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INVARIANT POLYGONAL DOMAINS FOR MULTIVALUED FUNCTIONAL DIFFERENTIAL EQUATIONS

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0. INTRODUCTION.

An important question concerning a wide class of functional differential equations is that regarding flow-invariance properties with respect to certain families of subsets of the ambient state space. As a classical result in this direction we must quote the 1942 Nagumo's theorem [15] (see also J.M.Bony [1], H.Brezis [2], M.G.Crandall [6]) dealing with flow-invariance problems for (univalued) ordinary differential equations with respect to closed subsets of a finite dimensional Banach space. An infinite dimensional extension of the above quoted results was performed by R.H.Martin Jr. [13] (see also N.Pavel [17]) and, respectively, by N.Pavel and I.Vrabie [19] in case of univalued and, respectively, multivalued ordinary differential equations acting on those spaces. Under these lines, it's the main objective of the present note to state and prove a flow-invariance theorem for a class of multivalued functional differential equations of the form

$$\text{(MFDE)} \qquad x'(t) \in k(x)(t)$$

with respect to a certain family of polygonal domains satisfying a "normality" hypothesis and, in this context, it must be emphasized that the basic tool in proving our main result is represented by a "multivalued" fixed point theorem comparable with a similar one due to H.Covitz and S.B.Nadler Jr. [5]. A number of extensions of these results, especially to time-dependent polygonal domains will be given elsewhere.

1. A "MULTIVALUED" FIXED POINT THEOREM.

Let (X,d) be a generalized metric space in Luxemburg-Jung's sense [12], [10], and let $C(X)$ denote the class of all (nonempty) closed subsets of X. For every $x \in X$, $Y \subset X$, $r > 0$ let $X(x)$ denote the x-component of X (the subset of all $y \in X$ with $d(x,y) < +\infty$), $d(Y,x)$ the usual distance between Y and x (the infimum of all distances $d(x,y)$ with $y \in Y$) and $S(Y,r)$ the open sphere with "center" Y and radius r

(the subset of all $x \in X$ with $d(Y,x) < r$). Let D be an extended real valued function from $(C(X))^2$ into $\overline{R}_+ = [0,+\infty]$ defined, for every $Y,Z \in C(X)$ by the convention

$$D(Y,Z) = \inf\{r > 0;\ Y \subset S(Z,r),\ Z \subset S(Y,r)\}\ ,\ \text{if } \{r > 0;\ Y \subset S(Z,r),\ Z \subset S(Y,r)\} \neq \emptyset$$

$$= +\infty \qquad\qquad ,\ \text{if } \{r > 0;\ Y \subset S(Z,r),\ Z \subset S(Y,r)\} = \emptyset.$$

A classical result (see, e.g., C.Kuratowski [11,p.106]) assures us that $(C(X),D)$ is a generalized metric space, the extended real valued function D being called the generalized Hausdorff metric of $C(X)$. The following facts about this metric are almost evident (see S.B.Nadler Jr. [14], for more details)

(A) *if* $Y,Z \in C(X)$ *and* $r > 0$ *are such that, for every* $u \in Y$ (*resp.,* Z) *there exists* $v \in Z$ (*resp.,* Y) *with* $d(u,v) \leqslant r$ *then* $D(Y,Z) \leqslant r$.

(B) *if the sequences* $(y_n;\ n \in N) \subset X$ *and* $(Y_n;\ n \in N) \subset C(X)$ *satisfy* $y_n \in Y_n$, $n \in N$ *and if* $y_n \xrightarrow{d} y$, $Y_n \xrightarrow{D} Y$ *as* $n \to \infty$, *for some* $y \in X$, $Y \in C(X)$, *then necessarily,* $y \in Y$.

Throughout this note, by a *multivalued mapping* from X into X we mean a mapping from X into $C(X)$. In this context, T being a multivalued mapping from X into X, a sequence $(x_n;\ n \in N) \subset X$ is said to be an *iterative sequence* generated by $x \in X$ and T iff it satisfies

$$x_0 = x \quad ,\ x_1 \in Tx_0\ ,\dots,\ x_{n+1} \in Tx_n\ ,\dots$$

and a point $z \in X$ is called a *fixed point* of T provided that $z \in Tz$.

