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ON THE ANNIHILATOR IDEAL OF AN INJECTIVE MODULE

Yasuji Takeuchi

1. INTRODUCTION.

Let A be a noetherian local ring and N a finitely generated A-module. In general it doesn't hold that a N-regular element is A-regular. However a N-regular element may be $A/\mathfrak{a}$-regular for some ideal $\mathfrak{a}$ of A. So we shall consider the following problem: Characterize a minimal ideal $\mathfrak{a}$ of A having the property that any N-regular element is $A/\mathfrak{a}$-regular.

First we shall determine the annihilator ideal of an injective module which has a finite set of associated prime ideals. Using this result the above problem will be solved. Finally a slight application to non-zero divisors will be given.

In the following discussion, $(A,\mathfrak{m},k)$ is a noetherian local ring and modules are always unitary. The unlabeled Hom means always Hom_A. For an A-module N $E(N)$ denotes an injective envelope of N.

2. THE ANNIHILATOR IDEAL OF AN INJECTIVE MODULE.

Let $\mathfrak{a}$ be an ideal of A and N an A-module. $\mathrm{Annih}(\mathfrak{a})$, $\mathrm{Annih}_N(\mathfrak{a})$ denote the ideal of elements α in A with $\alpha\mathfrak{a} = 0$, the submodule of elements z of N with $z\mathfrak{a} = 0$, respectively.

PROPOSITION 2.1. Let E be an injective A-module and x an element of A. Then $xE = E$ iff $\mathrm{Annih}(x) \subseteq \mathrm{Annih}(E)$.

Proof. The "only if" part is obvious. Assume $\mathrm{Annih}(x) \subseteq \mathrm{Annih}(E)$. From the exact sequence: $0 \longrightarrow \mathrm{Annih}(x) \longrightarrow A \longrightarrow xA \longrightarrow 0$, we obtain the exact sequence: $\mathrm{Hom}(xA,E) \longrightarrow \mathrm{Hom}(A,E) \longrightarrow \mathrm{Hom}(\mathrm{Annih}(x),E)$. This second map is zero by the hypothesis. So the map: $\mathrm{Hom}(xA,E) \longrightarrow \mathrm{Hom}(A,E)$ is surjective. The multiplication map by x: $\mathrm{Hom}(A,E) \longrightarrow \mathrm{Hom}(A,E)$ factorizes into two epimorphisms: $\mathrm{Hom}(A,E) \longrightarrow \mathrm{Hom}(xA,E)$ and $\mathrm{Hom}(xA,E) \longrightarrow \mathrm{Hom}(A,E)$, and so $xE = E$.

COROLLARY 2.2. Let E be an injective A-module with $\mathrm{Annih}(E) = 0$. Then, for $x \in A$, $xE = E$ iff x is non-zero divisor.

COROLLARY 2.3. For $x \in A$, $xE(k) = E(k)$ iff x is a non-zero divisor.

Proof. If is obvious from the following well-known lemma.

LEMMA 2.4. $\mathrm{Annih}(E(k)) = 0$.

Proof. $E(k)$ may be regarded as the injective envelope of the residue field $\hat{A}/m\hat{A}$ as an $\hat{A}$-module where $\hat{A}$ is the m-adic completion of A [c.f., 4]. So it is sufficient to show this lemma when A is complete. In this case, by Matlis duality, we have $\mathrm{Hom}(\mathrm{Hom}(A/a, E(k)), E(k)) \cong$ $\cong A/a$ for any ideal a of A. If $a = \mathrm{Annih}(E(k))$, $\mathrm{Hom}(A/a, E(k)) \cong E(k)$ and so $\mathrm{Hom}(\mathrm{Hom}(A/a, E(k)), E(k)) \cong A$ [c.f., 4]. Hence we obtain $\mathrm{Annih}(E(k)) = 0$.

PROPOSITION 2.5. Let p be a prime ideal of A. If $xE(A/p) = E(A/p)$ for $x \in A$, then there is an element t in $A-p$ with $t\mathrm{Annih}(x) = 0$.

Proof. If $\mathrm{Annih}(x) = 0$, it is trivial. Assume $\mathrm{Annih}(x) \neq 0$. Set $a = \mathrm{Annih}(x)$. Then we have $aE(A/p) = 0$. Since the injective envelope $E(A/p)_p$ of the A_p-module A_p/pA_p has a zero annihilator ideal, we obtain $a_p = 0$. So there is an element t in $A-p$ with $ta = 0$.

PROPOSITION 2.6. The annihilator ideal of a non-zero injective module consists of zero divisors.

Proof. Let E be a non-zero injective module. For any non-zero divisor x, we have $xE = E$ and so $xE \neq 0$.

