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AXIOMATIC APPROACH TO SPACE - TIME GEOMETRY

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ABSTRACT. The causal, differentiable, conformal, projective and riemannian structures of space-time are introduced, in that order, by independent sets of axioms. All the assumptions made are related to the more intuitive as possible physical properties of freely falling particles and light rays, joining different previous works on the subject (refs. 3,4,7) in a unique scheme.

1. INTRODUCTION.

The most common approach to the theory of General Relativity assumes a model for space-time representation which is obtained from the following hypotheses:

a) Space-time is a differentiable, four dimensional manifold.

b) It has a riemannian metric defined locally by a tensor g_{ij}, such that the time interval, measured by an standard clock between events x^i and $x^i + dx^i$ along its world line, is given by $d\tau = (-g_{ij}dx^i dx^j)^{1/2}$ (chronometric hypotheses [1]).

c) and d) Freely falling particles and light rays propagate, respectively, along timelike and null geodesics of the metric structure.

This model, having particles and standard clocks as fundamental concepts, introduces the riemannian metric as the primary geometric structure, from which the other structures present in space-time can be easily obtained: affine, projective, conformal and causal. This scheme is interesting when we desire to obtain the theory from the fewest possible axioms.

However, the less restrictive structures mentioned above are closely related to physical concepts, while these relations are not so clear when derived from the riemannian structure. In particular, the fact that free fall defines a projective structure on space-time, and that light rays propagation implies a conformal structure was already observed by Weyl [2]. But only much more recently [3,4] axiomatic theories for space-

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time geometry which take as fundamental concepts those of freely falling particles and light rays were developed. The riemannian character of space-time is deduced as a synthesis of its conformal and projective structures.

Another objection made to the current approach to general relativity is that proper time has an intrinsic meaning in this theory, and it can be measured by experiments carried out only with particles and light rays, as shown by Marzke and Wheeler [5] and Kundt and Hoffmann [6] in different ways. So the chronometric hypotheses is redundant or, if the word standard clock is to be interpreted as atomic clock (as is usually done, cf. e.g. [1]), then it implies a postulate of equality between "atomic time" and "gravitational time". As quoted in [4] (EPS from now on) it would be better to test this equality experimentally, or to deduce it from a more general theory embracing both kind of phenomena.

Different kind of objections were made later by Woodhouse [7]: it is usual to impose certain additional restrictions on physically reasonable space-times (cf. e.g. global techniques [8]), in such a way that local structures determine global properties. For example the causality conditions imply restrictions on the topology of space-time [9] which cannot be formulated until the metric has been defined, and yet the metric cannot be defined until the background manifold has been specified. In view of these arguments Woodhouse devoted to improve the axiomatization of the differentiable structure of space-time, showing that its continuum properties should not be taken as absolute, but derived from more intuitive and primitive concepts. The global restrictions are impossed right at the beginning, by means of the causal properties of space-time, and the topological and differentiable structures are related to particles and light rays propagation properties.

The purpose of the present work is to make a review of the three mentioned papers [3,4,7], in order to join in a unique scheme what we consider the more relevant features of each of them. The primitive concepts will be those of freely falling particles and light rays; and we will introduce independent sets of axioms for each different geometric structure. Causal, topological and differentiable structures, inspired on Woodhouse's ideas and on the paper of Kronheimer and Penrose on causal spaces [10], will be introduced in first place. Then it will be shown that if the causal properties of space-time are locally the same as in Minkowski space, then the causal structure determines a conformal one, having light rays as null geodesics.

The strong equivalence principle, applied to free particles motion, introduces a projective structure, with free particles as timelike geodesics.

It will be shown that these two last structures are compatible (in the sense of EPS) as a consequence of the causal structure. We conclude in such a way that the space-time must be represented by a Weyl space [11].

The next step will consist of relating the geometrical notions of parallel transport and affine parameters along timelike geodesics with physical facts. We will follow the ideas of a previous work, in which one of us (M.C.) developed, in a space-time more general than a Weyl space, a physical construction of parallelism [12]; and then, using this notion, the "geodesic clock" was constructed [3] (which had already been deviced by Marzke and Wheeler [5] to measure proper time in a Riemann space).

Using this connection between geometrical parameters and physical concepts, we formulate the last axiom, related to some properties of geodesic clock's readings; and then we derive the current point of view for space-time: that it has a riemannian structure.

