Revista de la
Unión Matemática Argentina
Volumen 30, 1981-82.

SIMPLE HOMOTOPY TYPE OF PSEUDO-LENS SPACES

A. Neme and C. Sánchez

1. INTRODUCTION.

In this paper we study actions of cyclic groups on spheres by considering their quotient spaces. We consider the important special case of "linear" actions.

Amongst all the quotient spaces of linear actions, we choose a special type which we call "Pseudo-Lens" spaces since, as we show in the subsequent sections, they have properties similar to those of the well known Lens spaces.

We believe that these spaces are interesting in themselves and also because they provide models for the study of quotient spaces of a special, but important, type of not-necessarily-free actions on spheres, namely those with well ordered set of isotropy groups.

In section two, we define pseudo-lens spaces and describe a cell structure for them and their universal coverings.

In section three, we give a necessary condition for simple homotopy equivalence and in section four, we see that in the case of actions of cyclic p-groups this condition is also sufficient.

2. PSEUDO-LENS SPACES.

Let $m \geqslant 3$ be an integer and $G = Z_m$ with a preferred generator t. Let $S^{2n-1} \subset C^n$ the unit sphere in complex n-space, with a standard orientation.

Let $\omega = \exp(i \frac{2\pi}{m})$ and consider the action of G on S^{2n-1} defined by

$$(2.1) \qquad g(z_1,\dots,z_n) = (\omega^{q_1} z_1,\dots,\omega^{q_n} z_n).$$

Our objective is to study the quotient space S^{2n-1}/G which in the case $(q_i.m) = 1$, $i = 1,\dots,n$ is a manifold and is called *lens space*.

We shall restrict our attention to effective actions without fixed points and we pose on the coefficients, the restriction $1 \leqslant q_i \leqslant m$ $i = 1,2,\dots,n$.

There is a natural partial ordering relation among the isotropy subgroups of the action (2.1) given by inclusion. If $H_1,\dots,H_k$ is the set of isotropy subgroups, we may think that the order is such that $H_j \supseteq H_i$ implies $j \leq i$.

If the resulting order is total, i.e. $H_1 \supseteq H_2 \supseteq \dots \supseteq H_k$, we shall call the quotient space S^{2n-1}/G a "*pseudo lens-space*" and denote it by $L_m(q_1,\dots,q_n)$ or $L(m;q_1,\dots,q_n)$.

Since the actions we consider are effective, we must have $H_k = \{e\}$, where e is the identity of G.

It is easy to provide these spaces with a CW-complex structure.

In order to describe this structure, we consider the following reindexing of the coefficients $q_i = 1,\dots,n$.

$$(2.2) \qquad i < j \quad \text{implies} \quad (q_j,m) \mid (q_i,m)$$

where (q_j,m) is the usual greatest common divisor. We have then

$$(q_i,m) \mid (q_{i-1},m) \qquad i = 2,\dots,n.$$

We can associate to each q_i another integer ℓ_i defined by

$$(2.3) \qquad \ell_i \frac{q_i}{(q_i,m)} \equiv 1 \pmod{\frac{m}{(q_i,m)}}$$

We can give now a cell decomposition of $L_m(q_1,\dots,q_n)$.

We shall have cells e^{2k-1} and $e^{2\ell}$, $1 \leq k \leq n$, $0 \leq \ell \leq n-1$. The cell e^{2k-1} will be image in the quotient of

$$\hat{e}^{2k-1} = \{(z_1,\dots,z_{k-1}, r_k e^{i\theta_k},0,\dots,0) : 0 \leq \theta_k \leq 2\pi \frac{(q_k,m)}{m}\}$$

and $e^{2\ell}$ the image of

$$\hat{e}^{2\ell} = \{(z_1,\dots,z_\ell, r_{(\ell+1)}, 0,\dots,0) : r_{\ell+1} \geq 0\}.$$

The boundary relations in S^{2n-1} take the form

$$(2.4) \qquad \begin{aligned} &\partial\hat{e}^{2k-1} = (t^{\ell_k} - 1)\, \hat{e}^{2k-2} && 1 \leq k \leq n \\ &\partial\hat{e}^{2s} = (1 + t + \dots + t^{(a_s-1)})\, \hat{e}^{2s-1} \\ &1 \leq s \leq n-1 && a_s = \frac{m}{(q_s,m)}. \end{aligned}$$

We easily obtain the homology and cohomology groups of $L_m(q_1,\dots,q_n)$. The homology groups are

$$(2.5) \qquad H_i(L_m,Z) = \begin{cases} 0 & i = 2k, \quad 1 \leq k \leq n-1 \\ Z & i = 0, \quad 2n-1 \\ Z_{a_k} & i = 2k-1, \quad 1 \leq k \leq n-1 \end{cases}$$

compare [3].