Let f be a given function from R_+ into itself. A multivalued mapping T from X into X is said to be a *multivalued contraction* with respect to f provided that it satisfies: if $x,y \in X$ and $r > 0$ are such that $d(x,y) \leqslant r$ then, for every $u \in Tx$ (resp.,Ty) there exists $v \in Ty$ (resp.,Tx) with $d(u,v) \leqslant f(r)$ (note that, by (A), we necessarily have in this case $D(Tx,Ty) \leqslant f(r)$). Finally, we shall indicate by Q the class of all functions $f\colon R_+ \longrightarrow R_+$ with $\sum_{n=0}^{\infty} f^{(n)}(t) < +\infty$, for all $t \in R_+$ (here $f^{(n)}$ denotes the n-th iterate of the function f, for all $n \in N$).

Suppose in what follows (X,d) is a complete g.m.s., and T is a multivalued mapping from X into itself. Concerning fixed points of this mapping, the main result of this paragraph is

THEOREM 1. *Suppose there exists a function* $f \in Q$ *such that*

(i) T *is a multivalued contraction with respect to* f

(ii) $X(T) = \{x \in X;\ X(x) \cap Tx \neq \emptyset\}$ *is not empty.*

Then, for every $x \in X(T)$ *there is an iterative sequence* $(x_n;\ n \in N) \subset X(x)$ *generated by* x *and* T , *a real number* $\rho > 0$ *and an element* $z \in X(x)$, *such that the following conclusions hold*

(C_1) $z \in Tz$ (*z is a fixed point of* T).

(C_2) $x_n \longrightarrow z$ *as* $n \longrightarrow \infty$ *in the sense* $d(x_n,z) \leqslant \sum_{m=n}^{\infty} f^{(m)}(\tau)$,
all $\tau \geqslant \rho$, $m \in N$.

Proof. Let $x \in X(T)$ be a given element and put $x_0 = x$. By (ii), there must be an element $x_1 \in Tx_0$ with $\rho = d(x_0,x_1) < +\infty$. In this case, taking into account (i), an element $x_2 \in Tx_1$ may be chosen with $d(x_1,x_2) \leqslant f(\rho)$, and so on. By induction, we get an iterative sequence $(x_n;\ n \in N) \subset X(x)$ generated by x and T, satisfying

$$d(x_n,x_{n+1}) \leqslant f^{(n)}(\rho), \text{ for all } n \in N$$

It follows at once that $(x_n;\ n \in N)$ is a Cauchy sequence so that, by completeness, $x_n \longrightarrow z$ as $n \longrightarrow \infty$, for some $z \in X(x) \subset X$. Moreover, taking into account that (A) gives

$$D(Tx_n,Tx_{n+1}) \leqslant f^{(n+1)}(\rho), \text{ for all } n \in N$$

we derive $Tx_n \xrightarrow{D} Tz$ as $n \longrightarrow \infty$, in which case, by (B), $z \in Tz$ and this completes, practically, the proof. Q.E.D.

As an important particular case, let the function $f\colon R_+ \longrightarrow R_+$ be defined by $f(t) = \lambda t$, $t \in R_+$, for some $\lambda \in (0,1)$; then, the above result may be compared with a similar one due to H.Covitz and S.B.Nadler Jr. [5] (see also S.B.Nadler Jr. [14]).

2. THE MAIN RESULTS.

In what follows, $(R^n,\|.\|)$ stands for the classical n-dimensional vector space endowed with an usual norm. Let X_n (resp.,X_0) denote the class of all continuous $x\colon R_+ \longrightarrow R^n$ (resp., $x\colon R_+ \longrightarrow R_+$). Define a mapping $x \longmapsto \|x\|$ from X_n into X_0 by $\|x\|(t) = \|x(t)\|$, $t \in R_+$, $x \in X_n$ and for every $g \in X_0$ let $\|.\|_g$ denote the *generalized norm* on X_n defined, for every $x \in X_n$ by

$$\begin{aligned} \|x\|_g &= \inf\{\lambda \geqslant 0;\ \|x\| \leqslant \lambda g\} \quad , \quad \text{if } \{\lambda \geqslant 0;\ \|x\| \leqslant \lambda g\} \neq \emptyset \\ &= +\infty \quad , \quad \text{if } \{\lambda \geqslant 0;\ \|x\| \leqslant \lambda g\} = \emptyset . \end{aligned}$$