Nevertheless, the geodesic clock is only an ideal experience, and so this last axiom, although very natural, is not well founded. So we will add an argument of plausibility for it, showing that the most natural generalization of the equation of motion for a photon to curved space-time can only be made if this axiom holds.

2. CAUSAL STRUCTURE.

The primitive concepts to be used along this work will be those of freely falling particles and light rays. We will call $\mathcal{P} = \{P,Q,\ldots\}$ and $\mathcal{L} = \{L,L',\ldots\}$ the sets of their respective world lines. Every intersection of two particles, or of a particle and a light ray, will be called an event. The set of all the events is called space-time: $M = \{x,y,z,\ldots\}$.

There is a time arrow defined along the history of each particle or light ray, which allows to distinguish between future and past events with respect to a given one. A natural way of representing this notion of "time elapsing" is by a continuous map of the particle or light ray onto the real line, $\mathbb{R}$. Let us introduce this property by our first axiom

> C_1: Particles and light rays are one-dimensional, real, differentiable manifolds class c^0.

The order of the real line corresponds to the chronological order of the events along the particle or light ray. Choosing a particular orientation between the two possible ones along these elements, relations between events expressing the notion of causality can be defined. We will call "future" the orientation along particles and light rays such that their parametrization is monotonically increasing. Then we define

DEFINITION 1. $x \ll y \iff \exists\, P \in \mathcal{P}$ with parametrization $t: P \longrightarrow \mathbb{R}$ / $x,y \in P$ and $t(x) < t(y)$.

$$x \to y \iff \exists\, L \in \mathcal{L} \text{ with parametrization } t\colon L \longrightarrow \mathbb{R} \;/\; x, y \in L \text{ and } t(x) < t(y).$$

$$x \Rightarrow y \iff x = y \text{ or } x \ll y \text{ or } x \to y.$$

The symbol $x \ll y$ reads "x chronologically preceds y over the world line of a material observer"; $x \to y$ means that a light pulse emitted at x may be absorbed at y; and the relation $\Rightarrow$, that can be read as "causally preceds", represents the ordinary cause-effect relation.

The following axioms will introduce physical properties connected to these relations. First of all we shall impose the natural restriction that no event may influence itself

$$C_2\colon \;\forall\, x,y \in M\colon\; x \Rightarrow y \;\text{ and }\; y \Rightarrow x \;\Rightarrow\; x = y.$$

There always exist a particle P' which can go "slower" than a given one, P, from an event x to another particle Q, i.e. that P' intersects Q later than P (fig. 1). This property can be expressed by the transitivity of $\ll$.

$$C_3\colon \;\forall\, x,y,z \in M,\; x \ll y \;\text{ and }\; y \ll z \;\Rightarrow\; x \ll z.$$

There is a limiting velocity for the propagation of particles: the velocity of light. Rigurously: the event where the light ray L emitted at x intersects the particle Q (event e in fig. 1) is the boundary (with respect to Q topology) of the events along Q that can be attained from x by means of particles. Before introducing this property we will make some previous definitions.

DEFINITION 2. Chronological future and past of an event x

$$I^+(x) = \{y \in M \;/\; x \ll y\} \quad ; \quad I^-(x) = \{y \in M \;/\; y \ll x\}$$

Analogously we can define null future and past, $C^+(x)$ and $C^-(x)$, and causal future and past, $J^+(x)$ and $J^-(x)$, by means of $\to$ and $\Rightarrow$ respectively.

DEFINITION 3. Alexandrov topology, τ.
Is the smallest topology in M in which the sets $I^+(x)$ and $I^-(x)$ are open sets, $\forall\, x \in M$.

Now, with the language of these definitions, we impose to space-time the above discussed property.

$$C_4\colon \;\forall\, x \in M\colon\; C^+(x) = \mathring{I}^+(x) \;\text{ and }\; C^-(x) = \mathring{I}^-(x).$$

where $^\circ$ denotes the boundary with respect to the Alexandrov topology. Equivalently we could state C_4 as $J^+(x) = \overline{I^+}(x)$ and $J^-(x) = \overline{I^-}(x)$, the symbol $^-$ meaning the closure with respect to τ topology.

