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NUMERICAL METHODS FOR INVERSE TRANSIENT HEAT CONDUCTION PROBLEMS *

Diego A. Murio

ABSTRACT. We show how to stabilize an improperly-posed inverse heat problem with slab symmetry by attempting to reconstruct a slightly "blurred" image of the unknowns. After introducing certain Inverse Kernels, the numerical problem is solved with an absolute minimum of computation.

1. INTRODUCTION.

In this paper we would like to consider a transient heat conduction problem with slab symmetry in which the temperature and heat flux histories f(t) and q(t) on the right hand side are desired and unknown, but the temperature and heat flux histories F(t) and Q(t) on the left hand side surface are approximately measurable for all t in $(-\infty,\infty)$.

This inverse problem is an improperly-posed problem in the sense of Hadamard [6]; that is, there are no decent norms for the data and solutions such that the solution depends continuously upon the data.

The inverse problem appears in many situations of considerable practical interest and may arise, for instance, in quenching studies, in developing transient calorimeters and in the measurements of aerodynamic heating.

The direct or well posed problems in this situation of course would involve specifying one of the two functions flux or temperature on the left surface and one on the right surface.

For the direct problems the well known solution formulae given in standart texts [3] may be applied in a relatively straight forward manner.

For the inverse problem, however, special methods must be employed. It is known that certain types of continuous dependence on data can usual

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ly be restored by restricting attention to those solutions satisfying certain prescribed global bounds, see Miller [8] and [9] for example. If the unknown function q (or f) is quite smooth, for instance, it is reasonable to assume that some high order derivative of q satisfies a known L^2 bound, Manselli and Miller [7].

However, for many problems of interest it is to be expected that the unknown function is not very smooth. In this case, the inverse problem can be stabilized if, instead of attempting to find the point values of q, we content ourselves with attempting to reconstruct a slightly "blurred" image of q. One natural functional is $J_\delta q$ the "δ-mollification" of q, that is, the convolution of q with the Gaussian kernel ρ_δ of "blurring radius" δ. Such an approach was also taken in [7] for a simplified special case of the present problem.

One of the first papers on inverse heat conduction was written by Stolz [11]. His procedure is an integral equation method which, when discretized allows a step by step recursive calculation of the solution, but which is unstable if the time intervals are made small. Integral equation methods with step by step solutions are also used by Sparrow et al. [10] and by Beck [1] who however adds the distinct improvement of allowing the least squares use of several future data points to compute the solution at the present time. Burggraf [2] uses a truncation of the classical power series method for attempting to solve the Cauchy initial value problem; however for this to be convergent F and Q would have to be analytic and the space interval sufficiently small; moreover the method would immensely amplify errors in the data. Other approaches, using finite difference methods, are studied by Frank [5] and Davies [4].

In the papers mentioned above, the assumptions on the solutions and on the choice of parameters which help restore stability are not usually clearly stated and the consequent continuity with respect to the data is not adequately studied.

In section 2 we consider the inverse problem in the slab with one insulated boundary, Q(t) = 0. We present the nondiscrete version of this problem with data specified on a continuum of times t and data error measured in the L^2 norm, and derive stability bounds for the inverse problem.

Section 3 is devoted to the discretized version of the problem of section 2, involving data at only a discrete sampling of times.

In section 4 we consider the general problem with approximate data for both temperature and flux, F(t) and Q(t), specified on the left hand surface of the slab. We proceed to solve this general problem by superposition of a direct and an inverse problem.

Finally, in section 5, the numerical method occupies our attention with the computation of certain inverse and direct convolution kernels with which we shall compute our numerical solution.

2. THE SLAB WITH ONE INSULATED BOUNDARY.

DESCRIPTION OF THE PROBLEM. We considerer a transient temperature conduction problem with slab symmetry where the left hand surface is insulated, so that $Q(t) \equiv 0$. We assume linear heat conduction and after appropriate changes in the space and time scales we may consider without loss of generality the normalized problem, with constant conductivity 1, heat capacity 1 and slab thickness 1.

The problem can be described mathematically as follows:

The unknown temperature $u(x,t)$ satisfies

(2.1) $u_t = u_{xx} \quad ; \quad 0 < x < 1 \quad ; \quad -\infty < t < \infty .$

(2.2) $u(0,t) = F(t)$, with corresponding approximate data function $\overline{F}(t)$.

(2.3) $u_x(0,t) = Q(t) = 0 \quad ; \quad -\infty < t < \infty .$

(2.4) $u(1,t) = f(t)$, the desired but unknown temperature function.

(2.5) $u_x(1,t) = q(t)$, the desired but unknown heat flux function.

Because $Q(t) = 0$, everything about $u(x,t)$ is uniquely determined by the single unknown function q (or f).

Considering the sinusoidal in time solutions of the heat equation (2.1) and the boundary conditions, we get

(2.6) $F(t) = Aq(t) = [(\mu+i\sigma\mu) \sinh (\mu+i\sigma\mu)]^{-1} q(t).$

where $\mu = \sqrt{|w|/2}$, and $\sigma = \text{sign}\,(w)$.

Thus, if $q(t) = e^{iwt}$, it follows from (2.6) that the operator A is strongly smoothing for high frequencies w. Conversely, however, the inverse problem attempting to go from Aq to q magnifies an error in a high frequency component by the gigantic factor $\approx \mu e^{\mu}/\sqrt{2}$, showing that this inverse problem is greatly ill-posed in the high frequency components.

THE STABILIZED INVERSE PROBLEM. For the moment, in order to use Fourier integral analysis, we are going to assume that all functions involved are L^2 functions on the whole line $(-\infty,\infty)$ and we will use the corresponding L^2 norm to measure errors. This is rather unnatural since in many applications one might expect the temperature and the flux to never tend to 0 as $t \to \pm\infty$, but to oscillate about in bounded fashion forever. Nevertheless, this assumption will be later loosened by switching to L^2 norms on bounded intervals of interest.

We assume only a known L^2 global error bound on q,

(2.7) $\|q\| \leqslant E$,

and since there is nothing that adequately forces down the high frequency part of $\hat{q}(w)$, we seek to reconstruct some useful functional of q

which strongly damps the high frequency part of $\hat{q}(w)$. One such functional is $J_\delta q$ the "δ-mollification" of q at time t, defined as

$$(2.8) \qquad J_\delta q(t) \equiv (\rho_\delta * q)(t), \text{ where}$$

$$(2.9) \qquad \rho_\delta(t) = (\delta\sqrt{\pi})^{-1} e^{-t^2/\delta^2}$$

is the Gaussian kernel of "blurring radius δ".

We thus have the following stabilized problem:

Attempt to find the linear function $J_\delta q(t)$ at some time t of interest and for some assigned blurring radius δ, given that q is a particular function satisfying

$$(2.10) \qquad \|Aq-\overline{F}\| \leqslant \varepsilon$$

$$(2.11) \qquad \|q\| \leqslant E.$$

We shall prove that the bound (2.11) is not necessary to stabilize this problem; the data error bound (2.10) by itself is sufficient to assure Lipschitz continuous dependence on the data as $\varepsilon \to 0$, provided we keep δ fixed.

However, the prescribed bound (2.11) can actually aid the stability in the case of ε which are not small, or in the case of data at only discrete sampling points in a limited data interval, as will occur in the numerical applications.

STABILITY ANALYSIS. The problem is now in a form that can be solved by the Method of Least Squares, see Miller [8] and [9].

This method is a "nearly-best-possible-method" in the sense that for any seminorm $\langle\cdot\rangle$ which might be used to measure the error, it gives an approximation $\overline{q}$ to q which satisfies the error bound

$$(2.12) \qquad \langle\overline{q}-q\rangle \leqslant 2M(\varepsilon,E), \text{ where } M(\varepsilon,E) \text{ is the "best-possible-stability-bound"}$$

$$(2.13) \qquad M(\varepsilon,E) = \sup\{\langle q\rangle : \|Aq\| \leqslant \varepsilon \ , \ \|q\| \leqslant E\}$$

If q satisfies (2.10) and (2.11), it also satisfies

$$(2.14) \qquad \|\overline{F}-Aq\|^2 + \left(\frac{\varepsilon}{E}\right)^2 \|q\|^2 \leqslant 2\,\varepsilon^2$$

and we have lost at most a factor of $\sqrt{2}$ going from the two constraints to the one.