It is a simple matter to verify that $(X_n,\|.\|_g)$ is a generalized Banach space (resp., a complete g.m.s., by the standard construction of its metric). For every $g \in X_0$ denote also $(X_n)_g = \{x \in X_n;\ \|x\|_g < +\infty\}$ and $C_g(X_n) = \{Y \subset X_n;\ Y \neq \emptyset,\ Y \text{ is } \|.\|_g\text{-closed}\}$.

Suppose $x \longmapsto k(x)$ is a given mapping from X_n into $P(X_n) = \{Y \subset X_n;\ Y \neq \emptyset\}$ and let $x^0 \in R^n$ be a given vector. Then, we may consider the *multivalued Cauchy problem* of a functional differential type

$$\text{(MCP)} \qquad x'(t) \in k(x)(t) \quad , \quad t \in R_+ \quad , \quad x(0) = x^0.$$

An important notion related to this class of Cauchy problems may be formulated as follows. Let H be a nonempty subset of R^n. H is said to possess a *flow-invariance property* with respect to (MCP) iff, for every $x^0 \in H$, the corresponding solution $x \in X_n$ of (MCP) will remain in H for all $t \in R_+$ or, in other words, (denoting by $X_n(H)$ the class of all $x \in X_n$ with $x(t) \in H$, $t \in R_+$) the corresponding solution $x \in X_n$ of (MCP) will belong to $X_n(H)$. In order to state and prove an useful flow-invariance result for the Cauchy problem (MCP) we have to introduce an appropriate terminology. Let $(R^n)^*$ denote the class of all linear functionals on R^n. For every $h \in (R^n)^*$, $\lambda \in R$, let $(h;\lambda)$ denote the *half-space* $h(x) \geq \lambda$. A family $((h_i;\lambda_i); i \in I)$ of half-spaces is said to be *admissible* iff its intersection $H = \cap((h_i;\lambda_i); i \in I)$ is nonempty, H being termed the *polygonal domain* generated by this family of half-spaces. Now, as a first flow-invariance result of the present note we have

THEOREM 2. *Let the mapping* $x \longmapsto k(x)$ *and the polygonal domain* H *be as above and suppose that, in addition,*

(IH) $\quad i \in I,\ t \in R_+,\ x \in X_n,\ \ h_i(x(t)) < \lambda_i$ *imply*

$\quad h_i(y(t)) \geq 0$ *for all* $y \in k(x)$

Then, necessarily, H *posesses a flow-invariance property with respect to the Cauchy problem* (MCP).

Proof. Suppose there is a vector $x^0 \in H$ such that, the corresponding solution $x \in X_n$ of (MCP) is not remaining in H. Then, there must exist $i \in I$ and $t_1 > 0$ such that $h_i(x(t_1)) < \lambda_i$. Put $t_2 = \sup(t \in [0,t_1]\ ;\ h_i(x(t)) \geq \lambda_i)$; clearly, $t_2 \in [0,t_1)$, $h_i(x(t_2)) = \lambda_i$, $h_i(x(t)) < \lambda_i$ for all $t \in (t_2,t_1]$ so that, from the classical mean value theorem, there must be $t_3 \in (t_2,t_1)$ with $h_i(x'(t_3)) < 0$. On the other hand, by (IH), $h_i(y(t_3)) \geq 0$, for all $y \in k(x)$ (since $t_3 \in (t_2,t_1)$ implies $h_i(x(t_3)) < \lambda_i$) and in particular, $h_i(x'(t_3)) \geq 0$, contradicting the above relation and proving our theorem. Q.E.D.

