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SOME RESULTS ON FIXED POINTS

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ABSTRACT. In this paper we have studied fixed point theorems for pair of contractive type mappings involving four points of the space. This result includes as special cases that of Reich [7], Kannan [2] and Maiti et al.[3] and also as a consequence is applicable to various discontinuous mappings.

1. INTRODUCTION.

In recent years many extensions and generalizations of Banach's fixed point theorem had been done by many authors. But in all the cases the mapping under consideration involves only two points of the space. Until recently, Pittnauer [4] and also Rhoades [6] studied contractive type mappings involving three points of the space. Pittnauer [5] also studied mappings involving four points of the space.

In this paper we have studied a fixed point theorem for pair of contractive type mappings involving four points of the space. We have then extended this result to family of mappings. Finally we have shown that our result contains as special cases that of Reich [7], Kannan [2] and Maiti et al.[3].

Let (X,d) be a complete metric space. Let $\psi_i: \overline{P} \underset{i=1,2,3}{\longrightarrow} [0,\infty)$ (P is the range of d and $\overline{P}$ is the closure of P) be upper semi-continuous functions from the right on $\overline{P}$ and satisfies the condition

$$\psi_i(t) < t/3 \quad \text{for} \quad t > 0 \quad \text{and} \quad \psi_i(0) = 0, \quad i=1,2,3 \qquad (1.1).$$

Also let f and g be mappings of X into itself such that

$$d(fu_1,gu_2) \leq \psi_1[d(u_1,u_2)] + \psi_2[d(u_1,fu_3)]+\psi_3[d(u_2,gu_4)] \qquad (1.2)$$

for u_1, u_2, u_3, $u_4 \in X$.

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2. FIXED POINT THEOREMS.

The following theorem is patterned after the result of Boyd and Wong [1] with necessary modifications as required for the more general settings.

THEOREM 1. *If* f *and* g *be mappings of* X *into itself satisfying* (1.2) , *then* f *and* g *have a unique common fixed point.*

Proof. Let $x,y \in X$ and we define

$$u_1 = gy \quad , \quad u_2 = fx \quad , \quad u_3 = x \quad , \quad u_1 = y.$$

Then (1.2) takes the form

$$d(f(gy),g(fx)) \leqslant \psi_1[d(fx,gy)] + \psi_2[d(fx,gy)] + \psi_3[d(fx,gy)] \qquad (2.1).$$

Let $x_0 \in X$ be arbitrary point and we construct a sequence $\{x_n\}$ defined by

$$fx_{n-1} = x_n \quad , \quad gx_n = x_{n+1} \quad , \quad fx_{n+1} = x_{n+2} \quad , \quad n=1,2,\ldots$$

Let us put $x = x_{n-1}$, $y = x_n$ in (2.1), then we have

$$d(f(gx_n),g(fx_{n-1})) \leqslant \psi_1[d(fx_{n-1},gx_n)] + \psi_2[d(fx_{n-1},gx_n)] + \psi_3[d(fx_{n-1},gx_n)] \quad (2.2)$$

or $\quad d(x_{n+2},x_{n+1}) \leqslant \psi_1[d(x_n,x_{n+1})] + \psi_2[d(x_n,x_{n+1})] + \psi_3[d(x_n,x_{n+1})].$

Let n be even and set $\beta_n = d(x_{n-1},x_n)$. Then

$$\begin{aligned}\beta_{n+2} &= d(x_{n+2},x_{n+1}) \leqslant \\ &\leqslant \psi_1[d(x_n,x_{n+1})] + \psi_2[d(x_n,x_{n+1})] + \psi_3[d(x_n,x_{n+1})] \leqslant \\ &\leqslant \psi_1(\beta_{n+1}) + \psi_2(\beta_{n+1}) + \psi_3(\beta_{n+1}) \qquad (2.3)\end{aligned}$$

From (2.3) it is clear that β_n decreases with n and hence $\beta_n \longrightarrow \beta$ say, as $n \longrightarrow \infty$. If possible, let $\beta > 0$. Then since ψ_i is upper semi-continuous, we obtain in the limit as $n \longrightarrow \infty$

$$\beta \leqslant \psi_1(\beta) + \psi_2(\beta) + \psi_3(\beta) < \beta$$

which is impossible unless $\beta = 0$.