Let our approximation $\overline{q}$ be chosen such as to minimize

$$(2.15) \qquad \{\|\overline{F}-Aq\|^2 + \left(\frac{\varepsilon}{E}\right)^2 \|q\|^2\}.$$

The cannonical equation for this minimization is given by

$$(2.16) \qquad \{A^*A + \left(\frac{\varepsilon}{E}\right)^2 I\}\,\overline{q} = A^*\overline{F}.$$

We can now derive an estimate for $M(\varepsilon,E)$ for the linear functional $J_\delta q(t)$. We may assume the time of interest to be t=0.

We want the supremum of

$$(2.17)\qquad |J_\delta q(0)| = |(2\pi)^{-1} \int_{-\infty}^{\infty} e^{-w^2\delta^2/4}\, \hat{q}(w)\, dw|$$

with respect to the two constraints (2.10) and (2.11). These constraints can also be written as

$$(2.18)\qquad \int_{-\infty}^{\infty} |(\mu+i\sigma\mu)|^{-2} |\sinh(\mu+i\sigma\mu)|^{-2} |\hat{q}(w)|^2\, dw \leq \varepsilon^2/2\pi$$

and

$$(2.19)\qquad \int_{-\infty}^{\infty} |\hat{q}(w)|^2\, dw \leq E^2/2\pi$$

However, it is sufficient to bound (2.17) by the single constraint (2.18) alone. Using the Cauchy inequality we have

$$(2.20)\qquad |J_\delta q(0)| \leq \frac{\varepsilon}{2\pi} \left(2 \int_0^{\infty} 2\mu^2\, \frac{e^{2\mu}}{2}\, e^{-w^2\delta^2/2}\, dw\right)^{1/2}$$

Since $e^{-w^2\delta^2/2}$ is about .6 for $w \leq w_1 \equiv 1/\delta$ and falls rapidly to zero for $w > w_1$, while on the other hand $\mu^2 e^{2\mu} = \frac{w}{2}\, e^{\sqrt{2w}}$ grows only slowly with w, it follows that

$$(2.21)\qquad |J_\delta q(0)| \leq \frac{\varepsilon}{\pi} \left\{\int_0^{w_1} \mu^2 e^{2\mu}\, dw\right\}^{1/2} \leq \frac{\varepsilon}{2\pi}\, \delta^{-1} \exp(1/\sqrt{2\delta})$$

which as $\varepsilon \to 0$ becomes the best possible bound but for a factor of two, for fixed δ.

This shows that the error can be guaranteed to go down in Lipschitz fashion as $\varepsilon \to 0$ for a fixed δ.

Finally, performing the same type of arguments, we can estimate the supremum of $|J_\delta f(0)|$ with respect to the constraint (2.18). We get

$$(2.22)\qquad |J_\delta f(0)| \leq \frac{\varepsilon}{\sqrt{2}\,\pi}\, \delta^{-1/2} \exp(1/\sqrt{2\delta}).$$

3. THE SLAB WITH ONE INSULATED BOUNDARY, DISCRETIZED PROBLEM.

In this section we assume that q is locally L^2 bounded, uniformly on every sufficiently long interval, and that the data for Aq is measured at a discrete set of equally spaced data points in some finite interval of lenght β; we then seek to reconstruct $J_\delta q\ (\xi)$ at some point ξ approximately opposite the middle of the data set. If we choose our point of interest to be $\xi = 0$, the data set consists of K points $d_1, \ldots, d_k$ in the $[-\beta/2, \beta/2]$ interval, with equal spacing $\Delta t = \beta/(k-1)$. The data function $\overline{F}$ is a discrete function measured at these sampling points. The interval $[-\beta/2, \beta/2]$ should contain all the data for Aq which might reasonably be expected to enter into the reconstruction of q at time

$\xi = 0$, provided we make β sufficiently large. If that is the case, since the operator A makes Aq so smooth, we have reason to believe that a discrete sampling of Aq in $[-\beta/2, \beta/2]$ contains just as much information as a continuous sampling, provided that the sampling interval Δt is made sufficiently small.

Aq is given by the integral

$$(3.1) \qquad Aq(t) = \int_{-\infty}^{t} P(t-s)\, q(s)\, ds$$

where the kernel function P(t) has the Fourier series

$$(3.2) \qquad P(t) = 1+2 \sum_{n=1}^{\infty} (-1)^n \exp(-n^2\pi^2 t).$$

The shape of the kernel shows that Aq(t) depends strongly on almost the entire "past history of q". This means that $\beta/2$ should be taken so large that the number of points in the discrete data sampling interval will become enormous when the sampling interval Δt is made sufficiently small.

In order to avoid this situation, we subtract from the actual data Aq(t) the influence of the past history of q. Of course, this implies the knowledge of q for all previous times. Because of this difficulty, we will lower our goal for the moment and assume as an alternative,

$$(3.3) \qquad q(t) = 0 \quad \text{for} \quad t \leqslant 0 \quad \text{and therefore}$$

$$(3.4) \qquad Aq(t) = 0 \quad \text{for} \quad t \leqslant 0.$$

Now, the choice of $\beta/2$ sufficiently large certainly allow us to rigorously approximate our problem by a completely discretized one with q replaced by a β-periodic Nth order trigonometric sum of the form

$$(3.5) \qquad \overline{q}(t) = \sum_{-N}^{N} x_j \exp(iw_j t) \quad , \quad \text{with} \quad w_j = (2\pi/\beta)j$$

Since the operators A and J_δ are very smoothing, it is easy to pick N sufficiently large such that $A(\overline{q}-q) < .1\, \varepsilon$ on the data set and $J_\delta(\overline{q}-q)$ is negligibly small at the reconstruction point of interest.

NUMERICAL METHOD. Given a function $\overline{q}$ on $[-\beta/2, \beta/2]$ of the form (3.5) satisfying

$$(3.6) \qquad \|Aq-\overline{F}\|_{[-\beta/2,\beta/2]} \quad , \quad \text{where } \overline{F} = 0 \quad \text{on} \quad [-\beta/2, 0],$$

and

$$(3.7) \qquad \|q\|_{[-\beta/2,\beta/2]} \leqslant E \ ,$$

we wish to approximately determine the linear function $J_\delta q(\xi)$ at the point $\xi = 0$ of interest.

The least squares problem becomes

(3.8) $$\text{minimize } \phi(x) = \|Hx-h\|_K^2 + (\tfrac{\varepsilon}{E})^2 \|Rx\|_K^2$$

where H is the Kx(2N+1) matrix with entries

(3.9) $$H_{kj} = k^{-1/2} e^{id_k w_j} [(\mu_j+i\sigma\mu_j) \sinh(\mu_j+i\sigma\mu_j)]^{-1}$$

$$k = 1,\ldots,K \quad ; \quad j = -N,\ldots,N.$$

R is the Kx(2N+1) matrix with entries

(3.10) $$R_{kj} = K^{-1/2} e^{id_k w_j}$$

$$k = 1,\ldots,K \quad ; \quad j = -N,\ldots,N.$$

h is the vector with elements

(3.11) $$h_k = F(d_k)/\sqrt{K} \quad ; \quad k = 1,\ldots,K.$$

The vector x^0 minimizing (3.8) is the solution of the normal equations

(3.12) $$Zx^0 = (H^*H+(\tfrac{\varepsilon}{E})^2 R^*R)\, x^0 = H^*h.$$

The desired linear functional can be written as

(3.13) $$J_\delta q(\xi) = (\rho_\delta * q)(\xi) = (x,v), \text{ where}$$

(3.14) $$v_j = \frac{1}{\delta\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(\xi-s)^2/\delta^2} e^{iw_j s}\, ds \ ,$$

and using (3.12) it follows that

(3.15) $$J_\delta q(\xi) = (Z^{-1}H^*h,v) = (h,HZ^{-1}v) \equiv (h,V).$$

The vector $V = HZ^{-1}v$ can be computed and stored once and for all for that time ξ of interest.

