r' = L\cos\psi/\sin(\psi-\phi') .$$

Finally, the marginal density of L results on integrating $f(L,\phi,\psi,\phi',\psi')$ over the (ϕ,ψ,ϕ',ψ')-set

$$[-\pi/2 \leqslant \phi \leqslant \psi \leqslant \pi/2] \cap [-\pi/2 < \phi' \leqslant \psi' < \pi/2] \cap [\phi' \leqslant \psi] .$$

In the $V(1,d)$ case, when $\rho(y) = \rho\sigma_{d-1}y^{d-2}$, $E\{L\}$ is given by (4.1) and, as may be anticipated from Fig.2, the integrations with respect to ϕ (from $-\pi/2$ to ψ) and ψ' (from ϕ' to $\pi/2$) are elementary, being finite series in closed form.

6. GENERALIZED VORONOI TESSELLATIONS.

The standard Voronoi tessellation V is defined in terms of proximity to a *single* particle of the underlying point process. However, each point of R^d (almost surely) has a well-defined set of nearest n particles (n = 2,3,...). Similarly the set of points with the same nearest n particles constitutes a simple convex polytope, and the aggregate of such polytopal cells is a generalized Voronoi tessellation V_n of R^d (see [5,10] for discussions of the case d=2); thus $V = V_1$. Like V and $V(s,d)$, V_n is a normal random tessellation.

One piece of the previous theory extends effortlessly to V_n. We have (cf. Section 3 for notation): for particles $x_0,\dots,x_{d-s}$ a point $y \in F$, lies in an associated s-facet of V_n iff int $Q_d(y,T)$ contains n-1 particles of $\mathbf{P}_\rho(0,d)$, an event of probability

$$(\rho\upsilon_d T^d)^{n-1} \exp(-\rho\upsilon_d T^d)/(n-1)! \ .$$

With this modification, the theory of Section 3 carries over, to yield (suffix n ~ V_n)

$$\frac{E_n\{L_s\}}{E_n\{V_d\}} = \frac{\Gamma(d+n-s-1+\frac{s}{d})}{(n-1)!\,\Gamma(d-s+\frac{s}{d})}\ \frac{E\{L_s\}}{E\{V_d\}} =$$

$$= \frac{2^{d-s+1}\ \pi^{\frac{d-s}{2}}\ \Gamma(\frac{d^2-sd+s+1}{2})\ \Gamma(\frac{d}{2}+1)^{d-s+\frac{s}{d}}\ \Gamma(d+n-s-1+\frac{s}{d})}{(n-1)!\ (d-s)!\ d\,\Gamma(\frac{d^2-sd+s}{2})\ \Gamma(\frac{d+1}{2})^{d-s}\ \Gamma(\frac{s+1}{2})}\ \rho^{1-\frac{s}{d}} \ .$$

Unfortunately, $E_n\{V_d\}$ is only known in two cases, viz.

$$E_n\{V_1\} = 1/\rho \ ,$$

$$E_n\{V_2\} = 1/(2n-1)\rho \qquad [5;\ \text{Theorem } 10.1]\ .$$

Apart from this it is known that, because of the one-to-one correspondence between (d-1)-facets of V and cells of V_2 ,

$$E_2\{V_d\} = 2\ E\{V_d\}/E\{N_{d-1}\} \ ,$$

where of course the value of $E\{N_{d-1}\}$ is also unknown.

We may also section V_n , and naturally write

$$V_n(s,d) = V_n(d,d) \cap F_s \ .$$

It is possible to write down an integral expression for the ergodic moments of V_s for $V_n(s,d)$, as we now show. Select an arbitrary point $0 = x_o$ as origin in F_s. It lies in a random polytope T_o of $V_n(s,d)$ whose distribution is that of a uniform random member of $V_n(s,d)$ weighted by V_s. Thus the kth order moment of $V_s(T_o)$ is

$$E_{n,V_s}\{V_s^k\} = E_n\{V_s^{k+1}\}/E_n\{V_s\} .$$

Now we may write

$$V_s(T_o) = \int_{F_s} I(x)\, dx$$

where $I(x)$ indicates that $x \in T_o$. Thus, in the usual way,

$$\begin{aligned} E_{n,V_s}\{V_s^k\} &= E \int_{F_s} \cdots \int_{F_s} I(x_1)\ldots I(x_k)\, dx_1 \ldots dx_k \\ &= \int_{F_s} \cdots \int_{F_s} E\{I(x_1)\ldots I(x_k)\}\, dx_1 \ldots dx_k \\ &= \int_{F_s} \cdots \int_{F_s} \Pr\{x_1,\ldots,x_k \text{ all } \in T_o\}\, dx_1 \ldots dx_k . \end{aligned}$$

Now

$$\Pr\{x_1,\ldots,x_k \text{ all } \in T_o\} = \Pr\{x_o,\ldots,x_k \in \text{some cell of } V_n\}$$

and the latter event occurs iff there are particles of $\mathbf{P}_\rho(0,d)$ at $y_1,\ldots,y_n$ and

$$U = \operatorname{int} \bigcup_{i=0}^{k} \bigcup_{j=1}^{n} Q_d(x_i, |y_j - x_i|_1)$$

contains no particles of $\mathbf{P}_\rho(0,d)$. Thus

$$E_{n,V_s}\{V_s^k\} = \int_{F_s} \cdots \int_{F_s} \int_{R^d} \cdots \int_{R^d} e^{-\rho|U|_d}\, dy_1 \ldots dy_n\, dx_o \ldots dx_k .$$

Further progress seems unlikely, given the complex nature of the ball union U.

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Department of Statistic, I.A.S.
Australian National University, Canberra.

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