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STATISTICAL DISTRIBUTION ON CONVEX NONOVERLAPPING PARTICLES IN THE EUCLIDEAN N-DIMENSIONAL SPACE

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SUMMARY. In this work we get an extension in the n-dimensional euclidean space of the Santaló's results [3] by using a recent work of M. Stoka [4].

1. Let E_n be the euclidean n-dimensional space, the kinematic density in E_n is [1]:

$$dK = dP \wedge dO_{n-1} \wedge \dots \wedge dO_1$$

where $dP = dx_1 \wedge \dots \wedge dx_n$ and dO_h is the area element of the h-dimensional unit sphere.

For a body K of E_n, σ denotes the (n-1)-dimensional volume of the random body intersection of K by a random hyperplane H.

Let $\varphi(\sigma)d\sigma$ be the probability distribution of the volume σ of $\{H \cap K\}$ when σ is between σ' and $\sigma'+d\sigma'$ with $\sigma' \in]0,\sigma_m[$ and σ_m denotes the maximal value of the volume, we have

$$\int_0^{\sigma_m} \varphi(\sigma)d\sigma = 1 \qquad (1)$$

Then the mean value of the random variable σ is [4, pag.58]:

$$E[\sigma] = \frac{\int_0^{\sigma_m} \sigma\varphi(\sigma)d\sigma}{\int_0^{\sigma_m} \varphi(\sigma)d\sigma} = \frac{\pi^{n/2}\ V_n(K)}{\Gamma(\frac{n}{2})\ M_{n-2}(K)} \qquad (2)$$

where $V_n(K)$ denotes the n-dimensional volume of body K and

$$M_h(K) = \frac{1}{\binom{n-1}{h}} \int_{\partial K} S_h(K)d\sigma \qquad (3)$$

with $S_h(K)$ the h-rt elementary symmetric function of the (n-1) principal curvatures of the hypersurface K, and $d\sigma$ the area element of ∂K.

Then for (1) and (2) we have

$$(4)\qquad \int_0^{\sigma_m} \sigma\, \varphi(\sigma)\, d\sigma = \frac{\pi^{n/2}\, V_n(K)}{\Gamma(\frac{n}{2})\, M_{n-2}(K)}$$

Let K_μ be a similar body to K with the radio of similitude μ; if $\varphi(\sigma,\mu)$ denotes the probability distribution of K_μ so that

$$\varphi(\sigma,1) = \varphi(\sigma)$$

we have

$$(5)\qquad \varphi(\sigma,\mu) = \mu^{1-n}\, \varphi\left(\frac{\sigma}{\mu^{n-1}}\right)$$

We consider a convex body Q that contains a certain number of convex nonoverlapping particles distributed at random and similar to K.

The probability that a random hyperplane H, that intersects the convex body Q, intersecs also K_μ is [4]

$$(6)\qquad p = \frac{M_{n-2}(K_\mu)}{M_{n-2}(Q)}$$

We denote with $F(\mu)d\mu$ the number of particles K_μ that are contained in the unit volume V(Q) of body Q whose ratio μ lies in the range μ' , $\mu'+d\mu'$, then the total number N_μ of particles in Q is:

$$(7)\qquad N_\mu = V(Q)F(\mu)d\mu$$

The mean value of the random variable N_μ is from (6) and (7)

$$(8)\qquad E[N_\mu] = \frac{M_{n-2}(K_\mu)}{M_{n-2}(Q)}\, V(Q)F(\mu)d\mu$$

The conditional mean value of N_μ with respect to the condition $\sigma' < \sigma < \sigma'+d\sigma'$ is

$$(9)\qquad \varphi(\sigma,\mu) \quad \frac{M_{n-2}(K_\mu)}{M_{n-2}(Q)}\, V(Q)F(\mu)d\mu$$

If K_μ contains as its part a particle (n-1)-dimensional with volume σ we have

$$\sigma \leqslant \sigma_m\, \mu^{n-1}$$

and then

$$\mu \geqslant \left(\frac{\sigma}{\sigma_m}\right)^{\frac{1}{n-1}}$$

We obtain for the mean value of the number of (n-1)-dimensional particles having volume in the range σ', $\sigma'+d\sigma'$

$$(10) \qquad \left[\int_{(\frac{\sigma}{\sigma_m})^{1/n-1}}^{\infty} \varphi(\sigma,\mu) \frac{M_{n-2}(K_\mu)}{M_{n-2}(Q)} V(Q)F(\mu)d\mu \right] d\sigma$$