Therefore,

(3.16) $$J_\delta q(\xi) = \sum_{k=1}^{K} h_k V_k.$$

If our data $\overline{F}$ is measured at a whole long sequence of sample points with equal spacing Δt, we can just translate our data set along the t axis by the multiples $T_j = j\Delta t$, j integer, and attempt to reconstruct our lineal functional $J_\delta q$ at those new points using the previous weights given by (3.15) if and only if we are able to repeat the conditions for the reconstruction at $\xi = 0$, which requires that F be = 0 for $t < T_j$. This can actually be achieved if we subtract the influence upon the data of the last reconstructed point. In doing so, we subtract the influence upon the data of the last $J_\delta q$ instead of q, but this is allowed since A is a smoothing operator and therefore the high frequencies die out very fast in Aq.

Hence, our approximation at T_j is given by the very cheap and simple discrete convolution against the data sequence, updated as mentioned:

$$J_\delta q(T_j) = \sum_{k=1}^{K} \overline{F}(T_j+d_k)\ (V_k/\sqrt{K}). \qquad (3.16)$$

4. GENERAL PROBLEM.

The general problem is obtained replacing (2.3) in the system (2.1)-(2.5) by

$$u_x(0,t) = Q(t), \text{ with corresponding approximate data function } \overline{Q}(t). \qquad (4.1)$$

The boundary functions are related now by

$$\begin{pmatrix} f(t) \\ q(t) \end{pmatrix} = \begin{pmatrix} \cosh(\mu+i\sigma\mu) & \sinh(\mu+i\sigma\mu)/(\mu+i\sigma\mu) \\ (\mu+i\sigma\mu)\ \sinh(\mu+i\sigma\mu) & \cosh(\mu+i\sigma\mu) \end{pmatrix} \begin{pmatrix} F(t) \\ Q(t) \end{pmatrix} \qquad (4.2)$$

The last equation clearly shows that this inverse problem is quite ill-posed in the high frequency components.

We recall that the direct or well-posed problems in this situation would involve specifying one of the two functions F(t) or Q(t) on the left surface and one of the two functions f(t) or q(t) on the right surface.

THE STABILIZED INVERSE PROBLEM. We decompose the general problem in two parts and proceed by superposition. The main idea is to perform the decomposition in such a way that only one of the new problems is ill-posed.

Let's consider the following well-posed problem in the slab:

PROBLEM A

$$u_t = u_{xx} \quad ; \quad 0 < x < 1 \quad ; \quad -\infty < t < \infty. \qquad (4.3)$$

$$u_x(0,t) = Q(t), \text{ with corresponding approximate data function } \overline{Q}(t). \qquad (4.4)$$

$$u_x(1,t) = 0 \quad , \text{ given.} \qquad (4.5)$$

$$u(0,t) = F_1(t) \quad , \text{ unknown.} \qquad (4.6)$$

$$u(1,t) = f_1(t) \quad , \text{ unknown.} \qquad (4.7)$$

If we denote by K_0 the solution for $F_1(t)$ with $Q(t) = \delta_0(t)$ and by K_1 the solution for $f_1(t)$ with $Q(t) = \delta_0(t)$, we get

$$K_0(t) = \frac{-1}{\sqrt{\pi t}}\,[1 + 2 \sum_{m=1}^{\infty} e^{-m^2/t}] \qquad (4.8)$$

and

(4.9) $$K_1(t) = \frac{-2}{\sqrt{\pi t}} \sum_{m=1}^{\infty} e^{-(2m-1)^2/4t}.$$

Of course for small t the m = 0 term completely dominates in these sums.

We now see how to give a stabilized solution of the general problem as a superposition of the well-posed problem A and the stabilized ill-posed insulated boundary problem (Q = 0) treated in sections 2 and 3. In order to solve the general problem, we first solve problem A with data $Q_A(t) = \overline{Q}(t)$ and $q_A(t) = 0$, obtaining $\overline{F}_1(t)$ and $J_\delta f_1(t)$.

Next, we solve the insulated boundary problem with data $\overline{F}_2(t) = \overline{F}(t) - \overline{F}_1(t)$ and $Q(t) = 0$, obtaining $J_\delta f_2$ and $J_\delta q(t)$.

Since f_1 (and even more so $J_\delta f_1$), $J_\delta f_2$ and $J_\delta q$ depend continuously upon the data, the superposition gives a stabilized solution of the general problem.

5. THE GENERAL PROBLEM. DISCRETIZED VERSION.

NUMERICAL METHOD. The transition to the discrete case for problem A is relatively straightforward.

In order to be consistent with the discussions of section 4, we add for the moment the assumption that the data $\overline{Q}(t) = 0$ for $t \leqslant 0$, and of course we assume $\overline{Q}$ to be locally L^2 bounded, uniformly on every sufficiently long interval of interest.

The data function $\overline{Q}(t)$ is a discrete function measured at the sampling points, and we consider $\overline{Q}(t)$ to be constant in each subinterval.

However, a discrete approximation of $K_0(t)$ or $K_1(t)$ by piecewise constant functions is very poor for small values of t since these kernels blow up at t = 0. Thus we would like to average the kernels K_0 and K_1 before considering the discrete versions of the convolutions against the data function $\overline{Q}(t)$. One natural way to accomplish this is to replace K_0 and K_1 by the piecewise constant functions $\overline{K}_0$ and $\overline{K}_1$ equal their mean values on each ith interval.

Thus the discrete convolutions $\overline{K}_0 * \overline{Q}$ and $\overline{K}_1 * \overline{Q}$ are actually the continuous convolutions that would be obtained if $\overline{Q}$ were extended to be a piecewise constant function.

After computing $\int_0^t K_0(s)\,ds$ for $t = (j \pm 1/2)\,\Delta t$; j integer, the mean value follows by difference and division by Δt. Using (4.8) and the Laplace transform, we get

(5.1) $$\int_0^t K_0(s)\,ds = \frac{-2}{\pi}\sqrt{t} + 4 \sum_{n=1}^{\infty} n \operatorname{erfc}\left(\frac{n}{\sqrt{\pi}}\right) - \frac{\sqrt{t}}{\sqrt{\pi}} e^{-n^2/t}$$

Similarly,

(5.2) $$\int_0^t K_1(s)ds = 2 \sum_{m=1}^{\infty} \{(2m-1)\operatorname{erfc}(\frac{2m-1}{2\sqrt{t}}) \frac{-2\sqrt{t}}{\sqrt{\pi}} e^{-(2m-1)^2/4t} .$$

The temperature solution for problem A at x = 0 is computed approximately by means of the discrete convolution

(5.3) $$\overline{F}_1(k\Delta t) = \sum_{i=0}^{K} \overline{Q}(i\Delta t)\ \overline{K}_0((k-i)\Delta t).\Delta t \ ;\ k = 0,\dots,N.$$

Similarly, the temperature solution at x = 1, could be computed by means of the discrete convolution

(5.4) $$f_1(k\Delta t) = (\overline{Q}*\overline{K}_1)(k\Delta t)$$

and

(5.5) $$J_\delta f_1(k\Delta t) = (\rho_\delta * f_1)(k\Delta t) = (J_\delta \overline{K}_1 * \overline{Q})\ (k\Delta t).$$

The weights $J_\delta \overline{K}_1$ are easily computed since $\overline{K}_1$ is a linear combination of the step function $\Psi_i(t)$, i.e.:

(5.6) $$\overline{K}_1(t) = \sum_{i=1}^{M} \overline{K}_1^i\ \Psi_i(t)$$

and it follows that

(5.7) $$J_\delta \overline{K}_1(t) = \sum_{i=1}^{M} \overline{K}_1^i\ \{\Phi(\frac{i\Delta t-t}{\sigma}) - \Phi(\frac{(i-1)\ \Delta t-t}{\sigma})\}$$

where Φ indicates the normalized Gaussian distribution function with $\sigma = \delta/\sqrt{2}$.

The data temperature $\overline{F}_2(t)$ for the insulated boundary problem (Q=0) is now computed as the difference between the measured data temperature $\overline{F}(t)$ and the computed temperature given by (5.3).

