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SURFACE INTEGRALS, SPHERICAL COORDINATES AND THE AREA ELEMENT OF S^{n-1}

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1. INTRODUCTION.

The object of this note is to discuss some geometrical aspects concerning the introduction of spherical coordinates in R^n: a simple way of computing the area element of the unit spherical surface S^{n-1}, the sets which have to be left outside the transformation to get a diffeomorphism between two open subsets of R^n and the manner in which this exclusion affects S^{n-1}.

It seems unnecessary to remark that the main formulas herein are well known and widely used, but the approach is new as far as we were able to check, and we think that it will be useful to advanced students in the field of Real Analysis, to whom it is addressed. In this connection see also [1].

Integrals are Lebesgue integrals and we assume familiarity with the formula for changing variables in a multiple integral and the subject (except for differential forms) of Spivak's book [3] whose notations we have adopted. In spite of this, some definitions and theorems are restated here in a slightly different form to fit the purpose we have in mind.

All subsets of R^n are endowed with the induced topology from that of R^n.

2. HYPERSURFACES.

Let U be an open subset of R^{n-1} and $\varphi\colon U \longrightarrow R^n$ a C^∞ mapping that we write naturally in the form $\varphi = (\varphi_1,\ldots,\varphi_n)$. Then φ is called an *immersion* provided that it is injective and the jacobian matrix

$$\varphi'(u) = (\partial\varphi_i/\partial u_k) \qquad (1 \leqslant i \leqslant n \quad , \quad 1 \leqslant k \leqslant n-1)$$

has rank n-1 at each point $u = (u_1,\ldots,u_{n-1})$ of U. We call a non void set $H \subset R^n$ a *hypersurface* if for each $p \in H$ there is an immersion $\varphi\colon U \longrightarrow R^n$ such that $S = \varphi(U)$ is an open subset of H with $p \in S$ and the mapping $\varphi\colon U \longrightarrow S$ is a homeomorphism. The set S of this definition is a "coordinate patch" and the ordered pair (U,φ) a *local chart* or local system of coordinates at the point p. We have the following theorem:

THEOREM 1. *A non void set* $H \subset R^n$ *is a hypersurface if and only if for each* $p \in H$ *there exist two open subsets* A *and* B *of* R^n *with* $p \in A$ *and a diffeomorphism* $f: A \longrightarrow B$ *of class* C^∞, *such that*

$$A \cap H = \{x \in A: f_n(x) = 0\},$$

where f_n *is the last component of the vector* f.

COROLLARY 1. *If* (U,φ) *and* (V,ψ) *are two local charts of a hypersurface, such that* $\varphi(U) = \psi(V)$, *then the mapping* $\psi^{-1} \circ \varphi: U \longrightarrow V$ *is a diffeomorphism of class* C^∞.

COROLLARY 2. *Let* $\Phi: E \longrightarrow R$ *be a* C^∞ *function defined on an open subset* E *of* R^n *with* grad $\Phi \neq 0$ *at each point of E. Then for every number* $t \in \Phi(E)$, *the set* $\Phi^{-1}(t)$ *is a hypersurface (by* grad Φ *we mean the vector function* $(D_1\Phi,\ldots,D_n\Phi)$, *where* $D_k = \partial/\partial x_k$).

For example, taking $E = R^n - \{0\}$ and $\Phi(x) = x_1^2 +\ldots+ x_n^2 - 1$, we see that $S^{n-1} = \Phi^{-1}(0)$ is a hypersurface.

3. SURFACE INTEGRALS.

We recall that if (U,φ) is a local chart of a hypersurface H, then the vectors $v_k(u) = \partial\varphi/\partial u_k = (\partial\varphi_1/\partial u_k,\ldots,\partial\varphi_n/\partial u_k)$, $k = 1,2,\ldots,n-1$, generate an (n-1)-dimensional vector subspace of R^n which depends only on the point $x = \varphi(u)$ and not on the local chart (U,φ) at the point x. We denote this subspace by H_x and call it the tangent space at x. If we denote by N(u) a unit vector of R^n in the orthogonal complement of H_x, then the area element of H is defined locally through the function

$$g(u) = \left| \det \begin{pmatrix} N(u) \\ v_1(u) \\ \vdots \\ v_{n-1}(u) \end{pmatrix} \right| .$$