We shall now prove some properties of the causal relations implied by the preceding axioms.

An antirreflexive linear ordering "$<$" is said to be complete (cf.[10]) when it satisfies the following conditions and its duals

i) $\forall\ x \in M, \exists\ y\ /\ y < x$.

ii) If $y_1 < x,\ y_2 < x \Rightarrow \exists\ z\ /\ z < x$ and $y_1 < z,\ y_2 < z$.

LEMMA 1. $\ll$ *is complete.*

i) There always exist $P \in \mathcal{P}\ /\ x \in P$, and being P homeomorphic with the real line (axiom C_1), obviously there exists $y \in P\ /\ y \ll x$.

ii) Given $P \in \mathcal{P}\ /\ x \in P$, and $y \in M\ /\ y \ll x$, because of axiom C_4, $I^+(x) \cap P$ will be an open set (and non void because of C_1). Then $\exists\ z \in I^+(y_1) \cap I^+(y_2) \cap P\ /\ z \ll x$.

LEMMA 2. $x \prec y \iff I^+(x) \supset I^+(y)$ *and* $I^-(x) \subset I^-(y)$.

This statement is equivalent, because of C_4, to the following two equalities:

$$\text{a)}\ \bar{I}^+(x) = \{y \in M\ /\ I^+(x) \supset I^+(y)\}$$

$$\text{b)}\ \bar{I}^-(x) = \{y \in M\ /\ I^-(x) \subset I^-(y)\}$$

We prove a): Suppose $y \in \bar{I}^+(x)$. Then every neighborhood of y contains at least one event of $I^+(x)$. For every $z \in I^+(y)$, $I^-(z)$ is a neighborhood of y, containing then events of $I^+(x)$. So $I^+(x) \supset I^+(y)$. Now suppose $y \in M\ /\ I^+(x) \supset I^+(y)$. $\ll$ being complete, then every neighborhood of y contains another one of the form $I^+(u) \cap I^-(v)$, with $u \ll y \ll v$, and $I^+(y) \cap I^-(v)$ is non empty and contains events of $I^+(x)$, because $I^+(x) \supset I^+(y)$. So $y \in \bar{I}^+(x)$.

Now it is an easy matter to prove that the seven axioms established by Kronheimer and Penrose as defining a causal structure are satisfied by our space-time. We recall that $(M,\ll,\prec,\rightarrow)$ is said to be a causal space, M being a point set, when the following axioms are verified

I) $\forall\ x \in M,\ x \prec x$.

II) If $x,y,z \in M$ and $x \ll y \ll z \Rightarrow x \ll z$.

III) If $x \prec y$ and $y \prec x \Rightarrow x = y$.

IV) Not $x \ll x$.

V) $x \ll y \Rightarrow x \prec y$.

VI) a) If $x \ll y \prec z \Rightarrow x \ll z$.

b) If $x \prec y \ll z \Rightarrow x \ll z$.

VII) $x \rightarrow y \iff x \prec y$ and not $x \ll y$.

Moreover, the property stated in Lemma 2 is the one required in [10] to define a particular class of causal space, named β-causal space. It is not difficult to prove also that $(M,\ll,\prec,\rightarrow)$ with the properties given by C_1 - C_4, is future and past reflecting, and future and past distinguishing, i.e. that $I^+(x) \supset I^+(y) \Rightarrow I^-(x) \subset I^-(y)$ and $I^+(x) = I^+(y) \Rightarrow x = y$, and its respective dual conditions.

LEMMA 3. *τ is Hausdorff* [13].

Suppose $x,y \in M$ and $x \neq y$. Then, because of C_2, necessarily one of the followingsis not verified: $x \nRightarrow y$ or $y \nRightarrow x$. Suppose not $x \nRightarrow y$. Then $\exists\ z_1 \in I^-(x)$, $\exists\ z_2 \in I^+(y)$ such that neither $z_1 \nRightarrow z_2$ nor also $z_2 \nRightarrow z_1$. So $I^+(z_1)$ and $I^-(z_2)$ are disjoint open neighborhoods of x and y respectively.

3. DIFFERENTIABLE STRUCTURE.

We have yet introduced a topological structure on space-time. We shall now show how a differentiable structure can be given to this topological space, related also to physical properties of particles and light rays. We define the neighborhood of a particle P, U_P, as the union of all the sets of the form $I^+(u) \cap I^-(v)$, with $u,v \in P$ and $u \ll v$.