INVERSE KERNELS. In order to investigate the stability of our numerical method, we would like to know the amplification factor associated with errors in the data when using numerical procedures.

We notice that any piecewise constant data function can be expresed by a discrete convolution against the numerical delta function N(t) defined as

(5.8) $$N(t) = \begin{cases} 1/\Delta t \text{ if } (-1/2)\ \Delta t \leqslant t < (1/2)\Delta t \\ 0 \text{ otherwise} \end{cases}$$

Moreover, if we know the solution for N(t) as data, our Inverse Kernel, it follows by linearity that the total error in the solution can be obtained as the discrete convolution of the data error against the inverse kernel.

For the general problem we have to consider four Inverse Kernels:

I) IKQf(t) , the temperature solution for the problem in the slab with data F(t) = 0 and Q(t) = N(t).

II) IKFf(t) , the temperature solution for the problem in the slab with data $F(t) = N(t)$ and $Q(t) = 0$.

III) IKQq(t) , the flux solution for the problem in the slab with data $F(t) = 0$ and $Q(t) = N(t)$.

IV) IKFq(t) , the flux solution for the problem in the slab with data $F(t) = N(t)$ and $Q(t) = 0$.

In figures 1,2,3 and 4 we plot the Inverse Kernels I), II), III) and IV) respectively, for different values of the parameters. In all cases the ordinate values have been scaled by the factor arcsinh(y).

The most important feature of the inverse kernels is the fact that they nearly have compact support. In Table 1 the apparent "support" indicates the interval in which the absolute value of the reconstructed function is greater than 10^{-4}. This allows us to actually compute the solutions by means of the discrete convolutions

$$\text{(5.9)} \qquad \begin{pmatrix} J_\delta f(t) \\ J_\delta q(t) \end{pmatrix} = \begin{pmatrix} IKFf & IKQf \\ IKFq & IKQq \end{pmatrix} * \begin{pmatrix} \overline{F}(t) \\ \overline{Q}(t) \end{pmatrix}$$

without needing the history of $\overline{F}$ and $\overline{Q}$ for very long times before or after the time t of interest; i.e.: the kernels are taken to be zero outside their "support".

The Inverse Kernels are computed once and for all for fixed Δt at the points $s_j = \pm j\Delta t$, $j = 0,1,\dots,m$ which include the "support" interval shown in Table 1. Then if we want to reconstruct the functionals $J_\delta f$ and $J_\delta q$ at any time $T_i = i\Delta t$, i integer, we read the data in the support interval with T_i as center and merely use (5.9) in the form

$$\text{(5.10)} \qquad J_\delta f(t_i) = \Delta t \sum_{j=-m}^{m} (\overline{F}(T_i - s_j) IKFf(s_j) + \overline{Q}(T_i - s_j) IKQf(s_j)$$

and

$$\text{(5.11)} \qquad J_\delta q(t_i) = \Delta t \sum_{j=-m}^{m} (\overline{F}(T_i - s_j) IKFq(s_j) + \overline{Q}(T_i - s_j) IKQq(s_j)).$$

DIRECT KERNELS. In order to test the accuracy of our method, we would like to approximately reconstruct a delta function at time $t = 0$ in $u(1,t)$ by solving the problem

$$\text{(5.12)} \qquad \begin{aligned} &u_t = u_{xx} \quad ; \quad 0 < x < 1 \quad ; \quad -\infty < t < \infty. \\ &u(0,t) = 0 \text{ , data.} \\ &u_x(0,t) = \text{data.} \\ &u(1,t) = \delta_0(t) \text{ , unknown.} \end{aligned}$$

We generate the exact data as the solution of the well-posed problem

$$\text{(5.13)} \qquad \begin{aligned} &u_t = u_{xx} \quad ; \quad 0 < x < 1 \quad ; \quad -\infty < t < \infty. \\ &u(0,t) = 0. \end{aligned}$$

$$u(1,t) = \delta_0(t).$$

Similarly, using (5.10) and (5.11) we reconstruct a delta function at time $t = 0$ in $u_x(1,t) = q(t)$ with exact data $Q(t)$ and $F(t) = 0$.

Several numerical solutions using different values of the parameters are shown in figures 5 and 6. Figures 7 and 8 show the reconstructed delta functions at $t=0$ in $u(1,t) = f(t)$ and $u_x(1,t) = q(t)$ respectively, corresponding to exact data $F(t)$ and $Q(t) = 0$.

In all cases, the Direct Kernels are symmetric, positive and the total integral is extremely well conserved.

Finally, figure 9 shows the reconstructed temperature solution (with $\beta=1$, $N=39$, $\Delta t=.0125$ and $\delta=2\Delta t$) which is the superposition of two solutions of simple structure; the first has $F(t) = 0$ and $f(t)$ = a step function of height 1 between $t = .1$ and $.3$; the second has $Q(t) = 0$ and the same $f(t)$. The reconstructed mollified temperature shows over and undershoots of only 5% when 1% random error is added to both $F(t)$ and $Q(t)$ (1% of max $|F(t)|$ for $F(t)$ and 1% of max $|Q(t)|$ for $Q(t)$).

β	1	1	1
N	9	19	39
Δt	.05	.025	.0125
ε/E	10^{-4}	10^{-4}	10^{-4}
δ	$4\Delta t$	$4\Delta t$	$4\Delta t$
"support" I	[-.5,.6]	[-.5,.6]	[-.5,.6]
"support" II	[-.5,.8]	[-.4,.4]	[-.3,.2]
"support" III	[-.5,.8]	[-.4,.4]	[-.3,.2]
"support" IV	[-.5,.9]	[-.5,1.]	[-.5,1.]

Table 1 for Figures 1, 2, 3 and 4

Same parameters as in Table 1,

Fig. 5:			
"support"	[-.4,.4]	[-.2,.2]	[-.15,.15]
Integral	.9993	.99952	.99999
Fig. 6:			
"support"	[-.4,.4]	[-.2,.2]	[-.15,.15]
Integral	.994	.99998	.99999
Fig. 7:			
"support"	[-.4,.4]	[-.2,.2]	[-.15,.15]
Integral	.994	.99998	.99999
Fig. 8:			
"support"	[-.4,.4]	[-.3,.5]	[-.35,.5]
Integral	1.02	1.002	1.0001