If $f(\sigma)d\sigma$ denotes the number of (n-1)-dimensional particles $H \cap K_\mu$, in the unit (n-1)-dimensional volume of the hyperplane with $\sigma' < \sigma < < \sigma'+d\sigma'$ the total number of particles whose volume lies between σ' and $\sigma'+d\sigma'$ averaged over all positions of the hyperplane is

$$(11) \qquad \frac{\pi^{n/2}}{\Gamma(\frac{n}{2})} \frac{V(Q)}{M_{n-2}(Q)} f(\sigma)d\sigma$$

If we observe that

$$(12) \qquad M_h(K_\mu) = \binom{n-1}{h} \int_{K_\mu} S_h \, d\sigma = \mu^{n-h-1} \binom{n-1}{h} \int_K S_h \, d\sigma = \mu^{n-h-1} M_h(K)$$

by making equal (10) and (11) we obtain

$$(13) \qquad \int_{(\frac{\sigma}{\sigma_m})^{1/n-1}}^{\infty} \mu\varphi(\sigma,\mu)M_{n-2}(K) \; F(\mu)d\mu = \frac{\pi^{n/2} f(\sigma)}{\Gamma(\frac{n}{2})}$$

that, using (5) becomes

$$\int_{(\frac{\sigma}{\sigma_m})^{1/n-1}}^{\infty} \mu^{2-n} \; \varphi(\frac{\sigma}{\mu^{n-1}})F(\mu)d\mu = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2})} \frac{f(\sigma)}{M_{n-2}(K)}$$

The last equality resolvs formally our problem, in fact in this integral equation it is possible to determine the function $F(\mu)$ if the function $f(\sigma)$ is known through measurements of the intersection of Q by random hyperplanes.

We now consider the particular case that the body K is the unit hypersphere K*.

The maximal (n-1)-dimensional volume σ_m of the body $H \cap K^*$ is

$$(15) \qquad \sigma_m = \frac{\pi^{\frac{1}{\alpha}}}{\Gamma(1+\frac{1}{\alpha})}$$

where $\alpha = 2/n-1$.

The probability that the (n-1)-dimensional hypersphere $H \cap K$ has radius in the range r, $r+dr$ is

$$|dx| = \frac{r \, dr}{\sqrt{1-r^2}}$$

with

$$x^2 = 1 - r^2$$

and since

(16) $$\sigma = \frac{\pi^{\frac{1}{\alpha}} r^{\frac{2}{\alpha}}}{\Gamma(1+\frac{1}{\alpha})}$$

we have

$$\varphi(\sigma) = \frac{\alpha\ \Gamma(1+\frac{1}{\alpha})\ [\Gamma(1+\frac{1}{\alpha})\sigma]^{\alpha-1}}{2\ \sqrt{\pi-[\Gamma(1+\frac{1}{\alpha})\sigma]^{\alpha}}}$$

and also for (15)

(17) $$\varphi(\sigma) = \frac{\alpha\ \sigma^{\alpha-1}}{2\ \sigma_m^{\alpha/2}\ \sqrt{\sigma_m^{\alpha} - \sigma^{\alpha}}}$$

From (5) we deduce that the probability distribution of the (n-1)-dimensional volume of the hypersphere $H \cap K^*$ with radius μ is

(18) $$\varphi(\sigma,\mu) = \frac{\alpha\ \sigma^{\alpha-1}}{2\mu\ \sigma_m^{\alpha/2}\ \sqrt{\mu^2\sigma_m^{\alpha}-\sigma^{\alpha}}}$$

Using (5) and (18), (13) becomes

(19) $$\int_{(\frac{\sigma}{\sigma_m})^{\alpha/2}}^{\infty} \frac{F(\mu)}{\sqrt{\mu^2\sigma_m^{\alpha}-\sigma^{\alpha}}}\, d\mu = \frac{\sqrt{\pi}\ f(\sigma)}{\alpha\ \sigma^{\alpha-1}\ \Gamma^{\alpha/2}(1+\frac{1}{\alpha})}$$

We make a change of variable putting

$$s = \mu^2\ \sigma_m^{\alpha}$$

then, the equality (19) becomes an integral equation of Abel's type

(20) $$\int_{\sigma^{\alpha}}^{\infty} \frac{F_1(s)}{(s-\sigma^{\alpha})^{1/2}}\, ds = f_1(\sigma)$$

with

(21) $$F_1(s) = \frac{F\left(\sqrt{\frac{s}{\sigma_m^{\alpha}}}\right)}{\sqrt{s}}$$

and

(22) $$f_1(\sigma) = \frac{2\pi\ f(\sigma)}{\alpha\ \sigma^{\alpha-1}\ \Gamma^{\alpha}(1+\frac{1}{\alpha})}$$