In terms of the jacobian determinants

$$J_k(u) = \frac{\partial(\varphi_1,\ldots,\hat{\varphi}_k,\ldots,\varphi_n)}{\partial(u_1,u_2,\ldots,u_{n-1})} \qquad (k = 1,2,\ldots,n)$$

where the function with circumflex is omitted, the function g can be written in the form

$$g(u) = (\sum_{k=1}^{n} J_k^2(u))^{1/2}.$$

A function $f: H \longrightarrow R$ is termed *measurable* if for each local chart (U,φ), the composition $f \circ \varphi$ is measurable on U. If f is a non negative measurable function with support contained in the coordinate patch $S = \varphi(U)$, the integral of f over H is defined by

$$\int_H f\, d\sigma = \int_U f(\varphi(u))\, g(u)\, du.$$

Here the symbol $d\sigma = g(u)\, du$ represents the area element of H. More generally, if f is any non negative measurable function, we select a countable family of coordinate patches $(S_i, i \in I)$ which cover H and - subordinate to this cover - a family of continuous functions $h_i: H \longrightarrow R$, $i \in I$, such that $0 \leqslant h_i \leqslant 1$, supp $h_i \subset S_i$ and $\sum h_i = 1$ at each point of H. Then the integral of f over H is defined by

$$\int_H f\, d\sigma = \sum_{i \in I} \int_H f h_i\, d\sigma.$$

A set $E \subset H$ is said to be measurable if its characteristic function is measurable and, in this case, the integral of the characteristic function over H is, by definition, the *area* of the set E under consideration.

Let us consider two particular cases: suppose that for a local system (U,φ) the function φ is of the form $\varphi(u_1,\ldots,u_{n-1}) = (u_1,\ldots,u_{n-1},\psi(u_1,\ldots,u_{n-1}))$. Then if we write $\psi_k = D_k\psi = \partial\psi/\partial u_k$, the area element of S takes the form

$$d\sigma = (\psi_1^2 + \ldots + \psi_{n-1}^2 + 1)^{1/2}\, du.$$

If, in addition to the last assumption, the points of $S = \varphi(U)$ satisfy a relation of the form $\Phi(x) = \Phi(x_1,\ldots,x_n) = c$, where c is a constant and Φ a C^∞ function such that $\Phi_n = D_n\Phi \neq 0$ at each point of S, then by differentiating with respect to u_k in the relation $\Phi(u_1,\ldots,u_{n-1},\psi(u_1,\ldots,u_{n-1})) = c$, we get $\psi_k = -\Phi_k/\Phi_n$ and the formula for the area element becomes

$$d\sigma = \frac{|\mathrm{grad}\ \Phi|}{|\Phi_n|}\, du\ ,$$

where all derivatives are evaluated at the point $(u,\psi(u))$.

Area is a σ-additive function of a set which has the important property of being invariant under isometries of the space R^n.

4. RESOLUTION OF MULTIPLE INTEGRALS.

The following theorem will be needed:

THEOREM 2. *Let* E *be an open subset of* R^n *and* Φ: E $\longrightarrow$ R *a* C^∞ *function which satisfies* grad $\Phi \neq 0$ *at each point of* E. *If* f: E $\longrightarrow$ R *is a non negative measurable function, then we have the formula*

$$\int_E f(x)\, dx = \int_{\Phi(E)} dt \int_{\Phi^{-1}(t)} \frac{f}{|\text{grad } \Phi|}\, d\sigma_t,$$

where $d\sigma_t$ *represents the area element of the hypersurface* $\Phi^{-1}(t)$.

Proof. At each point of E one at least of the derivatives $\Phi_k = D_k\Phi$ is different from zero and we may assume without loss of generality that at a given point p we have $\Phi_n(p) \neq 0$. Then the mapping $(x_1,\dots,x_n) \longrightarrow$ $\longrightarrow (u_1,\dots,u_{n-1},t)$ given by the equations

$$(1) \qquad u_1 = x_1,\dots,\ u_{n-1} = x_{n-1}\ ,\quad t = \Phi(x_1,\dots,x_n),$$

whose jacobian determinant Φ_n is non zero at p, establishes a diffeomorphism between a certain open neighborhood $V \subset E$ of p and an open set W of R^n, by virtue of the inverse mapping theorem; the inverse mapping being given by a set of equations of the form

$$(2) \qquad x_1 = u_1,\dots,\ x_{n-1} = u_{n-1}\ ,\quad x_n = \psi(u_1,\dots,u_{n-1},t).$$