Next, we shall show that the sequence $\{x_n\}$ is Cauchy. Suppose that it is not so. Then there exists an $\varepsilon > 0$ and sequences of positive integers $\{m(k)\}$, $\{n(k)\}$ with $m(k) > n(k) \geqslant k$ such that

$$d_k = d(x_{m(k)},x_{n(k)}) \geqslant \varepsilon \quad , \quad k=1,2,3,\ldots \qquad (2.4)$$

If m(k) is the smallest integer exceeding n(k) for which (2.4) holds, then from the well ordering principle, we have

$$d(x_{m(k)-1},x_{n(k)}) < \varepsilon \tag{2.5}$$

Then

$$d_k \leqslant d(x_{m(k)},x_{m(k)-1}) + d(x_{m(k)-1},x_{n(k)}) \leqslant \beta_{m(k)} + \varepsilon < \beta_k + \varepsilon$$

which implies that $d_k \to \varepsilon$ as $k \to \infty$. Now the following cases are to be considered

(a) m is even and n is odd,

(b) m and n are both odd,

(c) m is odd and n is even,

(d) m and n are both even,

Case (a)

$$d_k = d(x_m,x_n) \leqslant d(x_m,x_{m+1}) + d(x_{m+1},x_{n+1}) + d(x_{n+1}, x_n) \leqslant$$

$$\leqslant \beta_{m+1} + \beta_{n+1} + d(gx_m,fx_n) \leqslant$$

$$\leqslant \beta_{m+1} + \beta_{n+1} + \psi_1[d(x_m,x_n)] + \psi_2[d(x_n,fx_{n-1})] + \psi_3[d(x_m,gx_{m-1})] \leqslant$$

(by putting $u_1=x_n$, $u_2=x_m$, $u_3=x_{n-1}$, $u_4=x_{m-1}$ in (1.2))

$$\leqslant \beta_{m+1} + \beta_{n+1} + \psi_1(d_k).$$

Letting $k \to \infty$ we have

$$\varepsilon \leqslant \psi_1(\varepsilon) < \varepsilon/3.$$

This is a contradiction if $\varepsilon > 0$. In the case (b) we have

$$d_k = d(x_m,x_n) \leqslant d(x_m,x_{m+1}) + d(x_{m+2},x_{m+1}) + d(x_{m+2},x_{n+1}) + d(x_n,x_{n+1}) \leqslant$$

$$\leqslant \beta_{m+2} + \beta_{m+1} + \beta_{n+1} + d(gx_{m+1},fx_n) \leqslant$$

$$\leqslant \beta_{m+2} + \beta_{m+1} + \beta_{n+1} + \psi_1[d(x_n,x_{m+1})] + \psi_2[d(x_n,fx_{n-1})] + \psi_3[d(x_{m+1},gx_{m-1})] \leqslant$$

(by putting $u_1=x_n$, $u_2=x_{m+1}$, $u_3=x_{n-1}$, $u_4=x_{m-1}$ in (1.2))

$$\leqslant \beta_{m+2} + \beta_{m+1} + \beta_{n+1} + \psi_1(d_k+\beta_{m+1}) + \psi_3(\beta_{m+1}).$$

