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A CHARACTERIZATION OF PRÜFER RINGS

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ABSTRACT. Let R be a domain. In this paper we prove the following result: R is a Prüfer ring if and only if, for every finitely generated R-module M, t(M) (the torsion submodule) is a pure submodule of M. This has as corollary a theorem of Kaplansky characterizing Prüfer rings.

Throught this paper we assume R is a commutative ring with identity 1. We recall first, for reference, the definition and a characterization of pure submodules ([1], 2, ex. 24. a.) ([2], th. 2.4.).

DEFINITION. *An exact sequence of* R*-modules:*

$$0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$$

is called pure exact (also we say that M' *is a pure submodule of* M) *iff for every* R*-module* N*, the following sequence is exact:*

$$0 \longrightarrow N \otimes_R M' \xrightarrow{1\otimes i} N \otimes_R M \xrightarrow{1\otimes p} N \otimes_R M'' \longrightarrow 0$$

LEMMA. *Every direct summand is a pure submodule.*

THEOREM 1. *Let* M *be an* R*-module and* M' *a submodule of* M. M' *is a pure submodule of* M *if and only if, for every finite family* $(a'_j)_{j=1}^{n} \subset M'$ *such that* $a'_j = \sum_{i=1}^{m} r_{ij}\, a_i$ $(a_i \in M,\ r_{ij} \in R,\ j = 1,\dots,n)$, *there exist a family* $(b'_i)_{i=1}^{m} \subset M'$ *such that:*

$$a'_j = \sum_{i=1}^{m} r_{ij}\, b'_i \qquad j = 1,\dots,n$$

In this paper we give the following characterization of Prüfer rings.

THEOREM 2. *Let* R *be a domain.* R *is Prüfer if and only if for every* R-*module* M *of finite type, the torsion submodule* t(M) *is a pure submodule of* M.

Before proving theorem 2, let us show that it has as corollary the following theorem ([3]):

THEOREM 3. *Let* R *be a domain.* R *is a Prüfer ring if and only if, for* *every* R*-module* M *of finite type,* t(M) *is a direct summand of* M.

Proof. The non trivial part (the implication for the left) follows from the lemma and theorem 2.

Proof of theorem 2. If R is Prüfer, $\frac{M}{t(M)}$ is projective, then the exact sequence:

$$0 \longrightarrow t(M) \longrightarrow M \longrightarrow \frac{M}{t(M)} \longrightarrow 0$$

splits, so is pure exact.

For the converse we use the fact that R Prüfer is equivalent to say that every finitely generated ideal $I \neq 0$ of R is invertible. Let $I \neq 0$ an ideal of R generated by $a_1,\ldots,a_n$ with, say, $a_1 \neq 0$. Put:

$$M = \frac{R^n}{(a_1(a_2,\ldots,a_n))}$$

We have $\overline{(a_1,\ldots,a_n)} \in t(M)$, and $\overline{(a_1,\ldots,a_n)} = \sum_{i=1}^{n} a_i\, \overline{e}_i$ where (e_i) is the canonical base of R^n. Being, by hypothesis, t(M) a pure submodule of M, by theorem 1 there exist $p_{ij} \in R$ with $\overline{(p_{i1},\ldots,p_{in})} \in t(M)$, such that:

$$\overline{(a_1,\ldots,a_n)} = \sum_{i=1}^{n} a_i\, \overline{(p_{i1},\ldots,p_{in})}$$

In particular:

(1) $$a_1 = \sum_{i=1}^{n} a_i\, p_{i1} + r\,a_1 a_1 \qquad (r \in R)$$

We have $\overline{(p_{i1},\ldots,p_{in})} \in t(M)$, $i = 1,\ldots,n$, then there must exist r_i, $s_i \in R$, $r_i \neq 0$, $i = 1,\ldots,n$, such that:

(2) $$r_i\, p_{ij} = s_i\, a_1\, a_j \qquad i,j = 1,\ldots,n$$

Replacing $p_{i1} = \frac{s_i\, a_1\, a_1}{r_i}$ in (1) and using the fact that $a_1 \neq 0$, we obtain:

$$1 = \left(r + \frac{s_1\, a_1}{r_1}\right) a_1 + \sum_{i=2}^{n} \frac{s_i\, a_1}{r_i}\, a_i$$

To conclude the proof we only have to verify that:

$$\frac{s_i\, a_1}{r_i}\, a_j \in R \qquad i,j = 1,\ldots,n$$

but this is clear from (2).

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