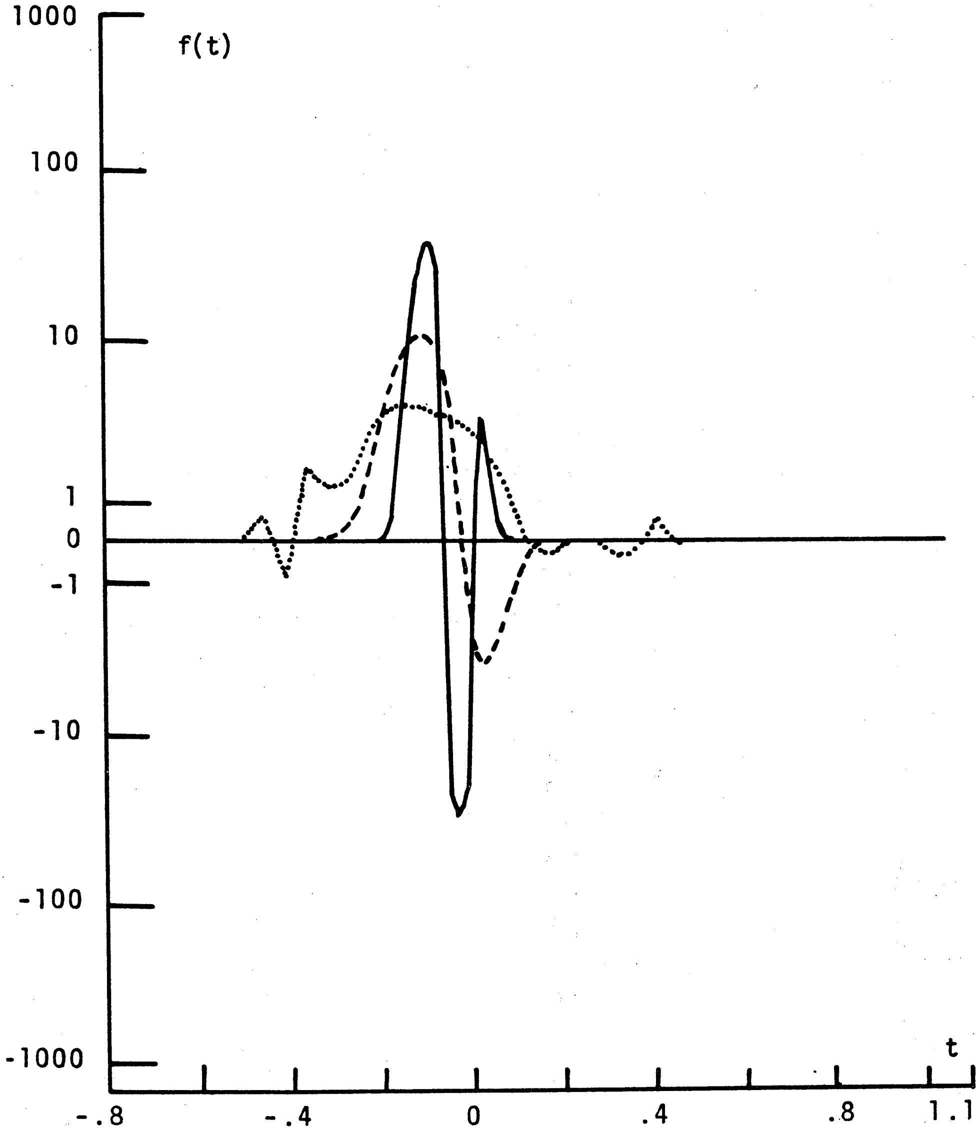

Figure 1 - Inverse Kernels IKQf; temperature f(t) corresponding to $\bar{F}(t) = 0$, $\bar{Q}(t)$ = numerical δ function.

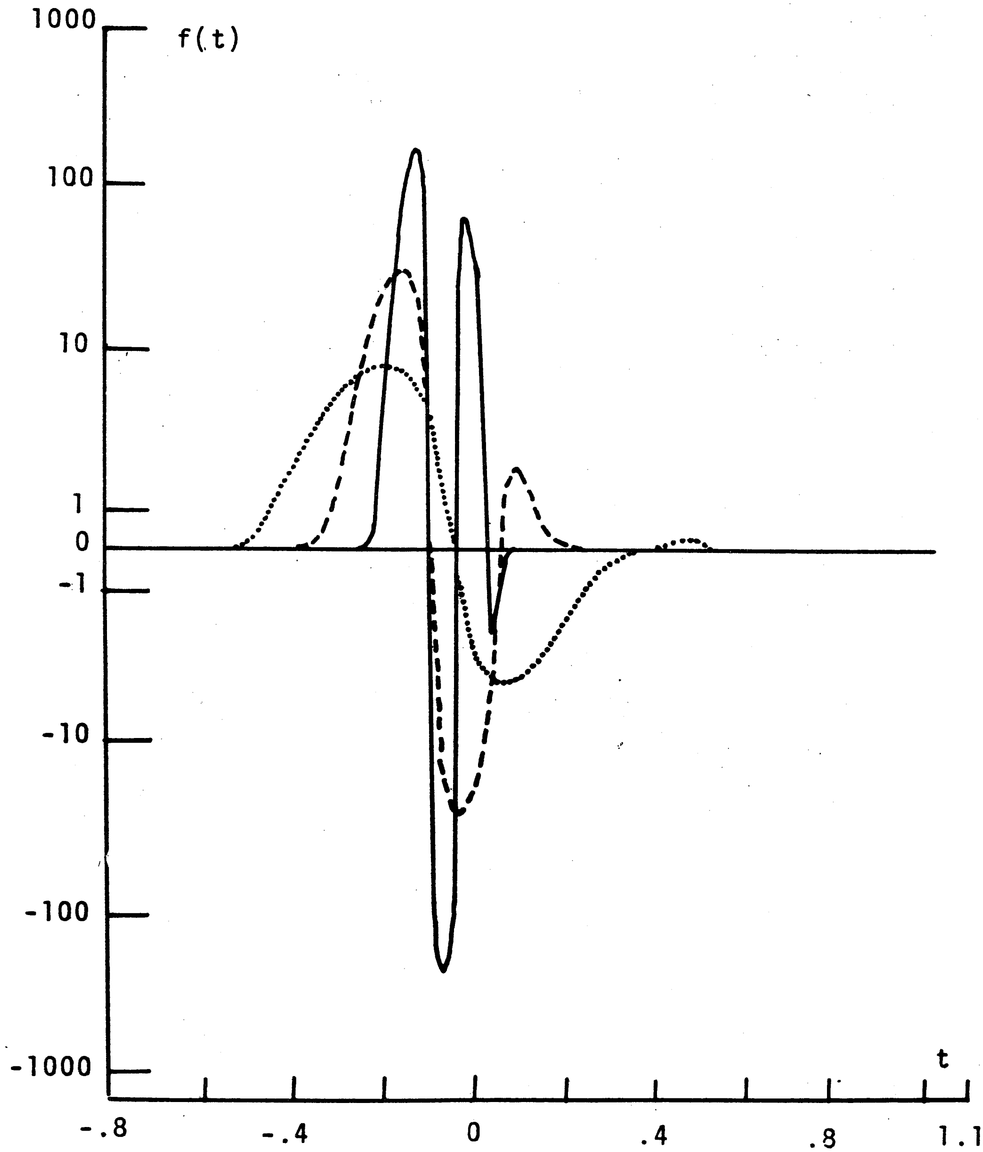

Figure 2 - Inverse Kernels IKFf; temperature f(t) corresponding to $\bar{F}(t)$ = numerical δfunction, $\bar{Q}(t)$ = 0.