A close analysis of the above invariance condition (IH) shows that it is, in fact, difficult to be manevrated because the whole class X_n is implicated there. It would be of interest to replace it by an invariance condition involving only the class $X_n(H)$, eventually under some restrictive assumptions about the considered polygonal domain H. On the other hand, for a number of practical reasons, it is important to connect flow-invariance conditions with existence uniqueness and approximation conditions about the considered (MCP). To this end, we have to introduce the notion of *normal polygonal domain*. Let H be defined as above. A mapping $\psi: R^n \longrightarrow R^n$ is said to be a *normal* H-*mapping* iff $\psi(R^n) \subset H$, $\psi/H = i_H$ (the identity mapping), $\|\psi(x)-\psi(y)\| \leq \|x-y\|$, all $x,y \in R^n$, and, in addition, $i \in I$, $x \in R^n$, $h_i(x) < \lambda_i$ imply $h_i(\psi(x)) =$

$= \lambda_i$. Every normal (polygonal domain generating) H-mapping will be denoted as $x \longmapsto \overline{x}$ and, the same convention will be used to denote the associated mapping from X_n into $X_n(H)$ defined by $\overline{x}(t) = \overline{x(t)}$, all $t \in R_+$, $x \in X_n$. Finally, a polygonal domain H is said to be a *normal polygonal domain* iff it possesses at least an associated normal H - mapping.

It is supposed henceforward H is a normal polygonal domain, $x \longmapsto k_H(x)$ a mapping from $X_n(H)$ into $P(X_n)$ and $x^0 \in H$ a given vector. Let us denote by $(MCP)_H$ the corresponding multivalued Cauchy problem (MCP) with k replaced by k_H. In this case, the main existence, uniqueness aproxi mation and flow-invariance result of this note is

THEOREM 3. *Suppose there exist a couple of functions* $g \in X_0$, $f \in Q$ *and a mapping* $x \longmapsto e(x)$ *from* X_0 *into* X_0 *such that*

(iii) *for every* $x \in X_n(H)$ *the set* $K_H(x)$ *of all* $y^* \in X_n$ *with* $y^*(t) = x^0 + \int_0^t y(s)ds$, $t \in R_+$, *for some* $y \in k_H(x)$ *is in* $C_g(X_n)$ *(resp., a* $\|\cdot\|_g$ *- closed subset of* X_n*).*

(iv) *if* $x, y \in X_n(H)$ *and* $a \in X_0$ *satisfy* $\|x-y\| \leqslant a$ *then, for every* $u \in k_H(x)$ *(resp.,* $k_H(y)$*) there exists* $v \in k_H(y)$ *(resp.,* $k_H(x)$*) satisfying* $\|u-v\| \leqslant e(a)$.

(v) $\int_0^t e(g.\tau)(s)ds \leqslant f(\tau)g(t)$, $\tau > 0$, $t \in R_+$.

(vi) *the set* $X_n^0 \subset X_n$ *of all* $y \in X_n$ *satisfying* $\|y(t) - x^0 - \int_0^t u(s)ds\| \leqslant \mu g(t)$, $t \in R_+$, *for some* $\mu > 0$, $u \in k_H(\overline{y})$, *is not empty.*

(vii) $i \in I$, $t \in R_+$, $x \in X_n(H)$, $h\,(x(t)) = \lambda_i$ *imply* $h_i(y(t)) \geqslant 0$ *for all* $y \in k_H(x)$.

Then, for every $y^0 \in X_n^0$ *there is a sequence* $(y_m;\ m \in N) \subset y^0 + (X_n)_g$, *and an element* $z \in (y^0 + (X_n)_g) \cap X_n(H)$ *such that*

(C_3) z *is a solution of* $(MCP)_H$ *in* $X_n(H)$

(C_4) $y_0 = y^0$, $y'_{m+1} \in k_H(\overline{y}_m)$, $m \in N$

(C_5) $(y_m;\ m \in N)$ *converges in the sense of* $\|\cdot\|_g$ *to* z *with an evaluation of the convergence given by*

$$\|y_m - z\| \leqslant \left(\sum_{p=m}^{\infty} f^{(p)}(\tau)\right)g \quad , \quad \tau \geqslant \rho(y^0) \quad , \quad m \in N$$

$\rho(y^0) > 0$ *being dependent only on* $y^0 \in X_n^0$.