We want to consider now the universal covering $\tilde{L}_m(q_1,\ldots,q_n)$ of $L_m(q_1,\ldots,q_n)$.

Let H be the subgroup of G generated by t^{a_1} and consider the quotient space S^{2n-1}/H.

Since H acts trivially on the first coordinate we clearly have:

$$S^{2n-1}/H \cong S^1 * L_{a_1}(h_2,\ldots,h_n)$$

where $h_i \equiv q_i \pmod{a_1}$ $\quad 1 \leqslant h_i \leqslant a_1$.

Now, since the 1-squeleton of $L_{a_1}(h_2,\ldots,h_n)$ is S^1, we have that the 3-squeleton of S^{2n-1}/H is S^3 and clearly then S^{2n-1}/H is simply-connected.

Furthermore, S^{2n-1}/H supports a free action of $G/H \cong Z_{a_1}$

$$[t].[(Z_1,\ldots,Z_n)] = [t(Z_1,\ldots,Z_n)]$$

and clearly $(S^{2n-1}/H)/(G/H) \cong L_m(q_1,\ldots,q_n)$.

We can obtain a "natural" cell structure on $\tilde{L}_m(q_1,\ldots,q_n) = S^{2n-1}/H$ by the same procedure as before i.e. by inducing the "quotient" cell structure from S^{2n-1}. In this way, we have a_1 cells in each dimension and by taking quotient under the action of Z_{a_1} we get the cell structure already described in $L_m(q_1,\ldots,q_n)$.

We shall take the following notation

(2.6)
$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{\quad p \quad} & L_m(q_1,\ldots,q_n) \\ & {\scriptstyle p_1}\searrow \quad \nearrow{\scriptstyle p_2} & \\ & \tilde{L}_m(q_1,\ldots,q_n) & \end{array}$$

and $\quad \bar{q}_i = \dfrac{q_i}{(q_i,m)} \quad$ so $\quad \ell_i \, \bar{q}_i \equiv 1 \pmod{a_i}$.

3. SIMPLE HOMOTOPY EQUIVALENCE.

In this section we study conditions under which two pseudo-lens spaces are simple homotopy equivalent.

(3.1) PROPOSITION. *Let* $L = L(m;q_1,\ldots,q_n)$ *and* $L' = L'(m;q_1',\ldots,q_n')$ *be speudo-lens spaces and let* $\pi_1(L) \cong Z_{a_1} = \langle [t] \rangle = \langle h \rangle$ *and* $\pi_1(L') \cong Z_{a_1'} = \langle [t'] \rangle = \langle h' \rangle$ *be the corresponding fundamental groups* (t *and* t' *are generators of* G).

Let $f\colon L \longrightarrow L'$ *be a simple homotopy equivalence such that* $f_*(h) = (h')^a$ $(a,a_1') = 1$. *Then*

i) $(m,q_i) = (m,q_i')$ $\quad i = 1,\ldots,n$

ii) *There is a permutation* σ *of* $(1,\ldots,n)$ *such that* $\varepsilon_k \, a \, \bar{q}'_{\sigma(k)} \equiv \bar{q}_k$

(mod a_1) *where* $\varepsilon_k = \pm 1$ *for each* $k = 1,\dots,n$.

Proof. We can clearly assume $t = t' = \exp(i\,\frac{2\pi}{m})$.

(i) is clear. Let us proceed to prove (ii).

Let $\tilde{L}$ and $\tilde{L}'$ be the corresponding universal coverings and $\tilde{f}$ the map such that the diagram

$$\begin{array}{ccc} \tilde{L} & \xrightarrow{\ \tilde{f}\ } & \tilde{L}' \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_2'} \\ L & \xrightarrow[\ f\ ]{} & L' \end{array}$$

commutes.