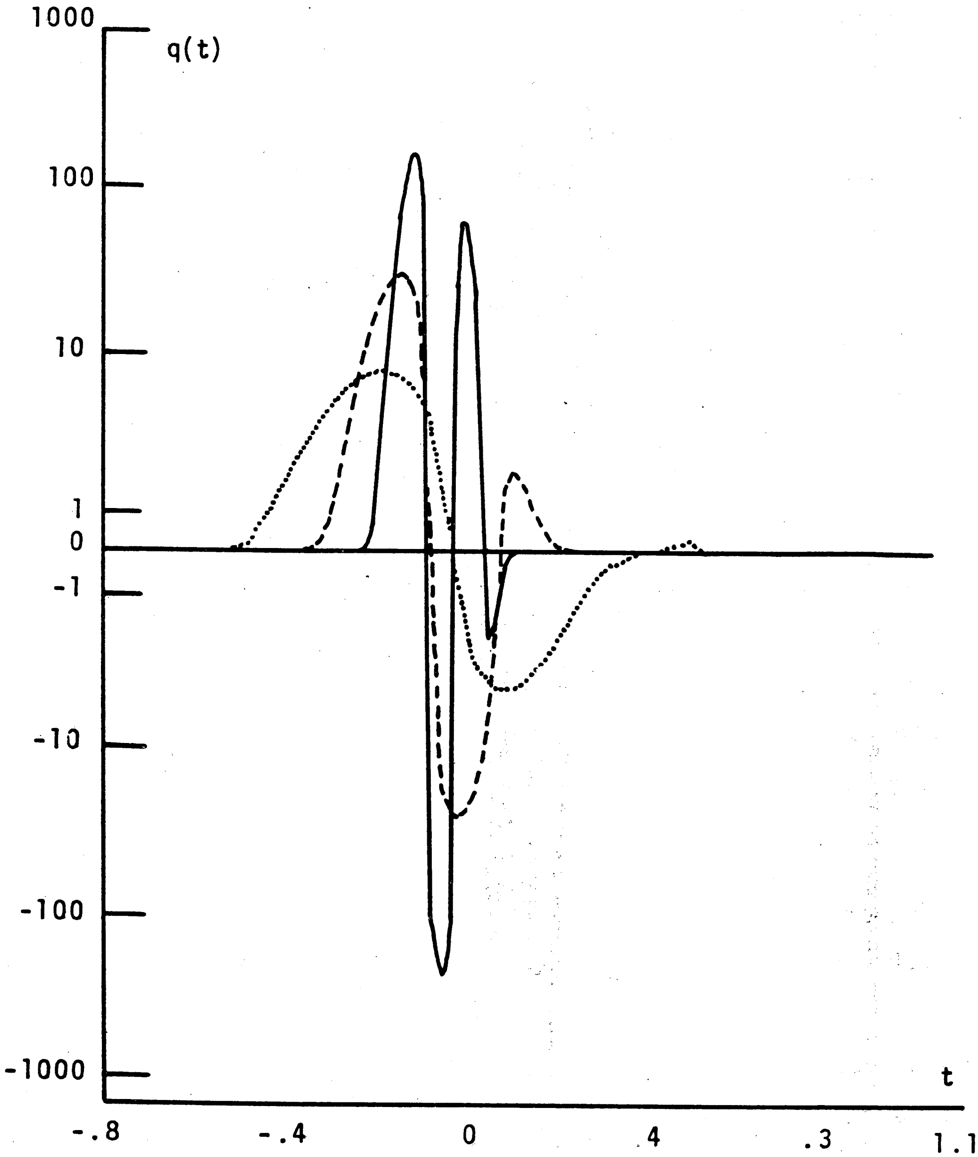

Figure 3 - Inverse Kernels IKQq; flux q(t) corresponding to $\bar{F}(t) = 0$; $\bar{Q}(t)$ = discrete δ function.

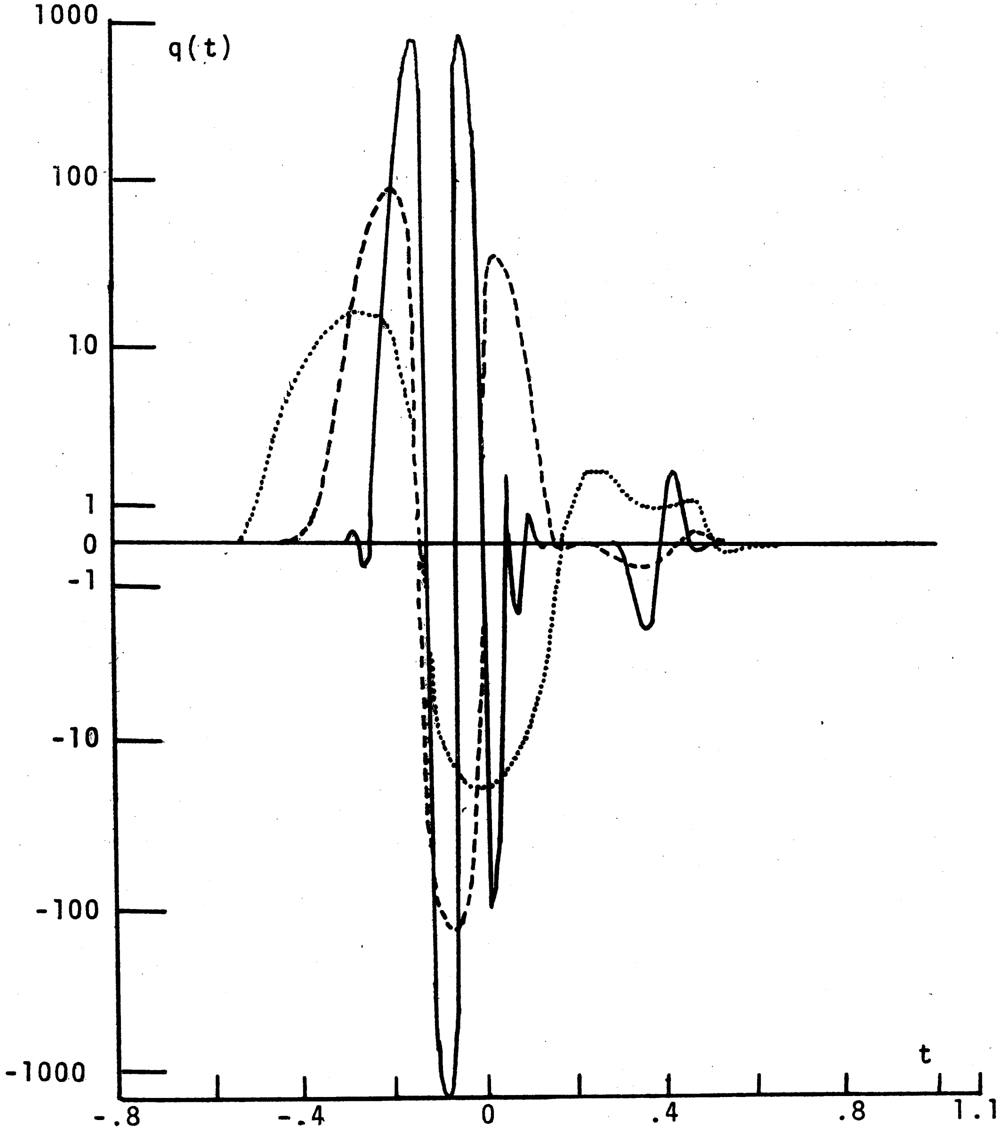

Figure 4 - Inverse Kernels IKFq; flux q(t) corresponding to $\bar{Q}(t) = 0$, $\bar{F}(t)$ = discrete δ function.

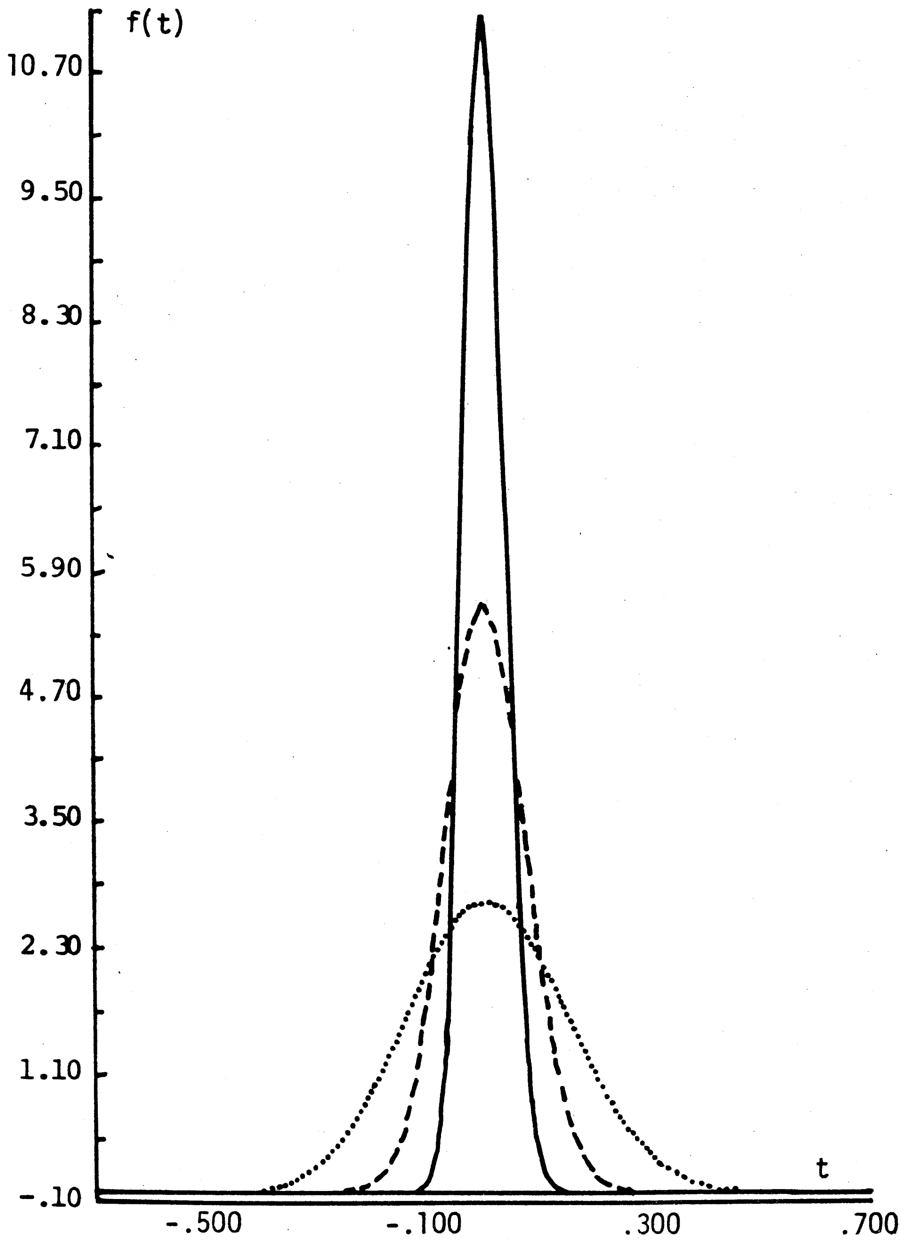

Figure 5 - Reconstructed temperature f(t) corresponding to $\bar{F}(t)=0$, $\bar{Q}(t)$=exact data.

