Revista de la
Unión Matemática Argentina
Volumen 30, 1982-83

DENSITY AND TOTAL MEASURE OF NORMAL CHAINS IN HERMITIAN SPACES

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1. INTRODUCTION.

Let K_n be a n-sphere bundle on $P_n(C)$ and there the n-normal chains defined originally by W.Blaschke, [1], in eliptic hermitian spaces.

The purposes of this note are: 1) to give a definition of normal chains of dimension n and r ($r < n$), both in $P_n(C)$, and to compute their invariants densities with respect to the hermitian group; 2) to obtain the invariants densities of normal chains by a fix point and of a point in a fix normal chain. Finally, we obtain an integral formula and from it we calculate the total measure of n-normal chains in $P_n(C)$.

Here and in the sequel we always mean invariant density.

Our notation will be C_r^n for r-normal chain in $P_n(C)$, dC_r^n their densi-ty, $C_r^n[P]$ indicate a C_r^n by a fix point P and $P(C_r^n)$ a point P in a fix normal chain.

I wish to thank Prof. Cristián Sánchez for his suggestions.

2. DEFINITIONS.

Let $P_n(C)$ be the n-complex projective space with the eliptic hermi-tian geometry. The inner product is

$$(u,v) = \sum_{j=0}^{n} u_j . \overline{v}_j \quad \text{if} \quad u = (u_0,\dots,u_n) \quad , \quad v = (v_0,\dots,v_n)$$

and $\overline{v}$ denote complex conjugate of v.

Over $P_n(C)$ we consider the manifold $F_0(C^{n+1})$ of unitary orthogonal frames $(z^0,\dots,z^n)$ and call π_1 the projection

$$\pi_1 : F_0(C^{n+1}) \longrightarrow P_n(C)$$
$$(z^0,\dots,z^n) \longrightarrow z^0$$

On $F_0(C^{n+1})$ the structure equations

$$\begin{cases} dz^i = \sum_{j=0}^{n} w_{ij}\, z^j & w_{ij} + \overline{w}_{ji} = 0 \\ dw_{ij} = \sum_{k=0}^{n} w_{ik} \wedge w_{kj} & \end{cases}$$

are valid, Griffiths [2].

It will sometimes be convenient to write

$w_{ij} = (dz^i, z^j)$ and with the notation $d^c = \frac{\sqrt{-1}}{4}(\partial - \overline{\partial})$

the standard Kähler form on $P_n(C)$

$$w = \frac{\sqrt{-1}}{2} \left\{\frac{(z,z)(dz,dz) - (dz,z)(z,dz)}{(z,z)^2}\right\} = d\, d^c \log\|z\|^2$$

We know, [2], that

$w = \frac{\sqrt{-1}}{2} \{\sum_{j=1}^{n} w_{0j} \wedge \overline{w}_{0j}\}$ is the pullback to $F_0(C^{n+1})$ of the standard

form $\partial\, \partial^c \log\|z\|^2$.

Now, let K_n be a n-sphere bundle over $F_0(C^{n+1})$, i.e., [5]

$$\pi_2 : K_n \longrightarrow F_0(C^{n+1})$$

$$C_n[z^0] \longrightarrow (z^0, \ldots, z^n)$$

where $C_n[z^0] = \sum_{j=0}^{n} a_j z^j$, $\sum_{j=0}^{n} a_j^2 = 1$, $a_j \in R$, $(z^0,\ldots,z^n) = \pi_1^{-1}(z^0)$

the fibre is the set of n-spheres that contain z^0.

We emphasize

$C_n[z^0]$ = is a sphere with respect to frame $(z^0,\ldots,z^n)$, varying $(z^0,\ldots,z^n)$ in the fibre $\pi_1^{-1}(z^0)$, is the set of spheres that pass by the fixed point z^0.

DEFINITION 1. (Blaschke, [1]). *We call* n-*dimensional normal chain* $C_n, (C_n^n)$ *to a section of bundle*

$$\pi_1 \circ \pi_2 : K_n \longrightarrow P_n(C)$$

i.e., to the set of points

$\sum_{j=0}^{n} a_j\, z^j$ *with* $\sum_{j=0}^{n} a_j^2 = 1$, $a_j \in R$ *and* $(z^0,\ldots,z^n)$ *fixed orthogonal unitary frame.*

DEFINITION 2. *If in definition 1,* $a_{r+1} = \ldots = a_n = 0$, *we obtain a* *r-dimensional normal chain* C_r^n *in* $P_n(C)$. *When no confussion can arise we shall abbreviate* C_r^n *by* C_r.