DEFINITION 1. Message functions

$$f^+: U_P \longrightarrow P \ / \ \forall z \in U_P,\ f^+(z) = C^+(z) \cap P.$$

$$f^-: U_P \longrightarrow P \ / \ \forall z \in U_P,\ f^-(z) = C^-(z) \cap P.$$

These functions represent the way in which a freely falling observer sees and is seen by its neighbouring events (fig.2). They can be related to observable physical properties, such as the red shift, and will be used for the introduction of the differentiable structure on space-time. The main idea is the following: every particle has a differentiable structure, associated to the continuous notion of time elapsing. Each of these structures can be related to the others by means of the message functions. Then, assuming differentiable properties of these functions, based on empirical facts (certainly idealized), we shall deduce the differentiable properties of the whole space-time.

Woodhouse has proved that these functions are continuous in the Alexandrov topology. This fact makes clearer the physical significance of τ: it is the smallest topology in which space-time "looks" continuous.

Our first step will be, following Woodhouse, to ensure the existence of local coordinates on space-time, which can be constructed by a method introduced in EPS and called "radar coordinates" (fig.3). They consist in the parametrization of the events of a neighborhood by means of the proper time of emission and reception along a particle of a light pulse that bounces at the considered event. The number of necessary parti cles depends of course on the dimension of the space, obviously four in our case.

D_1. Given $P_1 \in P$ there always exists $P_2 \in P$ such that the four message functions defined by P_1 and P_2, together with homeomorphisms $P_1 \longrightarrow \mathbb{R}$, $P_2 \longrightarrow \mathbb{R}$, define a one-to-one map from a neighborhood of every point in $(U_{P_1} \cap U_{P_2})-(P_1 \cup P_2)$ onto an open set in $\mathbb{R}^4$. Moreover, every event belongs to such a neighborhood for some pair of particles.

This axiom,together with the fact that f^+ and f^- are open maps, allows to conclude immediately that M *is a four-dimensional real differentiable manifold class* c^0.

In order to introduce a higher degree of differentiability in the way described at the beginning of this section, we must first assume the existence of a "particle fluid". i.e. that particles form a 4-dimensional continuum media (not actual particles, of course, but the whole of all possible paths for real particles). This is perhaps an idealized property, but is a natural assumption in every macroscopic theory, and is generally supossed to be valid in the problems with which classical general relativity deals.

D_2. $I^+(x)$ and $I^-(x)$ are four-dimensional submanifolds class c^0 for every $x \in M$.

Now we are able to talk about a continuously varying direction over a particle, and we can make questions about how it is seen by neighbouring observers. The information about the movement of a particle Q near an event e on its world line, as seen by an observer P, is contained in the functions $i^+(x)$ and $i^-(x)$ defined in D_3 below, evaluated at the event e.

It will be natural to assume that the information about the movement of particles is contained in smooth functions.

D_3. If Q is a particle, $x \in Q$, then for every particle P contained in U_Q the two following functions

$$i^+: Q \longrightarrow \mathbb{R} \;/\; i^+(x) = \frac{d}{dt_Q}\,(t_P \circ f_P^+ \circ t_Q^{-1})\Big|_{t_Q(x)}$$

$$i^-: Q \longrightarrow \mathbb{R} \;/\; i^-(x) = \frac{d}{dt_Q}\,(t_P \circ f_P^- \circ t_Q^{-1})\Big|_{t_Q(x)}$$

exist and are continuous.

t_P and t_Q are respectively the parametrizations of P and Q; f_P^+ and f_P^- are the message functions defined by P; and $\circ$ denotes composition of functions.

THEOREM 1. M *is a four-dimensional manifold class* C^1.

Because of axiom D_3, together with axiom D_2, the radar coordinates are C^1 functions in M. Moreover, the message functions $t_p \circ f_p^+ \circ t_Q^{-1}$ and $t_p \circ f_p^- \circ t_Q^{-1}$ are monotonically increasing, so they have inverse functions, also class C^1. If a coordinate patch is parametrized by means of two different pairs of particles, then the change of parametrization will be given by the composition of the inverse of one of them with the other. Being both of class C^1, the same will happen with the composition, and the theorem is proved.

