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ON PERFECT LIE ALGEBRAS

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Let L be a Lie algebra over a field K. We say that L is *perfect* if for every ideal I of L is [I,L] = I. Throughout this paper we will assume that K is a field of characteristic 0. Examples of perfect Lie algebras are the semisimple's and more generally the semi-direct product of a semisimple Lie algebra by a faithful finite representation of it (See [1], Exercise 4, §6).

In this Note we intend to construct for every non-negative integer m a perfect Lie algebra of dimension 3(m+1) essentially different from those mentioned above. To carry out the construction we make use of the split 3-dimensional simple Lie algebra $\underline{s}$ with bases e,f,h satisfying:

$$(0) \qquad [e,h] = 2e \qquad [f,h] = -2f \qquad [e,f] = h$$

and its irreducible representations.

As it is well known (See [2], Th.12, Chap.III) there exists, for every non-negative integer m, and in the sense of isomorphism, only one irreducible representation V of $\underline{s}$, of dimension m+1. V has a (let us call characteristic) basis x_i, $0 \leq i \leq m$ such that in the representation $e \longmapsto E \qquad f \longmapsto F \qquad h \longmapsto H$

we have

$$(1) \qquad \begin{aligned} Hx_i &= (m-2i)x_i \qquad 0 < i \leq m \\ Fx_i &= x_{i+1} \quad \text{if} \quad 0 \leq i < m \quad \text{and} \quad Fx_m = 0 \\ Ex_i &= i(-m+i-1)x_{i-1} \quad \text{if} \quad 0 < i \leq m \quad \text{and} \quad Ex_0 = 0. \end{aligned}$$

Let U be an irreducible representation of $\underline{s}$ of dimension 2m-1 with characteristic basis u_k , $0 \leq k \leq 2(m-1)$ satisfying:

$$(2) \qquad \begin{aligned} Hu_k &= 2(m-k-1)u_k \qquad 0 \leq k \leq 2(m-1) \\ Fu_k &= u_{k+1} \quad \text{if} \quad 0 \leq k < 2(m-1) \quad \text{and} \quad Fu_{2(m-1)} = 0 \\ Eu_k &= k(-2m+k+1)u_{k-1} \quad \text{if} \quad 0 < k \leq 2(m-1) \quad \text{and} \quad Eu_0 = 0. \end{aligned}$$

THEOREM. *For every non-negative integer* m, *there is a unique (in the sense of isomorphism) structure of perfect Lie algebra over the* K*-vec-*

tor space

$$L = \underline{s} \oplus V \oplus U$$

satisfying all the following conditions:

i) *The structure on* $\underline{s}$ *coincides with* (0)

ii) $\underline{s}$ *induces (by the adjoint representation) the representations given by* (1) *and* (2) *on* V *and* U *respectively*

iii) $[V,U] = [U,U] = 0$

iv) $[V,V] = U$.

Proof. We have to define the products $[x_i,x_j] \in U$

(3) $$[x_i,x_j] = \sum_{k=0}^{2(m-1)} c_{ij}^k \qquad 0 \leqslant i,j \leqslant m$$

consistently with the conditions

(a_{ij}) $$c_{ij}^k + c_{ji}^k = 0$$

(b_{ij}) $$H.[x_i,x_j] = [Hx_i,x_j] + [x_i,Hx_j]$$

(c_{ij}) $$F.[x_i,x_j] = [Fx_i,x_j] + [x_i,Fx_j]$$

(d_{ij}) $$E.[x_i,x_j] = [Ex_i,x_j] + [x_i,Ex_j] \ .$$

By a direct computation we get the equivalence

$$(b_{ij}) \text{ holds} \iff c_{ij}^k = 0 \quad \text{if} \quad i+j \neq k+1.$$

So, we set

(3') $$[x_i,x_j] = q(i,j)\, u_{i+j-1} \quad , \quad q(i,j) = c_{ij}^{i+j-1}$$

Now conditions (c_{ij}) are equivalent to conditions

(c'_{ij}) $$q(i+1,j) + q(i,j+1) = q(i,j).$$

LEMMA 1. *Conditions* (c'_{ij}) *are equivalent to*

$$(c''_{ij}): \quad q(i,j) = \sum_{k=0}^{i} (-1)^k \binom{i}{k} q(0,j+k) \ .$$

Proof. Assume (c''_{ij}). We have

$$q(i+1,j) + q(i,j+1) =$$

$$= \sum_{k=0}^{i+1} (-1)^k.\binom{i+1}{k}.q(0,j+k) + \sum_{k=0}^{i} (-1)^k.\binom{i}{k}.q(0,j+1+k) =$$