3. DENSITIES.

From the structure equations we observe that

$$(1)\qquad \begin{aligned} w_{jk} &= (dz^j, z^k) = \sum_{h=0}^{n} dz_h^j \, \bar{z}_h^k \\ w_{jk} &= \alpha_{jk} + \beta_{jk}\sqrt{-1} \end{aligned}$$

Calling L_r a linear subspace of dimension r it's known, [4], that their density is, up to a constant,

$$(2)\qquad dL_r = \bigwedge_{\substack{0\le j\le r \\ r+1\le k\le n}} w_{jk} \wedge \bar{w}_{jk}$$

Since [1], we know that the density for n-dimensional normal chains in $P_n(C)$ is

$$(3)\qquad dC_n = \bigwedge_{0\le j<k\le n} \beta_{jk} \bigwedge_{h=1}^{n} \beta_{hh}$$

and the density for r-normal chains in $P_n(C)$ is

$$(4)\qquad dC_r^n = dC_r^r \wedge dL_r^n$$

From this it follows our interest by dC_n.

We can prove that:

The degree of density of r-dimensional normal chains in $P_n(C)$ is

$$N(r,n) = \frac{4n(r+1) - r(3r+1)}{2}$$

Proof. A C_r is determined by the L_r that contained it and then C_r in that subspace.

The L_r depends of $2(n-r)(r+1)$ real parameters, [4]; fixed L_r, to fix the r-normal chain C_r we determine the orthogonal unitary frame $(z^0,\ldots,z^r)$, from [3], pag.340, we know that we need $r(r+2)$ real parameters.

But z^0 can vary in the C_r ($\dim C_r = r$), z^1 can vary in the C_{r-1} obtained by omission of z^0, with generality, z^j can vary in the C_{r-j} obtained by omission of $z^0, z^1, \ldots, z^{j-1}$.

Finally,

$$N(r,n) = 2(n-r)(r+1)+r(r+2) - \frac{r(r+1)}{2} = \frac{4n(r+1)-r(3r+1)}{2}$$

we have arrived at the desired result.

Following the method given in [3], chap.10, we verify that density of n-normal chains that their pass by a fixed point P, that we consider as z^0, without loss of generality, is, up to a constant,

$$(5) \qquad dC_n[P] = \bigwedge_{i\le j<k\le n} \beta_{jk} \bigwedge_{h=1}^{n} \beta_{hh}$$

and fixing a n-normal chain C_n, the density of a point P in C_n is,

$$(6) \qquad dP(C_n) = \bigwedge_{j=1}^{n} w_{0j}$$

From (6) we want to obtain other expression of $dP(C_n)$.

Let C_r be the r-normal chain determined by $(x^0,\dots,x^r)$. We take r+1 orthonormal points in C_r, i.e.

$$y^j = \sum_{s=0}^{r} \alpha_s^j x^s \quad , \quad (\alpha_s^j) \text{ is a unitary matrix of order } r+1.$$

It's known from (6) that if $P = y^0$, up to a constant, we have

$$dP(C_r) = \bigwedge_{j=1}^{r} (dy^0,y^j)$$

As C_r^n is fixed, the x^s are fixed, then

$$dy^0 = \sum_{s=0}^{r} d\alpha_s^0.x^s \qquad \text{and by orthogonality}$$

$$(dy^0,y^j) = \sum_{s=0}^{r} d\alpha_s^0 .\alpha_s^j \qquad j = 1,\dots,r$$

with this

$$\bigwedge_{j=1}^{r} (dy^0,y^j) = \bigwedge_{j=1}^{r} \sum_{s=0}^{r} d\alpha_s^0.\alpha_s^j =$$

$$= \sum_{\substack{i_m \varepsilon \pi \\ m\varepsilon\{0,1,\dots,\hat{j},\dots,r\}}} \sum_{j=0}^{r} (-1)^{sg\,\pi} \alpha_{i_1}^1 \dots \alpha_{i_r}^r \, d\alpha_0^0 \wedge \dots \wedge \widehat{d\alpha_j^0} \wedge \dots \wedge d\alpha_r^0$$

