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THE KERNEL OF A REGULARIZING OPERATOR

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ABSTRACT. Let R be a linear continuous operator on L^2, such that $D^\alpha R$, RD^α are also continuous operators on L^2, for $|\alpha| = N$, given N. Following [1], we call R a regularizing operator of order N. Under the hypothesis $N>j+n/2, j = 0,1,2,...,n$ the dimension of R^n, we prove here that R is actually an integral operator with a kernel h(x,y) satisfying

$$\begin{array}{ll} D_x^\alpha h \in L^\infty(R_x^n ; L_y^2) & \\ & |\alpha| \leqslant j \\ D_y^\alpha h \in L^\infty(R_x^n ; L_y^2) & \end{array}$$

In [1] we have considered regularizing operators R of order N. This means that R is a linear operator such that R, $D^\alpha R$, RD^α are continuous on L^2, for $|\alpha| = N$, where D^α denotes any partial derivative of order $|\alpha|$.

Roughly speaking, a regularizing operator is the error term in the representation of a pseudo-differential operator as

$$\sum_{j=0}^{N-1} \int e^{-2\pi i x\xi}\, p_j(x,\xi)\, \hat{f}(\xi)\, d\xi + Rf.$$

The sum in the first term, gives the main part of the operator and generally, the form of the term R is not completely determined. (See [1]).

However, at least under certain conditions, it is possible to write the operator R as an integral operator,

$$Rf = \int h(x,y)\, f(y)\, dy \qquad f \in L^2.$$

The conditions to be imposed to the kernel h(x,y) will be especified later.

Let R be a linear continuous operator in L^2, such that $D^\alpha R$ is also continuous in L^2 for $|\alpha| = N$; we suppose that $N > n/2$; so, given $f \in L^2$, it is known that Rf coincides almost everywhere with a bounded continuous function. Moreover, if we fix an $x \in R^n$, the map

$$\begin{array}{ccc} L^2 & \longrightarrow & C \\ f & \longrightarrow & Rf(x) \end{array}$$

is linear and continuous.

So, there exists a function $h_x \in L^2$ such that

$$Rf(x) = \int h_x(y)\, f(y)\, dy.$$

The next lemma allows us to represent the operator R with a measurable kernel. This lemma was proved in [2], but we include it here, adapted to our situation.

LEMMA. *Let* Φ *be a strongly measurable map*

$$R^n \longrightarrow L^2$$
$$x \longrightarrow \Phi(x)$$

Then, there exists a complex Lebesgue measurable function $\varphi(x,y)$ *and a set* $X \subset R^n$ *of zero measure, such that for each* $x \in R^n \setminus X$, $\Phi(x) = \varphi(x,y)$ *ae.*

Proof. We call a map strongly measurable if there exists a sequence $\{\Phi_k\}$ such that

a) $$\Phi_k(x,y) = \sum_{j=1}^{H(k)} h_j^{(k)}(y)\, \chi_j^{(k)}(x)$$

where $h_j^{(k)} \in L^2$; $\chi_j^{(k)}$ are the characteristic functions of disjoint measurable subsets of R^n.

b) There exists

$$\lim_{k\to\infty} \int_{R^n} |\Phi_k(x,y) - \Phi(x)|^2\, dy = 0 \qquad \text{ae.}$$

Now, let

$$C_k = \{x \in R^n \,/\, \|\Phi_k(x,\ )\|_{L^2} \leqslant 2\|\Phi(x)\|_{L^2}\}$$

We set

$$\Psi_k(x,y) = \Phi_k(x,y).\chi_{C_k}(x)$$

where χ_{C_k} denotes the characteristic function of C_k.

The sequence $\{\Psi_k\}$ satisfies the same conditions as $\{\Phi_k\}$ and moreover,

$$\|\Psi_k(x,\ )\|_{L^2} \leqslant 2\|\Phi(x)\|_{L^2} \qquad \text{for all } x.$$

Given m = 1,2,..., let

$$B_m = \{x \in R^n \,/\, |x| \leqslant m \quad , \quad \|\Phi(x)\|_{L^2} \leqslant m\}.$$

R^n is the increasing union of the sets B_m.

We assert that $\{\Psi_k\}$ is a Cauchy sequence in $L^2(B_m \times R^n)$.

In fact, since

$$\int_{R^n} |\Psi_k(x,y) - \Phi(x)|^2 \, dy \xrightarrow[k\to\infty]{} 0 \qquad \text{ae.}$$

we get also

$$\int_{R^n} |\Psi_k(x,y) - \Psi_l(x,y)|^2 \, dy \xrightarrow[k,l\to\infty]{} 0 \qquad \text{ae.}$$

Furthermore,

$$\int_{R^n} |\Psi_k(x,y) - \Psi_l(x,y)|^2 \, dy \leqslant 8m \qquad \text{for} \quad x \in B_m.$$

So, the dominated convergence theorem tells us that

$$\int_{B_m} \int_{R^n} |\Psi_k(x,y) - \Psi_l(x,y)|^2 \, dy \, dx \xrightarrow[k,l\to\infty]{} 0$$

Thus, there exists a function $\varphi^{(m)}(x,y)$ such that

$$\Psi_k \longrightarrow \varphi^{(m)} \qquad \text{in} \qquad L^2(B_m \times R^n).$$

Moreover,

$$\varphi^{(m+1)} / \, B_m \times R^n \; = \; \varphi^{(m)} \qquad \text{ae.}$$

Let $\varphi(x,y)$ the function defined as $\varphi^{(m)}$ over $B_m \times R^n$. We assert that this function satisfies the desired conditions.