COROLARIO 2.7. For any injective A-module E, $\mathrm{Annih}(E)$ is contained in an associated prime ideal of A.

LEMA 2.8. Let p be a prime ideal of A. Then $\mathrm{Annih}(t)E(A/p) = 0$ for any $t \in A - p$.

Proof. It follows from $tE(A/p) = E(A/p)$ for any $t \in A - p$.

COROLLARY 2.9. For any $t \in A - p$, $\mathrm{Annih}(t) \subseteq \mathrm{Annih}(E(A/p))$.

LEMMA 2.10. Let $a = \mathrm{Annih}(E(A/p))$ with p a prime ideal of A. Then there exists an element t in $A - p$ with $a = \mathrm{Annih}(t)$ and so $a = \max\{\mathrm{Annih}(t) : t \in A - p\}$.

Proof. Since the injective envelope $E(A/p)_p$ of an A_p-module A_p/pA_p has a zero annihilator ideal, there is an element t in $A - p$ with $ta = 0$ and so $a \subseteq \mathrm{Annih}(t)$. The statement follows from the above corollary.

The following corollary is immediate from the above lemma:

COROLLARY 2.11. Let p,q be two prime ideals of A with $p \subseteq q$. Then Annih(E(A/p)) contains always Annih(E(A/q)).

COROLLARY 2.12. Let $0 \longrightarrow N \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \ldots$ be a minimal injective resolution for a finitely generated A-module N. If Annih(E^i) $\neq 0$, then Annih(E^{i-1}) $\supseteq$ Annih(E^i), where $E^{-1} = N$.

Proof. It is trivial if $i=0$. Assume $i > 0$. Put a = Annih(E^i). Let p be any prime ideal of A with $\mu^{i-1}(p,N) > 0$. Then we have ht(p) $\leqslant$ n-2 where n = dim A. For, if ht(p) > n-2, then $\mu^i(m,N) > 0$ [c.f.,2,3], which is a contradiction to $a \neq 0$. So there is a prime ideal q of A such that $p \subset q$ are distinct with no prime ideal between them. In this case, we have $\mu^i(q,N) > 0$ [c.f.,2] and so aE(A/p) = 0 because aE(A/q) = 0. This completes the proof.

PROPOSITION 2.13. Let E be an injective A-module. If there is an associated prime ideal p of A which is not contained in the union of the associated prime ideals of E, then Annih(E) $\neq 0$.

Proof. For $t \in p - \bigcup_{q \in \mathrm{Ass}(E)} q$, we have $tE = E$, and so Annih(t) $\subseteq$ $\subseteq$ Annih(E).

THEOREM 2.14. Let E be an injective A-module such that Ass(E) = = $\{p_1,p_2,\ldots,p_n\}$ is a finite set. Then
Annih(E) = max{Annih(t): $t \in A - \bigcup_i p_i$}.

Proof. Put a = max{Annih(t): $t \in A - \bigcup_i p_i$}. Then we have obviously $a \subseteq$ Annih(E(A/p_i)) for i = 1,2,...,n and so $a \subseteq$ Annih(E). Let x be any element of Annih(E). Then xE(A/p_i) = 0 for i = 1,2,...,n. So we may take elements t_i in $A - p_i$ with $t_i x = 0$ for i = 1,2,...,n [c.f.,(2.4)]. Let us denote $\{p_1,p_2,\ldots,p_m\}$ the set of associated prime ideals of E except those contained in another of them.

Choose any element u_i of $p_i - \bigcup_{j \neq i} p_j$ for i = 1,2,...,m and set v_i = $= u_1 \ldots u_{i-1} t_i u_{i+1} \ldots u_m$. Then we have $v_i \in p_k$ for $i \neq k$ and $v_i \notin p_i$, and so $v_1 + v_2 + \ldots + v_m \notin p_i$ for i = 1,2,...,m. Since $x(v_1 + v_2 + \ldots + v_m) = 0$, we obtain $x \in$ Annih($v_1 + v_2 + \ldots + v_m$) $\subseteq a$ and so Annih(E) $\subseteq a$. This completes the proof.

COROLLARY 2.15. Let E be as above. Then there exists a principal ideal I of A such that E is faithfull over I.

Proof. We have Annih(t) = Annih(E) for some $t \in A - \bigcup_i p_i$.
Set $I = tA$. If $atE = 0$ for $a \in A$, a belongs to Annih(E) since $tE = E$, and so $at = 0$. This means that E is faithfull as an I-module.

3. NON-ZERO DIVISORS.

LEMMA 3.1. Let N be an A-module with Annih(E(N)) = 0. Then a non-zero divisor on N is a non-zero divisor.