Letting $k \to \infty$ in the above inequality we obtain $\varepsilon < \varepsilon/3$, which is a contradiction if $\varepsilon > 0$. Similarly the cases (c) and (d) may be disposed of. This leads us to conclude that $\{x_n\}$ is Cauchy. Since X is complete so $\lim_{n\to\infty} x_n = z \in X$. We shall now show that $gz = z = fz$. Putting $u_1 = x_{n-1}$, $u_2 = z$, $u_3 = x_{n+1}$, $u_4 = x_n$ in (1.2) we get

$$d(fx_{n-1},gz) \leqslant \psi_1[d(x_{n+1},z)] + \psi_2[d(x_{n-1},fx_{n+1})] + \psi_3[d(z,gx_n)]$$

letting $n \longrightarrow \infty$ we get $d(z,gz) \leqslant 0$, which is a contradiction and hence $gz = z$. In the same way it is possible to show that $fz = z$. Thus z is a common fixed point of f and g. If possible let there be another point w ($\neq z$) such that $fw = w = gw$. Then putting $u_1 = u_4 = z$ and $u_2 = u_3 = w$ in (1.2) we have

$$d(z,w) = d(fz,gw) \leqslant \psi_1[d(z,w)] + \psi_2[d(z,fw)] + \psi_3[d(w,gz)] < d(z,w)$$

which is a contradiction. Hence $z=w$. This completes the proof of the theorem.

THEOREM 2. *Let* f_k $(k=1,2,\ldots,n)$ *be a family of mappings of* X *into itself. If* $\{f_k\}$ *satisfy the conditions*

$$f_1 f_2 \ldots f_n \text{ commutes with every } f_k , \qquad \text{(i)}$$

$$d(f_1f_2\ldots f_n u_1, f_n f_{n-1}\ldots f_1 u_2) \leqslant \psi_1[d(u_1,u_2)] + \psi_2[d(u_1,f_1f_2\ldots f_n u_3)] + \psi_3[d(u_2,f_n f_{n-1}\ldots f_1 u_4)] \qquad \text{(ii)}$$

for u_1, u_2, u_3, $u_4 \in X$ *and* ψ_i $(i=1,2,3)$ *satisfies the condition* (1.1), *then* $\{f_k\}$ *have a unique common fixed point.*

Proof. Let $f = f_1f_2\ldots f_n$ and $g = f_nf_{n-1}\ldots f_1$, then (ii) takes the form

$$d(fu_1,gu_2) \leqslant \psi_1[d(u_1,u_2)] + \psi_2[d(u_1,fu_3)] + \psi_3[d(u_2,fu_4)] \qquad \text{(iii)}$$

By Theorem 1, f and g have a unique common fixed point z. Then $fz = gz = z$. For any f_k, $f_k(fz) = f_kz$. By the assumption, $f(f_kz) = f_kz$. So f_kz is a fixed point of f and z is a fixed point of g. By putting $u_1 = u_3 = f_kz$ and $u_2 = u_4 = z$ in (iii) we have

$$d(f_kz,z) = d(f(f_kz),gz) \leqslant \psi_1[d(f_kz,z)] + \psi_2[d(f_kz,z)] + \psi_3[d(f_kz,z)] < d(f_kz,z)$$

which is a contradiction. Hence $f_kz = z$, $(k=1,2,\ldots,n)$. This means that z is the common fixed point of the family $\{f_k\}_{k=1}^{n}$. It can be easily shown that z is the unique common fixed point of $\{f_k\}$. This completes the proof.

3. SOME SPECIAL CASES.

In this section we shall show that our result contains some well-known fixed point theorems as special cases.

If we define the functions $\psi_i(t)$ by $\psi_1(t) = a.t$, $\psi_2(t) = b.t$, $\psi_3(t) = c.t$ with $0 < a+b+c < 1$, then we have the following results as special cases:

(a) By putting a=0, b=c=α and $u_4 = u_1$, $u_3 = u_2$ in the Theorem 1, we get the result of Maiti et al. [3].

(b) Putting f=g and $u_3 = u_1$, $u_4 = u_2$ in Theorem 1, we have the results of Reich [7].

(c) By putting f=g and a=0, b=c=α, $u_3 = u_1$, $u_4 = u_2$ in Theorem 1, we get the result of Kannan [2].

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