Let us consider now the chain-complexes $C(\tilde{L}')$ and $[C(\tilde{L})]_{f\#}$ (see [1] p.74 for notation)

$$C(L') \qquad 0 \longrightarrow \tilde{C}'_{2n-1} \longrightarrow \tilde{C}'_{2n-2} \longrightarrow \dots \longrightarrow \tilde{C}'_1 \longrightarrow \tilde{C}'_0 \longrightarrow 0$$

with the boundary relations

$$\tilde{\partial}'(p_1'(\hat{e}^{2k-1})) = [(h')^{\ell_k'} - 1]\; p_1'(\hat{e}^{2k-2})$$

$$\tilde{\partial}'(p_1'(\hat{e}^{2s})) = \frac{a_s'}{a_1'}\,[1 + h' + \dots + h'^{(a_1'-1)}]\;\; p_1'(\hat{e}^{2s-1})$$

$$[C(L)]_{f\#} \qquad 0 \longrightarrow \tilde{C}_{2n-1} \longrightarrow \tilde{C}_{2n-2} \longrightarrow \dots \longrightarrow \tilde{C}_1 \longrightarrow \tilde{C}_0 \longrightarrow 0$$

with boundaries

$$\tilde{\partial}(p_1'(\hat{e}^{2k-1})) = [(h')^{a\ell_k} - 1]\; p_1'(\hat{e}^{2k-2})$$

$$\tilde{\partial}(p_1'(\hat{e}^{2s})) = \frac{a_s}{a_1}\,[1 + (h')^{a} + \dots + (h')^{a(a_1-1)}]\; p_1'(\hat{e}^{2s-1}).$$

Now f is a simple homotopy equivalence and therefore $\tau(f) = 0$ and by [1] p.74, Th.(22.8) it is possible to construct a $W(\pi_1(L'))$ complex C such that $\tau(C) = \tau(f)$ and C can be put into a *based* short exact sequence of $W(\pi_1(L'))$ complexes

$$0 \longrightarrow C(\tilde{L}') \longrightarrow C \longrightarrow \overline{C}(\tilde{L}) \longrightarrow 0$$

where $C(\tilde{L})$ is $[C(\tilde{L})]_{f\#}$ shifted in dimension by one and with its boundary multiplied by (-1) [1] p.74.

C is acidic, as can be verified from the definition and now we complete the proof of the theorem as in [1] (30.1).■

4. CYCLIC P-GROUPS.

We shall restrict ourselves now to the case $m = p^s$ with p *odd* prime.

If $L(p^s;q_1,\dots,q_n)$ is a pseudo-lens space and $\tilde{L}(p^s;q_1,\dots,q_n)$ is the universal covering, then

$$\tilde{L} = \underbrace{S^1 * S^1 * \dots * S^1}_{k} * L(p^j;q_{k+1},\dots,q_n)$$

where $k \geqslant 1$ is the greatest natural number such that $(p^s,q_1) = (p^s,q_2) = \dots = (p^s,q_k)$ and $p^j = (p^s,q_1)$. Also $\pi_1(L) \cong Z_{p^{(s-j)}} = Z_{a_1}$.

Let us take now pseudo-lens spaces $L(p^s;q_1,\dots,q_n)$ and $L(p^s;q_1',\dots,q_n')$. Let ℓ_k $k = 1,\dots,n$ be as in (2.3) and $\bar{q}_i'$ as in (2.6); then we have the following

(4.1) LEMMA. *If* $a^n \ell_1 \dots \ell_n \bar{q}_1' \dots \bar{q}_n' \equiv \pm 1 \pmod{p^{(s-j)}}$ *then there exists a cellular map* $f_3\colon \tilde{L} \longrightarrow \tilde{L}'$ *such that* $f_3(h(x)) = (h')^a f_3(x)$ *and* $\deg f_3 = \pm 1$.

Proof. Let us start by defining a function $F\colon S^{2n-1} \longrightarrow S^{2n-1}$ equivariant respect to the two actions of $G \cong Z_{p^s}$

$$F(z_1,\dots,z_n) = F(r_1 \exp(i2\pi\phi_1),\dots,r_n \exp(i2\pi\phi_n)) =$$
$$= (r_1 \exp(i2\pi a\ell_1 \bar{q}_1'\phi_1),\dots,r_n \exp(i2\pi a\ell_n \bar{q}_n'\phi_n))$$

we clearly have

$$F(g(z_1,\dots,z_n)) = (g')^a F(z_1,\dots,z_n)$$

F is cellular and $\deg F = a^n \ell_1 \dots \ell_n \bar{q}_1' \dots \bar{q}_n' = d$ with $d \equiv \pm 1 \pmod{p^{(s-j)}}$. (In order to simplify notation we shall consider from now on that $d \equiv 1 \pmod{p^{(s-j)}}$. The proof will be the same in the other case). F induces a new cellular function $f\colon \tilde{L} \longrightarrow \tilde{L}'$ and $\deg f = d$.