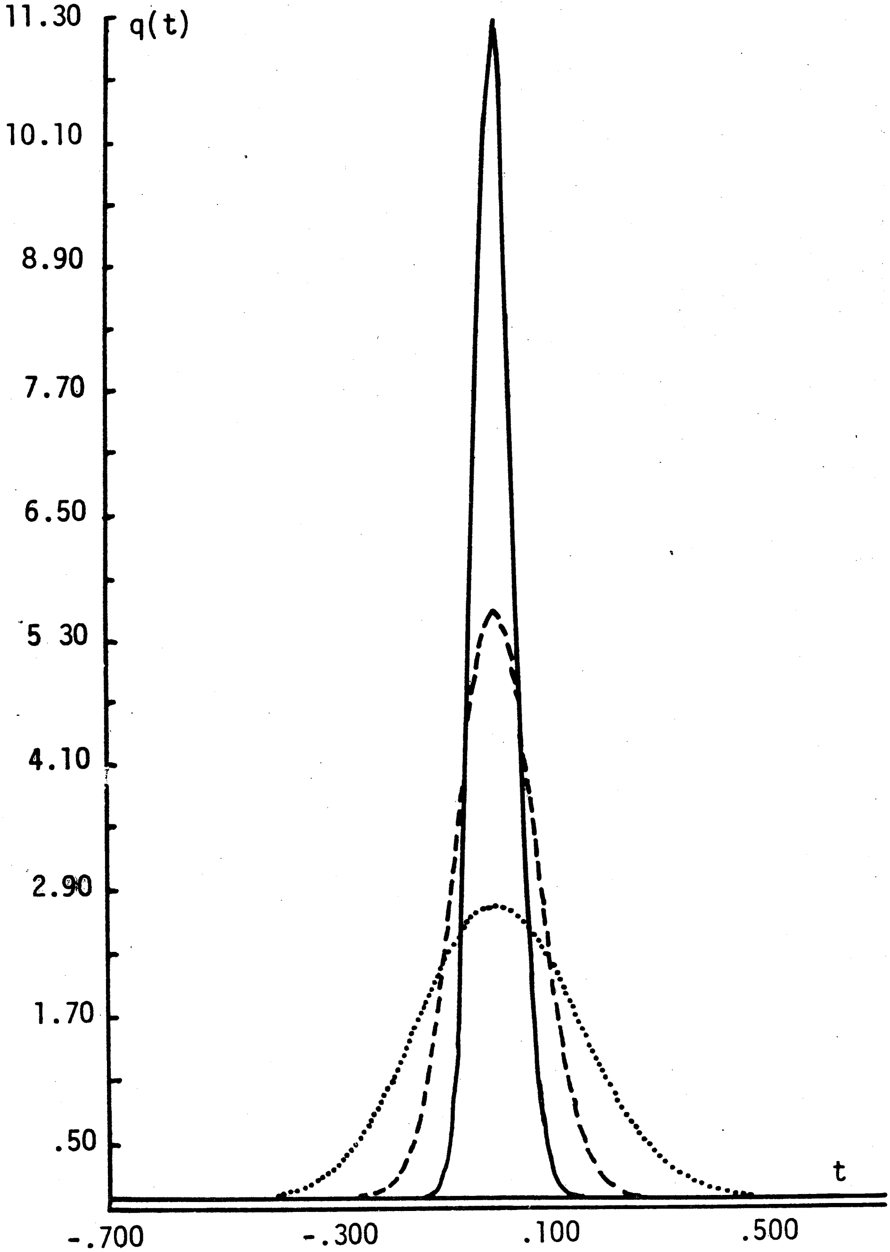

Figure 6 - Reconstructed flux q(t) corresponding to $\bar{F}(t)=0$, $\bar{Q}(t)$=exact data.

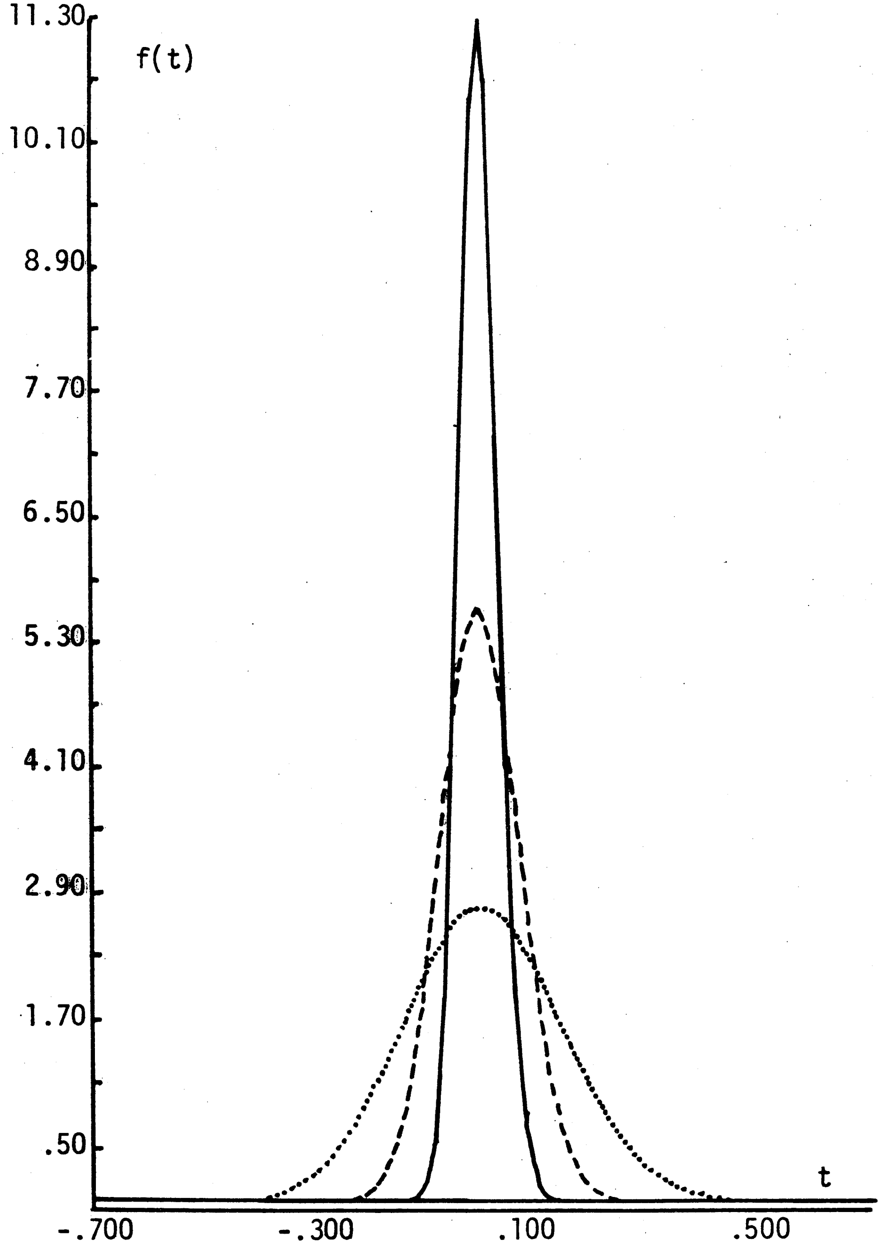

Figure 7 - Reconstructed temperature f(t) corresponding to $\bar{Q}(t)=0$, $\bar{F}(t)$=exact data.