Proof. Let $x \longmapsto k(x)$ be a mapping from X_n into $P(X_n)$ given by $k(x) = k_H(\overline{x})$, all $x \in X_n$ and let T be a multivaled mapping from $(X_n, \|\cdot\|_g)$ into itself given by the convention $Tx = K_H(\overline{x})$, all $x \in X_n$ (note that, by (iii), T is well defined). We claim that T is a multivalued contraction with respect to f. Indeed, let $x, y \in X_n$ and $\tau > 0$ be such that $\|x-y\|_g \leqslant \tau$. From the definition of $\|\cdot\|_g$ combined with the definition of a normal mapping, $\|\bar{x}-\bar{y}\| \leqslant \|x-y\| \leqslant g\tau$. Let $u^* \in Tx$ (resp., Ty) be given; clearly, $u^*(t) = x^0 + \int_0^t u(s)ds$, $t \in R_+$, for some $u \in k_H(\bar{x})$ (resp., $k_H(\bar{y})$). Now, let $v \in k_H(\bar{y})$ (resp., $k_H(\bar{x})$) be the associated element given by (iv), then $\|u-v\| \leqslant e(g\tau)$ so that, denoting $v^*(t) = x^0 + \int_0^t v(s)ds$, $t \in R_+$ - clearly, $v^* \in Ty$ (resp., Tx) - we have

$$\|u^*(t)-v^*(t)\| \leqslant \int_0^t \|u(s)-v(s)\|ds \leqslant \int_0^t e(g\tau)(s)ds \leqslant f(\tau)g(t) \quad , \quad t \in R_+$$

that is, $\|u^*-v^*\|_g \leqslant f(\tau)$, showing that T is a multivalued contraction with respect to f. On the other hand, clearly, (vi) says that, for every $y^0 \in X_n^0$ we have (denoting $u^*(t) = x^0 + \int_0^t u(s)ds$, $t \in R_+$ and observing that, evidently, $u^* \in T(y^0)$) $\|y^0-u^*\|_g \leqslant \mu < +\infty$, i.e., $y^0 \in X_n(T)$, showing that $X_n(T)$ is not empty. Finally, we claim that the invariance condition (IH) holds. Indeed, let $i \in I$, $t \in R_+$ and $x \in X_n$ with $h_i(x(t)) < \lambda_i$ then, $\bar{x} \in X_n(H)$ and $h_i(\bar{x}(t)) = \lambda_i$ so that, by (vii), $h_i(y(t)) \geqslant 0$, for all $y \in k_H(\bar{x}) = k(x)$, proving our assertion. In this case both Theorem 1 and Theorem 2 are applicable and this completes the proof. Q.E.D.

A very important particular case of our theorem is that corresponding to the choice $H = R_+^n$. Here, obviously, $H = \cap\,((h_i;\lambda_i);\ 1 \leqslant i \leqslant n)$ with $h_i(x) = x_i$, $\lambda_i = 0$, for all $x = (x_1,x_2,\dots,x_n) \in R^n$, $1 \leqslant i \leqslant n$, and a standard normal H - mapping is $x \longmapsto \bar{x} = \max(x,0)$, the function "max" being usually defined by the lattice structure of R^n. Then, the invariance condition (vii) becomes

(vii)' $i \in \{1,\dots,n\}$, $t \in R_+$, $x \in X_n(R_+^n)$, $x_i(t) = 0$ *imply*

$y_i(t) \geqslant 0$ *for all* $y \in k_H(x)$

in which case, the corresponding variant of Theorem 3 above may be considered as being a "multivalued" extension of a result due to the author [21] (see also, N.Pavel and M.Turinici [18]).