Proof. For any associated prime ideal p of A, there is an associated prime ideal q of E(N) containing p. For, otherwise we have Annih(E(N)) $\supseteq$ Annih(t) $\neq 0$ for $t \in p - \bigcup_{q \in \mathrm{Ass(E}(N))} q$. Since Ass($N$) = = Ass(E($N$)), our statement holds.

THEOREM 3.2. Let N be an A-module, a = Annih(E(N)). Then any non-zero divisor on N is a non-zero divisor on A/a. If N is finitely generated, then a is the unique minimal ideal of A with respect to this property.

Proof. E(N) may be considered as the injective envelope of N as an A/a-module. Moreover an A/a-module E(N) has a zero annihilator ideal. From the above lemma, x is a non-zero divisor on A/a. Let b be an ideal of A such that any non-zero divisors on N are non zero-divisors on A/b.Now there is $t \in A$ such that Annih(t) = Annih(E(N)) and t is a non-zero divisor on N. Then t is a non-zero divisor on A/b. So we have $a \subseteq b$. For, if $a \not\subseteq b$, ta = 0 for a $\in a - b$, which is absurd.

COROLLARY 3.3. Let N be an A-module. Put a = Annih(E(N)). Then, if an element x of A is a non-zero divisor on N, $\mathrm{inj.dim}_{A/a}\, N/xN =$ $= \mathrm{inj.dim}_{A/a}\, N - 1$.

Proof. It is obvious from the fact that x is a non-zero divisor on A/a.

COROLLARY 3.4. Let N be a finitely generated A-module and $x_1, x_2, \ldots, x_r$ an N-sequence. Then $x_1, x_2, \ldots, x_r$ form an A-sequence if $\mathrm{Annih}(\mathrm{Annih}_{E^i} \mathbf{x}_i) = \mathbf{x}_i$ for i = 0,1,...,r where $0 \longrightarrow N \longrightarrow E^0 \longrightarrow E^1 \longrightarrow$ $\longrightarrow \ldots$ is a minimal injective resolution of N and $\mathbf{x}_0 = 0$, $\mathbf{x}_i = (x_1, x_2, \ldots, x_i)$.

Proof. By the hypothesis we have Annih(E(N)) = 0 and so x_1 is a non-zero divisor. Assume $x_1, x_2, \ldots, x_i$ is an A-sequence. Then $\mathrm{Hom}(A/\mathbf{x}_i, E^i)$ is the injective envelope of $N/\mathbf{x}_i N$ as an $A/\mathbf{x}_i$-module [c.f.,1, Theorem 2.2]. By the hypothesis we obtain that the annihilator ideal of $\mathrm{Hom}(A/\mathbf{x}_i, E^i)$ as an $A/\mathbf{x}_i$-module is zero. So $\mathbf{x}_{i+1}$ is a non-zero divisor on $A/\mathbf{x}_i$.

COROLLARY 3.5. The following conditions are equivalent;

a) A is Cohen-Macaulay.

b) For a s.o.p. $x_1, x_2, \ldots, x_n$, for A there exists a finitely generated

A-module N such that $x_1, x_2, \ldots, x_n$ is an N-sequence and $\mathrm{Annih}(\mathrm{Annih}_{E^i}\mathbf{x}_i) = \mathbf{x}_i$ $(i = 0,1,\ldots n)$ where $\mathbf{x}_0 = 0$, $\mathbf{x}_i = (x_1, x_2, \ldots, x_i)$ and $0 \longrightarrow N \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \ldots$ is a minimal injective resolution of N.

Proof. b) $\Rightarrow$ a). It is obvious from the above corollary.

a) $\Rightarrow$ b). Consider A as N. Then the condition b) is satisfied.

COROLLARY 3.6. Let N be a finitely generated A-module and $0 \longrightarrow N \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \ldots$ a minimal injective resolution of N. Then any N-sequence forms an A-sequence iff $\mathrm{Annih}(E^0) = 0$ and for any N-sequence $\mathrm{Annih}(\mathrm{Annih}_{E^i}\mathbf{x}) = \mathbf{x}$ where i is the length of the N-sequence and $\mathbf{x}$ is the ideal generated by the N-sequence.

Proof. The "if part" follows from the above corollary. We shall show the converse. We obtain $\mathrm{Annih}(E^0) = 0$ from the above theorem. Moreover $\mathrm{Hom}(A/\mathbf{x}, E^i)$ is the injective envelope of $N/\mathbf{x}N$ as an $A/\mathbf{x}$-module. By the hypothesis and the above theorem, the annihilator ideal of $\mathrm{Hom}(A/\mathbf{x}, E^i)$ as an $A/\mathbf{x}$-module is zero and so $\mathrm{Annih}(\mathrm{Annih}_{E^i}\mathbf{x}) = \mathbf{x}$.

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Kobe University
Japan

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