$$= q(0,j) + \sum_{k=1}^{i+1} (-1)^k.\binom{i+1}{k}.q(0,j+k) + \sum_{k=0}^{i} (-1)^k.\binom{i}{k}.q(0,j+h+1) =$$

$$= q(0,j) + \sum_{k=0}^{i} (-1)^{k+1}.\binom{i+1}{k+1}.q(0,j+k+1) + \sum_{k=0}^{i} (-1)^{k}.\binom{i}{k}.q(0,j+k+1) =$$

$$= q(0,j) + \sum_{k=0}^{i-1} (-1)^{k+1}.\binom{i}{k+1}.q(0,j+k+1) =$$

$$= q(0,j) + \sum_{k=1}^{i} (-1)^{k}.\binom{i}{k}.q(0,j+k) = \sum_{k=0}^{i} (-1)^{k}.\binom{i}{k}.q(0,j+k) = q(i,j).$$

Conversely, assume (c'_{ij}). Notice that (c''_{0j}) holds for every j. We can proceed inductively assuming (c''_{ij}). A computation as in the first part of Lemma 1 gives $(c''_{(i+1)j})$ and so we are done.

As a consequence of Lemma 1 we have that the q(i,j)'s are uniquely determined by the q(0,j)'s consistently with (b_{ij}) and (c_{ij}).

Next we see that the q(0,j)'s, $0 < j$, are uniquely determined by conditions (d_{0j}). In fact,

$E.[x_0,x_j] = q(0,j).Eu_{j-1} = q(0,j)(j-1)(-2m+j)u_{j-2}$, and

$[Ex_0,x_j] + [x_0,Ex_j] = [x_0,Ex_j] = j(-m+j-1).[x_0,x_{j-1}] =$

$= j(-m+j-1)q(0,j-1)u_{j-2}$ that is $(j-1)(2m-j)q(0,j) = j(m-j+1)q(0,j-1) \quad 0 < j.$

Therefore

$$(4) \qquad q(0,j) = j.\frac{(2m-j-1)!}{(m-j)!}.\frac{(m-1)!}{(2m-2)!}q(0,1) \qquad 0 < j$$

$$= j.\frac{(2m-j-1)!}{(m-j)!}.a \qquad \left(a = \frac{(m-1)!}{(2m-2)!}q(0,1)\right)$$

Next we define

$q(0,j)$, $0 < j \leqslant m$ according to (4)

$q(0,0) = 0$

$q(j,0) = -q(0,j)$, $0 \leqslant j \leqslant m$.

The coefficients q(i,j)'s are so defined in a unique way (up to the constant factor q(0,1) consistently with conditions (a_{0j}), (a_{j0}), (b_{ij}), (c_{ij}), (d_{0j}), (d_{j0}) for all $0 \leqslant i,j \leqslant m$.

We have to verify the consistency with the remaining conditions.

LEMMA 2. i) (a_{ij}) *holds for every* i,j.

ii) (d_{ij}) *holds for every* i,j.

Proof. i) Induction over i . $c_{0j}^{j-1} = q(0,j) = -q(j,0) = -c_{j0}^{j-1}$.