As a final remark, let $t \longmapsto \hat{t}$ be a mapping from R_+ into $P(R_+)$. In this case, for every $x \in X_n$, $k(x)(t)$ may be considered as being completely defined only by the values of the vector function $x \in X_n$ taken on the subset $\hat{t} \subset R_+$. Note that, in the case $\hat{t} \subset [0,t]$, $t \in R_+$, the Cauchy problem (MCP) is a *nonanticipative* one (for a number of general results in this direction see C.Corduneanu [4] and, respectively, A.Cellina [3], H.Hermes [9], T.Wazewski [22], A.F.Filippov [8], D.T.Dočev and D.D.Bajnov [7] for the univalued and, respectively, multivalued case) while, in the case $\hat{t} \cap (t,+\infty) \neq \emptyset$, $t \in R_+$, the Cauchy problem (MCP) appears as an *anticipative* one, being considered as a multivalued extension of a similar problem introduced by the author [20] (see also in this direction Oberg's paper [16]).

REFERENCES

[1] J.M.BONY, *Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés*, Ann. Inst. Fourier, 19, 277-304, (1969).

[2] H.BREZIS, *On a characterization of flow-invariant sets*, Comm. Pure Appl. Math., 23, 261-263, (1970).

[3] A.CELLINA, *Multivalued differential equations and ordinary differential equations*, SIAM J. Appl. Math., 18, 533-538, (1970).

[4] C.CORDUNEANU, *Sur certain équations fonctionnelles de Volterra*, Funkc. Ekv., 9, 119-127, (1966).

[5] H.COVITZ and S.B.NADLER Jr., *Multi-valued contraction mappings in generalized metric spaces*, Israel J. Math., 8, 5-11, (1970).

[6] M.G.CRANDALL, *A generalization of Peano's existence theorem and flow-invariance*, Proc. Amer. Math. Soc., 36, 151-155, (1972).

[7] D.T.DOČEV and D.D.BAJNOV, *On the existence of solutions of a class of multivalued differential equations with deviating argument* (in Russian) Rev. Roum. Math. Pures Appl., 23, 1497-1503, (1978).

[8] A.F.FILIPPOV, *The existence of solutions of multivalued differential equations*, Mat. Zametki, 10, 307-313, (1971).

[9] H.HERMES, *The generalized differential equation* $x' \in R(t,x)$, Adv. in Math., 4, 149-169, (1970).

[10] C.F.K.JUNG, *On generalized complete spaces*, Bull. Amer. Math.Soc., 75, 113-116, (1969).

[11] C.KURATOWSKI, *Topologie*, vol.I, P.W.N., Warszawa, 1958.

[12] J.S.W.LUXEMBURG, *On the convergence of successive aproximations in the theory of ordinary differential equations*, Indag. Math., 20, 540-546, (1958).

[13] R.H.MARTIN Jr., *Differential equations on closed subsets of a Banach space*, Trans. Amer. Math. Soc., 179, 399-414, (1973).

[14] S.B.NADLER Jr., *Multi-valued contraction mappings*, Pacific J.Math., 30, 475-488, (1969).

[15] M.NAGUMO, *Über die Lage Integralkurven gewöhnlicher Differentialgleichungen*, Proc. Phys.-Math. Soc. Japan, 24, 551-559, (1942).

[16] R.J.OBERG, *On the local existence of solutions of certain functional differential equations*, Proc. Amer. Math. Soc., 20, 295-302, (1969).

[17] N.PAVEL, *Invariant sets for a class of semi-linear equations of evolution*, Nonlinear Analysis, TMA, 1, 187-196, (1977).

[18] N.PAVEL and M.TURINICI, *Positive solutions of a system of ordinary differential equations*, An. St. Univ. "Al.I.Cuza" Iaşi, 24, 63-70, (1978).

[19] N.PAVEL and I.VRABIE, *Semi-linear evolution equations with multivalued right hand side in Banach spaces*, An. St. Univ. "Al.I.Cuza" Iaşi, 25, 137-157, (1979).

[20] M.TURINICI, *Volterra functional equations with transformed argument*, Bull. Math. Soc. Sci. Math. Roum., 22(70), 99-107, (1978).

[21] M.TURINICI, *Invariant polygonal domains for a class of functional

differential equations, Bull. Math. Soc. Sci. Math. Roum.,24(72), (1980) to appear.

[22] T.WAZEWSKI, *Sur une généralisation de la notion des solution d'une équation au contingent*, Bull. Acad. Pol. Sci., 10, 11-15, (1962).

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