In order to fix ideas, we assume for a moment that V = E. Then the section $W_t = \{u \in R^{n-1}: (u,t) \in W\}$ has the property that for each $u \in W_t$ the point $x = (u,\psi(u,t))$ satisfies the relation $\Phi(x) = \Phi(u,\psi(u,t)) = t$. This shows that equations (2) provide a global system of coordinates for the hypersurface $\Phi^{-1}(t)$, as u varies inside W_t. Therefore, from the formula for changing variables and the last equation of the preceding section, we get

$$\int_E f(x)\, dx = \int_W f(u,\psi(u,t))\, \frac{1}{|\Phi_n|}\, du\, dt =$$

$$= \int dt \int_{W_t} \frac{f(u,\psi(u,t))}{|\text{grad } \Phi|}\, \frac{|\text{grad } \Phi|}{|\Phi_n|}\, du =$$

$$= \int_{\Phi(E)} dt \int_{\Phi^{-1}(t)} \frac{f}{|\text{grad } \Phi|}\, d\sigma_t .$$

If $V \neq E$, the formula of the theorem holds, provided that the support of f is contained in V. In the general case, we select a countable open cover $(V_i, i \in I)$ of E with the property that the formula holds for any $f \geqslant 0$ whose support is contained in some V_i, and the rest of the job is done by taking a continuous partition of the unity on E subordinate to this cover.

5. SPHERICAL COORDINATES.

Let us consider in R^n a mapping S given by the set of equations

$$
(S)\qquad
\begin{aligned}
x_1 &= r\cos\theta_1\\
x_2 &= r\sin\theta_1\cos\theta_2\\
x_3 &= r\sin\theta_1\sin\theta_2\cos\theta_3\\
&\dots\quad\dots\quad\dots\quad\dots\\
x_{n-1} &= r\sin\theta_1\dots\sin\theta_{n-2}\cos\theta_{n-1}\\
x_n &= r\sin\theta_1\dots\sin\theta_{n-2}\sin\theta_{n-1}\,,
\end{aligned}
$$

the domain of S being the infinite parallelepiped P defined by

$$r>0\,,\quad 0<\theta_1<\pi\,,\ \dots\,,\ 0<\theta_{n-2}<\pi\,,\quad 0<\theta_{n-1}<2\pi.$$

By induction on n we prove that (i) S is injective; (ii) $r^2 = x_1^2 + \dots + x_n^2$; (iii) the jacobian of S has the value $J = r^{n-1}\sin^{n-2}\theta_1\sin^{n-3}\theta_2 \cdots \sin\theta_{n-2}$; (iv) if we allow the variables $r,\theta_1,\dots,\theta_{n-1}$ to take up the values 0 and π, then the image of S is all of R^n.

All these assertions are readily verified when n=2. In the general case, we decompose S as the product of two transformations S_1 and S_2 thus:

$$
(S_1)\qquad
\begin{aligned}
x_1 &= x_1\\
x_2 &= r_1\cos\theta_2\\
x_3 &= r_1\sin\theta_2\cos\theta_3\\
&\dots\quad\dots\quad\dots\\
x_n &= r_1\sin\theta_2\dots\sin\theta_{n-1}
\end{aligned}
\qquad\qquad
(S_2)\qquad
\begin{aligned}
x_1 &= r\cos\theta_1\\
r_1 &= r\sin\theta_1\\
\theta_2 &= \theta_2\\
&\dots\quad\dots\\
\theta_{n-1} &= \theta_{n-1}
\end{aligned}
$$

Obviously, S_1 is essentially a transformation of the same type as S in an (n-1)-dimensional space, while S_2 is the familiar polar coordinates transformation in the plane. So that if we assume that our assertions hold in R^{n-1}, we get their validity in R^n from the preceding decomposition.

From assertion (iv) and the equations defining S, it follows that $R^n - S(P)$ is contained in the hyperplane $x_n = 0$, a set of measure zero. Since $J \neq 0$ in the open set P, it follows from the inverse mapping theorem that S(P) is an open subset of R^n and that S is a diffeomorphism of class C^∞ between P and S(P). It is illuminating to draw diagrams to figure out what these sets are in the plane and in three dimensional space.

The mapping S may be written more concisely in the form

$$x = S(r,\theta_1,\dots,\theta_{n-1}). \tag{3}$$

Writing r=1 in the preceding equation , we obtain the new mapping

$$x' = S(1,\theta_1,\dots,\theta_{n-1}) \tag{4}$$

which defines a local chart of the unit sphere S^{n-1} as the point $\theta =$ $= (\theta_1,\ldots,\theta_{n-1})$ varies inside the (n-1)-dimensional interval

(5) $$L = \{\theta: 0 < \theta_1 < \pi , \ldots , 0 < \theta_{n-2} < \pi , 0 < \theta_{n-1} < 2\pi\}.$$

Among other things, we shall see in the next section that the coordinate patch corresponding to (4) covers all of S^{n-1} except for a set of area zero.