Assume (a_{ij}). Then $q(i+1,j) + q(i,j+1) = q(i,j)$

$q(j+1,i) + q(j,i+1) = q(j,i)$

and adding we get $q(i+1,j) + q(j,i+1) = 0$, that is $(a_{(i+1)j})$.

ii) Induction over i. Case (d_{0j}) is true. Assume $(d_{(i-1)j})$. Then,

$E.[x_i,x_j] - [Ex_i,x_j] - [x_i,Ex_j] = E.[Fx_{i-1},x_j] - [EFx_{i-1},x_j] - [x_i,Ex_j]$

$= E(F.[x_{i-1},x_j] - [x_{i-1},Fx_j]) - [EFx_{i-1},x_j] - [x_i.Ex_j] =$

$= FE.[x_{i-1},x_j] + H.[x_{i-1},x_j] - E.[x_{i-1},Fx_j] - [EFx_{i-1},x_j] - [x_i,Ex_j] =$

$= F([Ex_{i-1},x_j] + [x_{i-1},Ex_j]) + H.[x_{i-1},x_j] - [Ex_{i-1},Fx_j] - [EFx_{i-1},x_j] -$

$- [x_{i-1},EFx_j] - [x_i,Ex_j] =$

$= [FEx_{i-1},x_j] + [Ex_{i-1},Fx_j] + [Fx_{i-1},Ex_j] + [x_{i-1},FEx_j] + H.[x_{i-1},x_j] -$

$- [Ex_{i-1},Fx_j] - [FEx_{i-1},x_j] - [Hx_{i-1},x_j] - [x_{i-1},FEx_j] - [x_{i-1},Hx_j] -$

$- [x_i,Ex_j] = [x_i,Ex_j] - [x_i,Ex_j] = 0.$

We now pass to define a Lie algebra structure on $L = \underline{s} \oplus V \oplus U$.

We define products $\underline{s} \times \underline{s} \longrightarrow \underline{s}$, $\underline{s} \times V \longrightarrow V$, $\underline{s} \times U \longrightarrow U$, $V \times U \longrightarrow 0$ $U \times U \longrightarrow 0$ such that conditions i), ii) and iii) of Theorem are satisfied. We define a product $V \times V \longrightarrow U$ by (3') where the $q(i,j)$'s are determined by (c_{ij}), (4) and (5) giving to $q(0,1)$ any non-zero value.

In this way we get a Lie algebra structure on L and we claim that condition iv) of the Theorem is satisfied. In fact, $\underline{n} = V \oplus U$ is an ideal of L (the radical) satisfying $\underline{n}^2 \subset U$. Moreover $\underline{n}^2 \neq 0$ since it contains the products $[x_0,x_j] = q(0,j)\, u_{j-1}$ and $q(0,j) \neq 0$, $0 < j$. Since $\underline{n}^2$ is an ideal of L is stable under $\underline{s}$. But the representation U is irreducible, so $V^2 = \underline{n}^2 = U$. The uniqueness in the Theorem follows clearly from the uniqueness in which coefficients $q(i,j)$ are determined.

We have finally to prove that L so defined is a perfect Lie algebra. To this end observe that the adjoint representation of $\underline{s}$ on L is faithful and completely reducible. Therefore if I is an ideal of L we have $[\underline{s},I] = I$ and, a fortiori, $[L,I] = I$.

REMARKS. We now add some remarks on perfect Lie algebras and their Lie algebra of derivations.

1) Let L be a perfect Lie algebra. Then its radical is nilpotent. In fact, let $\underline{r}$ and $\underline{n}$ denote respectively the radical and the nilpotent radical. For any x in L we have $ad_L(x)(\underline{r}) \subset \underline{n}$. Therefore $\underline{r} = [\underline{r},L] \subset$ $\subset \underline{n}$, that is $\underline{r} = \underline{n}$.

2) Let D(L) denote the Lie algebra of derivations of a perfect Lie algebra. Then D(D(L)) = D(L). In fact, observe that the center of L is 0. Our claim follows from the Schenkman's derivation tower theorem (See [2], Chap.II, Ex.16).

3) Let L be one of the perfect Lie algebras constructed above. Then

$L \neq D(L)$, that is, L has outer derivations. In fact, the radical of L is a (nilpotent) quasy-cyclic Lie algebra in the sense of Leger (See [3], pag.145). Therefore Theorem 5 of [3] applies and we have then our claim.

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