where π = perm $\{0,1,\dots,\hat{j},\dots,r\}$ and the simbol ^ over an index indicates it was omitted. Equivalently,

$$dP(C_r) = \sum_{\substack{j=0 \\ i_p\varepsilon\{0,\dots,\hat{j},\dots,r\} \\ i_p < i_q \text{ if } p<q}}^{r} \alpha_{i_1}^1 \dots \alpha_{i_r}^r \, d\alpha_0^0 \wedge \dots \wedge \widehat{d\alpha_j^0} \wedge \dots \wedge d\alpha_r^0$$

In the second member of before equality, each coefficient is the determinant of the matrix obtained from (α_s^j) by omitting the 0-th row and s-th column of (α_s^j) with s = 0,1,...r.

Then we remember that α_s^0 = Adj (0/s) and now we can write

$$dP(C_r^n) = \sum_{j=0}^{r} \alpha_j^0 \, d\alpha_0^0 \wedge \ldots \wedge \widehat{d\alpha_j^0} \wedge \ldots \wedge d\alpha_r^0$$

By the other hand, we consider the projections over each cartesian axis α_j^0 with j = 0,1,...r.

Calling ds^r the element of volumen of S^r at the point y^0 we have

$$d\alpha_0^0 \wedge \ldots \wedge \widehat{d\alpha_j^0} \wedge \ldots \wedge d\alpha_{r-1}^0 = ds^r . \alpha_j^0 \qquad j = 0,\ldots,r$$

multiplying each expression by α_j^0 and adding, since $\sum_{j=0}^{r} (\alpha_j^0)^2 = 1$ we have

$$ds^r = \sum_{j=0}^{r} \alpha_j^0 \, d\alpha_0^0 \wedge \ldots \wedge \widehat{d\alpha_j^0} \wedge \ldots \wedge d\alpha_r^0$$

and we obtain

$$(7) \qquad dP(C_r^n) = ds^r$$

Considering (2), (3), (5) and (6) we can verify

$$(8) \qquad dP \wedge dC_n[P] = dC_n \wedge dP(C_n)$$

4. INTEGRAL FORMULA.

The total measure of n-normal chains in $P_n(C)$ is obtained integrating over all space, the density of n-dimensional normal chain. From (8) we remark that varying the point P and then integrating over the fibre, we compute the n-normal chain (n+1) times, consequently

$$(9) \qquad \int dP . \int dC_n[P] = (n+1) \int dC_n . \int dP(C_n)$$

We observe that

$$dC_n[P] = dC_{n-1}^{n-1} \wedge \beta_{nn}$$

and we know, [4] , that by integration over the fibre $\int \beta_{nn} = 2\pi$ and dP = measure of volumen of n-dimensional eliptic hermitian

$$\text{space} = \frac{(2\pi)^n}{n!}$$

From this, (7) and (9)

$$\int dC_n = \frac{2^n . \Gamma((n+1)/2) . \pi^{(n+1)/2}}{(n+1)!} . \int dC_{n-1}^{n-1}$$

equivalently

$$(10)\quad \text{tot.measure } C_n = 2^{n(n+1)/2} \cdot \prod_{j=3}^{n+1} \frac{\Gamma(j/2)}{j!} \cdot \pi^{((n+2)(n+1)-2)/4}$$

REFERENCES

[1] W.BLASCHKE, *Densita negli spazi di Hermite*, Rendiconti della R. Accademia dei Lincei 29, (1939), 105-108.

[2] P.GRIFFITHS, *Complex diff. and integral geometry and curvature integrals associated to singularities of complex analytic varieties*, Duke Math. Journal, 45, (1978), 427-512.

[3] L.A.SANTALO, *Integral geometry and geometric probability*, Addison-Wesley, Reading, 1976.

[4] -----------, *Integral geometry in Hermitian spaces*, 74, American J. of Math. (1952), 423-434.

[5] N.STEENROD, *Topology of fibre bundles*, Princeton Univ. Press., Princeton, New Jersey (1965).

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Recibido en marzo de 1981.