The new f satisfies

$$f(h(x)) = (h')^a f(x).$$

Now, since $d \equiv 1 \pmod{p^{(s-j)}}$, we have $d = 1 + \alpha p^{(s-j)}$ and since, $\underbrace{S^1 * \dots * S^1}_{k}$ is the $(2k-1)$-squeleton of $\tilde{L}$ and we can define

$$f_1\colon S^1 * \dots * S^1 \longrightarrow S^1 * \dots * S^1 \qquad \text{(k factors)}$$

such that it is cellular, $f_1(h'(x)) = h'f(x)$ and $\deg f_1 = 1 + \alpha p^{(s-j)}$. Such an f_1 exists since $S^1 * \dots * S^1 = S^{2k-1}$ and the actions of $\pi_1(L)$ and $\pi_1(L')$ are free (see [1] (29.4)).

Let us take now $f_1^\ell\colon S^1 * \dots * S^1 \longrightarrow S^1 * \dots * S^1$. It is cellular, $f_1^\ell(h'(x)) = h'f_1^\ell(x)$ and $\deg f_1^\ell = (1 + \alpha p^{(s-j)})^\ell$.

It is clear then, that for each $\ell \geqslant 1$ we can construct $f_\ell\colon \tilde{L}' \longrightarrow \tilde{L}'$ cellular equivariant ($f_\ell(h'(x)) = h' f_\ell(x)$) and such that

$$\deg f_\ell = (1 + \alpha p^{(s-j)})^\ell$$

since it suffices to define $f_\ell = (f_1)^\ell * \mathrm{Id}$.

Let us define now f_2: $\tilde{L} \longrightarrow \tilde{L}'$ by $f_2 = f_\ell \circ f$. Then f_2 is cellular, $f_2(h(x)) = (h')^a f_2(x)$, and $\deg f_2 = (1 + \alpha\, p^{(s-j)})^{\ell+1}$ i.e.
$\deg f_2 = 1 + (\ell + 1)\, \alpha\, p^{(s-j)} + \ldots + \alpha^{(\ell+1)}\, p^{(m-j)(\ell+1)}$.

We can clearly choose $\ell+1$ as a sufficiently high power of p in order to obtain $\deg f_2 = 1 + \beta\, p^s$.

Now we can change the degree of f_2 by multiples of p^s since there are functions of arbitrary degree, say $(-\beta)$,

$$\sigma\colon (D^{2n-1}, S^{2n-2}) \longrightarrow (S^{2n-1}, *)$$

and then if we define

$$\theta = p_1' \circ \sigma\colon (D^{2n-1}, S^{2n-2}) \longrightarrow (\tilde{L}', *)$$

we have $\deg \theta = -\beta\, p^j$. Furthermore there is a standard procedure to modify the degree of f_2 by using θ ([1] p.95, [2] p.98) and it gives a function f_3: $\tilde{L} \longrightarrow \tilde{L}'$ cellular, such that $f_3(h(x)) = (h')^a f_3(x)$ and $\deg f_3 = 1$.■

We want to show now that f_3 is in fact a homotopy equivalence and to that end it is enough to check that it induces isomorphisms in homology. Furthermore, we see that it suffices to show that for each $1 \leqslant r \leqslant n-1$

$$(\deg\, (f_3 \mid (\tilde{L})^{2r-1}), p) = 1$$

and this follows from the fact that

$$f_3 \mid (\tilde{L})^{2r-1} = f_2 \mid (\tilde{L})^{2r-1} \qquad 1 \leqslant r \leqslant n-1$$

and $$f_2 \mid (\tilde{L})^{2r-1} = (f_\ell \mid (\tilde{L}')^{2r-1}) \circ (f \mid (\tilde{L})^{2r-1})$$

since $$\deg\, (f \mid (\tilde{L}')^{2r-1}) = a^r\, \ell_1 \ldots \ell_r\, \overline{q}_1' \ldots \overline{q}_r'$$

and $$(\deg\, (f_\ell \mid (L')^{2r-1}), p) = 1.$$

We have then

(4.2) *The function* f_3 *in* (3.2) *is a homotopy equivalence.*■

We can prove now

(4.3) THEOREM. *Let* $L = L(p^s; q_1, \ldots, q_n)$ *and* $L' = L'(p^s; q_1', \ldots, q_n')$ *be pseudo-lens spaces and let* $\pi_1(L) \cong Z_{a_1} = \langle [t] \rangle = \langle h \rangle$ *and* $\pi_1(L') \cong$ $\cong Z_{a_1'} = \langle [t'] \rangle = \langle h' \rangle$ *be the corresponding fundamental groups* (t *and* t' *are generators of* G).