In fact, according to the definition, it is clear that φ is a measurable function.

Moreover, there exists

$$\lim_{k\to\infty} \int_{B_m} \int_{R^n} |\Psi_k(x,y) - \varphi(x,y)|^2 \, dy \, dx = 0$$

for each m = 1,2,... .

Thus, we can construct a subsequence $\{\Psi_j\}$ such that

$$\lim_{j\to\infty} \int_{R^n} |\Psi_j(x,y) - \varphi(x,y)|^2 \, dy = 0 \qquad \text{ae in } x.$$

Since there also exists

$$\lim_{j\to\infty} \int_{R^n} |\Psi_j(x,y) - \Phi(x)|^2 \, dy = 0 \qquad \text{ae in } x,$$

we deduce that there exists a measurable subset X of R^n of measure zero, such that

$$\Phi(x) = \varphi(x,y) \qquad \text{ae in } y, \text{ for } x \in R^n \setminus X.$$

This concludes the proof of the lemma.

Turning to our situation, we claim that the map

$$\begin{aligned} R^n &\longrightarrow L^2 \\ x &\longrightarrow h_x \end{aligned}$$

is strongly measurable.

In fact, since L^2 is a separable Hilbert space, it suffices to show that it is a weakly measurable map. But given $f \in L^2$, the function

$$\int h_x(y)\ f(y)\ dy$$

is a measurable one, because according to the hypothesis, it depends continuously on x.

So, applying the lemma, there exists a measurable function h(x,y) such that

$$Rf(x) = \int h(x,y)\ f(y)\ dy \qquad f \in L^2$$

Now, we assert that h(x,y) belongs to $L^\infty(R^n_x;L^2_y)$; this means that for each $x \in R^n$, h(x,y) is a square integrable function of y and

$$\operatorname*{ess\ sup}_{x \in R^n} \|h(x,y)\|_{L^2_y} < \infty$$

In fact, if B_c denotes the space of bounded continuous functions with the L^∞- norm, the linear map

$$\begin{aligned} L^2 &\longrightarrow B_c \\ f &\longrightarrow Rf \end{aligned}$$

is continuous.

Thus, there exists a constant $M > 0$ such that

$$\|f\|_{L^2} \leqslant 1 \quad \text{implies} \quad \|Rf\|_{L^\infty} \leqslant M.$$

So, we can write

$$\left(\int_{R^n} |h(x,y)|^2\ dy\right)^{1/2} = \sup_{\|f\|_{L^2} \leq 1} \left|\int h(x,y) f(y)\ dy\right| = \sup_{\|f\|_{L^2} \leq 1} |Rf(x)| \leqslant M.$$

Now, we are going to give the main result.

PROPOSITION. *Let* R *be a linear continuous operator in* L^2 *such that* $D^\alpha R$, RD^α *are also continuous operators in* L^2 *for* $|\alpha| = N$ *where* $N > j+n/2$ *for some* $j = 0,1,\ldots$. *Then, there exists a function* h(x,y) *satisfying the following conditions*

$$\left.\begin{aligned} D^\alpha_x\ h(x,y) &\in L^\infty(R^n_x\ ;\ L^2_y) \\ D^\alpha_y\ h(x,y) &\in L^\infty(R^n_y\ ;\ L^2_x) \end{aligned}\right\} \quad \textit{for} \quad |\alpha| \leqslant j$$

$$Rf = \int h(x,y)\ dy \quad \textit{for} \quad f \in L^2.$$

Proof. According to what we have said above, there exists a function

$h(x,y)$ in $L^\infty(R^n_x;L^2_y)$ such that

$$Rf = \int h(x,y)\ f(y)\ dy$$

Moreover, for each $|\alpha| \leq j$, there exists a function $h_\alpha(x,y)$ in the same class, such that

$$D^\alpha Rf = \int h_\alpha(x,y)\ f(y)\ dy$$

Now, given two test functions f,g, we can write in the sens of distributions

$$\int h_\alpha(x,y)\ f(y)\ g(x)\ dy\ dx = (D^\alpha Rf,g) = (-1)^{|\alpha|}\ (Rf,D^\alpha g) =$$

$$= (-1)^{|\alpha|} \int h(x,y)\ f(y)\ D^\alpha g(x)\ dy\ dx$$

Thus, we deduce that $D^\alpha_x\ h(x,y)$ belongs to $L^\infty(R^n_x;L^2_y)$, for $|\alpha| \leq j$.

On the other hand, the adjoint operator R^t satisfies the same conditions as R. So, we obtain that $D^\alpha_y\ h(x,y) \in L^\infty(R^n_y;L^2_x)$, for $|\alpha| \leq j$.

This completes the proof of the proposition.

REMARK. The proposition provides necessary conditions on the function h(x,y) to define a regularizing operator. It remains open the problem to characterize that function.

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