The solution of integral equation (20) is

$$F_1(s) = \frac{1}{\pi}\int_s^{\infty} \frac{f_1'(\sigma)}{\sqrt{\sigma^{\alpha}-s}}\, d\sigma$$

from (21) and (22) we deduce

$$(23) \qquad F(\mu) = -\frac{2\pi^{1/2}\,\mu}{\alpha\, \Gamma^{\frac{3}{2}\alpha}(1+\frac{1}{\alpha})}\int_{\mu^2\sigma_m^{\alpha}}^{\infty} \frac{f'(\sigma)\sigma^{1-\alpha} - (\alpha-1)\sigma^{-\alpha}f(\sigma)}{(\sigma^{\alpha}-\mu^2\sigma_m^{\alpha})^{1/2}}\, d\sigma$$

Consequentily we have the

THEOREM 1. *Let* Q *be a convex body in the euclidean* n*-dimensional space that contains a certain number of nonoverlapping hyperspheres distributed at random. Suppose that all hyperspheres are similar to unit hypersphere and let* μ *be the radio of similitude. Let* H *be a random hyperplane and* $f(\sigma)d\sigma$ *the number of sections per unit volume* (n-1)*-dimensional in* $H \cap Q$ *of hyperspheres that have volume between* σ' *and* $\sigma' \pm d\sigma'$. *Then the number of hyperspheres whose ratio lies in the range* μ', $\mu'+d\mu'$ *is given by* $F(\mu)d\mu$, *where* $F(\mu)$ *is defined by* [23].

In particular case n=3 we find the result due to Santaló [3].

Let g(r)dr be the number of intersected hyperspheres per unit (n-1)-dimensional volume $Q \cap H$ whose intersections have radii in the range r, r+dr, we have

$$(24') \qquad g(r)dr = f(\sigma)d\sigma$$

From (16) and (24') we obtain

$$f(\sigma) = \frac{\alpha\ \Gamma(1+\frac{1}{\alpha})}{2\pi^{1/\alpha}\ r^{\beta}}\, g(r)$$

and

$$f'(\sigma) = \left(\frac{g(r)}{r^{\beta}}\right)' \frac{\alpha\Gamma^2(1+\frac{1}{\alpha})}{4\pi^{2/\alpha}\ r^{\beta}}$$

where $\beta = \frac{2-\alpha}{\alpha}$

and (23) becomes

$$F(\mu) = -\frac{\pi^{-1/\alpha}\ \mu}{\alpha\Gamma^{\alpha}(1+\frac{1}{\alpha})}\int_{\mu^{\alpha}}^{\infty} \frac{\left(\frac{g(r)}{r^{\beta}}\right)' \Gamma(1+\frac{1}{\alpha}) - 2\pi^{1/\alpha}(\alpha-1)\sigma^{-\alpha}g(r)}{(r^2 - \mu^2)^{1/2}}\, dr$$

In particular case n=3 we find the formula due to Wicksell [5].

2. We now consider as probe a moving random line G. Let G be the set of the random lines, then the lenght λ of the chord intersection of body K by line G is a random variable.

Let λ' be a value in the range $]0,\lambda_m[$ where λ_m denotes the maximal value of λ, and let $\varphi(\lambda)d\lambda$ be the probability distribution when λ lies between λ' and $\lambda'+d\lambda'$, then

(24) $$\int_0^{\lambda_m} \varphi(\lambda)d\lambda = 1$$

The mean value of the random variable λ is [4,pag.59]

(25) $$E[\lambda] = \int_0^{\lambda_m} \lambda\varphi(\lambda)d\lambda = \frac{2\sqrt{\pi}\ \Gamma(\frac{1}{\alpha})\ V_n(K)}{\Gamma(\frac{1}{2}+\frac{1}{\alpha})\ V_{n-1}(\partial K)}$$

Let K_μ be a convex body similar to K. If $\varphi(\lambda,\mu)$ denotes the probability distribution of $K \cap G$ so that $\varphi(\lambda,1) = \varphi(\lambda)$, we have

$$\varphi(\lambda,\mu)\ d(\lambda\mu) = \varphi(\lambda/\mu)d\lambda$$

therefore

(26) $$\varphi(\lambda,\mu) = \frac{1}{\mu}\varphi(\frac{\lambda}{\mu})$$

With the same notation as above we consider the probability that a line G, wich intersects Q, intersecs also K, namely

$$p = \frac{V_{n-1}(\partial K)}{V_{n-1}(\partial Q)}$$

If N_μ is the total number of particles contained in Q we have

$$N_\mu = V(Q)\ F(\mu)d\mu$$

The mean value of number of particles K_μ having a ratio in the range μ', $\mu'+d\mu'$ which are intersected by the line G is

$$\frac{V_{n-1}(\partial K_\mu)}{V_{n-1}(\partial Q)}\ V(Q)\ F(\mu)d\mu$$

We observe that for μ we have the condition $\mu \geqslant \frac{\lambda}{\lambda_m}$.