There exists a simple homotopy equivalence f: L ⟶ L' *such that* $f_*(h) = (h')^a$ $(a, a_1') = 1$ *if and only if*

i) $(p^s, q_i) = (p^s, q_i')$ $\quad i = 1, \ldots, n$

ii) *There is a permutation* σ *of* $(1, \ldots, n)$ *such that* $\varepsilon_k\, a\, \overline{q}'_{\sigma(k)} \equiv \overline{q}_k$ (mod a_1) *where* $\varepsilon_k = 1$ *for each* $k = 1, \ldots, n$.

Proof. The conditions are necessary by (3.1). In order to show that they are sufficient we notice that our hypothesis imply those on (3.2) and (3.3) and therefore we have a function f: $L \longrightarrow L'$ such that f is a homotopy equivalence and $f_*(h) = (h')^a$ $(a,a') = 1$.

It remains to be shown that f is simple. To that end we first notice that the complexes L and L' are *special* in the sense of [4] p.404 since they are finite complexes which fundamental groups are finite abelian and operate trivially on the rational homology groups of the universal covering spaces.

Now according to lemma 12.5 [4] p.405 we just have to prove

$$f_* \Delta(L) \sim \Delta(L')$$

(see [4] p.405 for notation).

Now a short calculation shows that $\Delta(L) \sim \prod_{i=1}^{n} (h^{\ell_i} - 1)$ and since $f_*(h) = (h')^a$ we have

$$f_*(\Delta(L)) \sim \prod_{i=1}^{n} ((h')^{a\ell_i} - 1)$$

and accordingly

$$\Delta(L') = \prod_{i=1}^{n} ((h')^{\ell'_i} - 1).$$

Now since we have $\pm a\, \bar{q}'_{\sigma(k)} \equiv \bar{q}_k \pmod{p^{(m-j)}}$

we can write $\pm a\, \bar{q}'_{\sigma(k)}\, \ell'_{\sigma(k)}\, \ell_k \equiv \bar{q}_k\, \ell'_{\sigma(k)}\, \ell_k \pmod{p^{(m-j)}}$

i.e. $\pm a\, \ell_k \equiv \ell'_{\sigma(k)} \pmod{p^{(m-j)}}$.

We must consider two cases:

i) $a\, \ell_k \equiv \ell'_{\sigma(k)} \pmod{p^{(m-j)}}$ then $((h')^{a\ell_k} - 1) = ((h')^{\ell'_{\sigma(k)}} - 1)$.

ii) $-a\, \ell_k \equiv \ell'_{\sigma(k)} \pmod{p^{(m-j)}}$ then $((h')^{a\ell_k} - 1).(-(h')^{-a\ell_k}) =$

$= ((h')^{\ell'_{\sigma(k)}} - 1)$.

Therefore we conclude

$$\prod_{i=1}^{n} ((h')^{a\ell_i} - 1) \sim \prod_{i=1}^{n} ((h')^{\ell'\sigma(i)} - 1)$$

and this finishes the proof of (4.3).■

(4.4.) FINAL REMARKS.

It follows from the proof of (4.3) that one can extend this result to the following more general situation:

If $m = p_1^{\alpha_1}\, p_2^{\alpha_2} \dots p_j^{\alpha_j}$ is the decomposition of m into prime factors then

i) All the p_i s are odd.

ii) $a_1 = \frac{m}{(q_1,m)} = p_1^{\beta_1}.p_2^{\beta_2} \ldots p_j^{\beta_j}$ for some $1 \leqslant \beta_i \leqslant \alpha_i$, $i = 1,\ldots,j$.

REFERENCES

[1] COHEN,M.M., *A Course in Simple-Homotopy Theory*, Springer, (1970).

[2] de RHAM, G.; MAUMARY,S. and KERVAIRE,M.A., *Torsion et Type Simple d-Homotopie*, Lecture Notes in Math, 48, Springer.

[3] KAWASAKI,T., *Cohomology of Twisted Projective Spaces and Lens Complexes*, Math. Ann., 206, 243-248, (1973).

[4] MILNOR,J., *Whitehead Torsion*, Bull of A.M.S. 72, 1966; 358-426.

Escuela de Matemática
Universidad Nacional de San Luis.

I.M.A.F.
Universidad Nacional de Córdoba.

Recibido en abril de 1979.
Versión final agosto de 1980.