Note that S may still be written in the form $x = S(r,\theta) = r\, S(1,\theta) =$ $= rx'$, and that the jacobian of S has the concise expression

(6) $$J = r^{n-1}\, g(\theta),$$

where $g(\theta) = g(\theta_1,\ldots,\theta_{n-1})$ represents the trigonometrical monomial $\sin^{n-2}\theta_1 \ldots \sin\theta_{n-2}$ which is positive inside L.

6. THE AREA ELEMENT OF S^{n-1}.

Keeping the notations of the preceding section, let M be an arbitrary subinterval of L defined by

$$M = \{\theta: \alpha_1 < \theta_1 < \beta_1 , \ldots , \alpha_{n-1} < \theta_{n-1} < \beta_{n-1}\}.$$

The Lebesgue measure of the domain

$$\omega_R = \{x: x = S(r,\theta),\ r < R,\ \theta \in M\}$$

is

(7) $$|\omega_R| = \int_{\omega_R} dx = \int_0^R dr\, r^{n-1} \int_M g(\theta)\, d\theta.$$

If we wish to express the area element of the hypersurface σ_R given by the single local chart

$$x = Rx' = RS(1,\theta) \qquad (\theta \in M),$$

we form the jacobians

$$J_i = \frac{\partial(x_1,\ldots,\hat{x}_i,\ldots,x_n)}{\partial(\theta_1,\theta_2,\ldots,\theta_{n-1})} = R^{n-1}\, T_i \quad (i = 1,2,\ldots,n),$$

the variable with circumflex being omitted and $T_i = T_i(\theta)$ being a certain trigonometric polymonial. Hence, the area element of σ_R is

$$d\sigma_R = \Big(\sum_{i=1}^{n} J_i^2\Big)^{1/2} d\theta = R^{n-1}\, h(\theta)\, d\theta,$$

where h is a continuous function. So that the area of σ_R is given by

$$|\sigma_R| = R^{n-1} \int_M h(\theta)\, d\theta = R^{n-1} |\sigma_1|.$$

On the other hand, if we apply theorem 2 with $E = \omega_R$, $\Phi(x) = |x|$ (the euclidean norm of x) and $f = 1$, taking into account that in this case $|\text{grad}\ \Phi| = 1$, then we get

$$|\omega_R| = \int_{\omega_R} 1\, dx = \int_0^R dt \int_{\sigma_t} d\sigma_t = \int_0^R |\sigma_t|\, dt.$$

Therefore

$$\frac{d|\omega_R|}{dR} = |\sigma_R| = R^{n-1} \int_M h(\theta)\, d\theta.$$

But from (7) we also have

$$\frac{d|\omega_R|}{dR} = R^{n-1} \int_M g(\theta)\, d\theta.$$

This shows that the continuous functions g and h have the same integral over any subinterval M of L and we get $g = h$; that is, the area element of S^{n-1} as computed through the local chart (4) is $d\sigma_1 = g(\theta)\, d\theta$.

Moreover, from the previous relations we obtain $|\omega_R| = |\sigma_1|\ (R^n/n)$, which shows that any set of area zero on the sphere S^{n-1} generates a cone with vertex at the origin of measure zero and conversely. Area and measure being invariant under rotations, our last assertion holds for any set $E \subset S^{n-1}$.

Since $R^n - S(P)$ is contained in a hyperplane, we conclude that the coordinate patch corresponding to (4) covers all of S^{n-1} except for a set of area equal to zero.

Writing dx' for $d\sigma_1$, from the formula for changing variables we obtain a useful equation which holds for any non negative measurable function f on the space R^n, namely

$$\int_{R^n} f(x)\, dx = \int_0^\infty dr\, r^{n-1} \int_{S^{n-1}} f(rx')\, dx'.$$

It is well known that the area of S^{n-1} can be obtained by taking in this equation $f(x) = \exp[-(x_1^2 + \ldots + x_n^2)]$ and by applying Fubini's theorem to the left hand member.

7. FINAL COMMENTS.

The computations of the last section were developed in [2]. The formula for changing variables in a multiple Lebesgue integral is masterly explained in [4].

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