Then for number of chords whose lenght lies in the range λ', $\lambda'+d\lambda'$ we have the expectation

(27) $$\left[\int_{\frac{\lambda}{\lambda_m}}^{\infty} \varphi(\lambda,\mu)\ \frac{V_{n-1}(\partial K_\mu)}{V_{n-1}(\partial Q)}\ V(Q)\ F(\mu)d\mu\right] d\lambda$$

Let $f(\lambda)d\lambda$ the number of particles K_μ per unit lenght $Q \cap G$ in the range λ', $\lambda'+d\lambda'$, then the total number of chords of lenght averaged over all intersection $Q \cap G$ is given by

$$(28) \qquad \frac{2\sqrt{\pi}\ \Gamma(\frac{1}{\alpha})\ V_{n-1}(Q)}{\alpha\ \Gamma(\frac{1}{2}+\frac{1}{\alpha})\ V_{n-1}(\partial Q)}\ f(\lambda)d\lambda$$

Making equal (27) and (28) we have

$$\int_{\lambda/\lambda_m}^{\infty} \varphi(\lambda,\mu)\ V_{n-1}(\partial K_\mu)F(\mu)d\mu = \frac{2\sqrt{\pi}\ \Gamma(\frac{1}{\alpha})}{\alpha\ \Gamma(\frac{1}{2}+\frac{1}{\alpha})}\ f(\lambda)$$

that, for (26) and since

$$V_{n-1}(\partial K_\mu) = \mu^{n-1}\ V_{n-1}(\partial K)$$

we get the integral equation

$$(29) \qquad \int_{\lambda/\lambda_m}^{\infty} \mu^{\beta}\ \varphi(\frac{\lambda}{\mu})\ V_{n-1}(\partial K\ )F(\mu)d\mu = \frac{2\sqrt{\pi}\ \Gamma(\frac{1}{\alpha})}{\alpha\ \Gamma(\frac{1}{2}+\frac{1}{\alpha})}\ f(\lambda)$$

We resolve this integral equation in particular case. For hypersphere particles we have

$$\varphi(\lambda) = \frac{\lambda^{\beta}\ \Gamma(\frac{1}{\alpha}+\frac{1}{2})}{2^{\beta-1}\ \Gamma(\frac{1}{\alpha})}$$

and

$$\lambda_m = 2 \qquad\qquad V_{n-1}(K) = \frac{2\pi^{\frac{1}{\alpha}+\frac{1}{2}}}{\Gamma(\frac{1}{\alpha}+\frac{1}{2})}$$

the (29) becomes

$$\int_{\lambda/2}^{\infty} F(\mu)d\mu = \frac{\Gamma^2(\frac{1}{\alpha})\ f(\lambda)}{2\alpha\pi^{\beta/2}\ \Gamma(\frac{1}{\alpha}+\frac{1}{2})}\ (\frac{2}{\lambda})^{\beta}$$

which has the solution

$$(30) \qquad F(\mu) = -\ \frac{\Gamma^2(\frac{1}{\alpha})}{2\mu\pi^{\beta/2}\ \Gamma(\frac{1}{\alpha}+\frac{1}{2})}\ (\frac{f(2\mu)}{\mu^{\beta}})'$$

Then the following result holds

THEOREM 2. *Let* Q *be a convex body in the euclidean* n*-dimensional space that contains a certain number of nonoverlapping hyperspheres distributed at random. Suppose that all hyperspheres are similar to the unit hypersphere and let* μ *be the ratio of similitude. Let* G *be a random line and* $f(\lambda)d\lambda$ *the number of intersected hyperspheres per unit length of* $Q \cap G$ *whose chords have their length between* λ' *and* $\lambda'+d\lambda'$*, then* *the number of hyperspheres whose ratio is in the range* μ' *and* $\mu'+d\mu'$ *is given by* $F(\mu)d\mu$*, where* $F(\mu)$ *is defined by* (30).

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