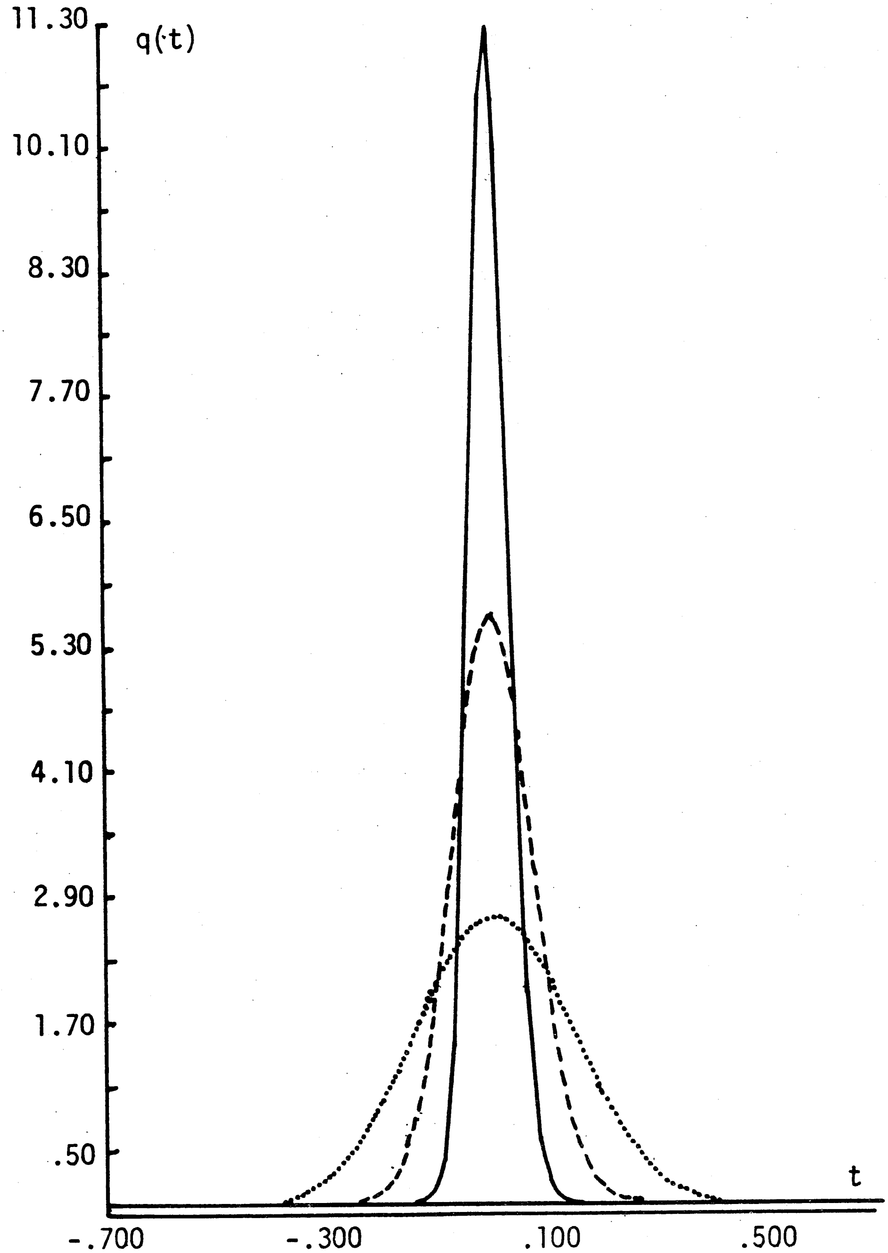

Figure 8 - Reconstructed flux function q(t) corresponding to $\bar{Q}(t)=0$, $\bar{F}(t)$=exact data.

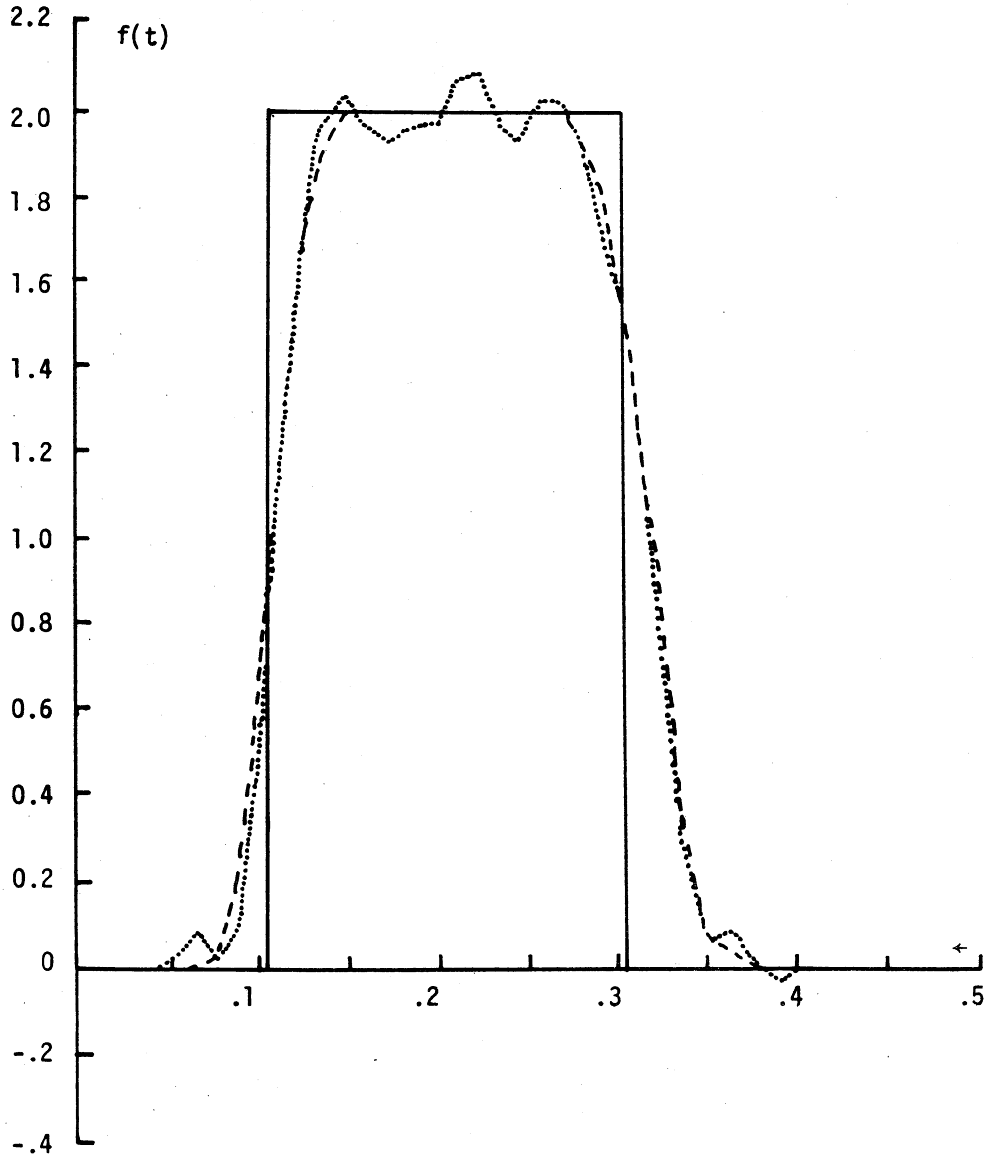

Figure 9 - Reconstructed temperature f(t) with exact (--) and perturbed (··) data $\bar{F}(t)$ and $\bar{Q}(t)$. $\delta = 2\ \Delta t = .025$.

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Department of Mathematical Sciences
University of Cincinnati
Cincinnati, Ohio 45221

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