4. CONFORMAL STRUCTURE.

The causal structure of space-time will determine a conformal structure, with the light rays as null geodesics, provided we assume that the causal properties of special relativity are locally verified, for each point of M. We define, following EPS, an implicit parametrization of the light conoids $C^+(x)$ and $C^-(x)$: let P be a particle passing through x, parametrized so that t(x) = 0. Then consider the function

$$g(z) = t(f^+(z)).t(f^-(z))$$

Obviously g(z) = 0 if and only if z lies in the null future or past of x. $C^+(x)$ and $C^-(x)$ are smooth hipersurfaces in a neighborhood of x, except at x itself, because if parametrized with the radar coordinates defined by means of P and another adequate particle, then one of the coordinates has the constant value zero (remember t(x) = 0) while the other three are smoothly varying. We don't know if it is smooth at x or not. Suppose it is smooth: then the same vector of the tangent space at x could be tangent to a future directed light ray and to another one , past directed. We would have in such a case the causal structure of the Galilean space, instead of that of Minkowski space. So we demand

K: The set of tangent vectors to the light rays passing through any event has two connected components.

The $C^+(x)$ and $C^-(x)$ aren't smooth at x, because if they were, it would be possible to pass smoothly from a future directed vector to a past directed vector, violating axiom K.

So, because of the implicit functions theorem, $g,_i(z)\big|_{z=x} = 0$, and then $g_{ij}(x) = g,_{ij}(z)\big|_{z=x}$ defines a tensor at x.

Deriving twice the equation g(z) = 0, valid along a light ray through x, we obtain, after evaluating at x

$$g_{ij}\ L^iL^j = 0 \qquad (1)$$

where L^i is the tangent vector to the light ray at x. So g_{ij} defines a conformal structure on M such that the light rays are null curves. We say there is a conformal structure because a change like $g_{ij} \longrightarrow \Omega^2(x).\tilde{g}_{ij}$ doesn't change the physics impossed until now: eq. (1) remains valid.

EPS shows that this structure does not depend on the particular choice of P, used to define g(z). The only possible signature for g_{ij}, in order to satisfy axiom K, is (+,+,+,-) (or, of course, the same with the opposite signs).

Light rays are not only null curves, but also geodesics of the conformal structure (C-geodesics). In effect, let $x \in M$ and $z \in C^+(x)$. Then $I^+(x) \supset I^+(z)$, and so the tangent hiperplane to $C^+(x)$ at z cannot contain timelike vectors. It can only contain spacelike ones, or null vectors if they are parallel to L^i (the tangent vector to the light ray passing through x and z). So, it is a null hiperplane with normal vector L^i, and then $C^+(x)$ is a null hipersurface. Being contained in a null hipersurface , light rays are null geodesics.

5. PROJECTIVE STRUCTURE.

The strong equivalence principle [14] states that: "There exist local coordinates for each event of space-time such that the gravitational effects are eliminated at that point". Such a coordinate system is called freely falling system. In particular,referred to the free particles motion this principle means that freely falling particles are mapped onto straight lines, in the freely falling system of coordinates. So we assume

P: For each event $e \in M$ there always exist local coordinates $\tilde{x}^i$ of a neighborhood of it, such that every particle through e obeys the equation

$$\left.\frac{d^2\tilde{x}^i}{dt^2}\right|_{x=e} = C\left.\frac{dx^i}{dt}\right|_{x=e}$$

C being a constant.

Changing to arbitrary coordinates x^i this equation becomes

$$\frac{d^2x^i}{dt^2} + \pi^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = C\frac{dx^i}{dt} \qquad (2)$$

where $\pi^i_{jk} = \theta^i_{k'}\,\theta^{k'}_{k,j}$ being $\theta^i_j = \dfrac{\partial x^i}{\partial x^j}$

π^i_{jk} is a linear affine connection that takes the zero value in the freely falling coordinate systems. It was obtained from the equation of freely falling particles motion, so any transformation letting them invariant will be physically equivalent, although leading to a different connection. Such transformations are of the form

$$\bar{\pi}^i_{jk} = \pi^i_{jk} + \delta^i_{(j}\, p_{k)} \qquad (3)$$

with p_k an arbitrary covector and $\delta^i_{(j}\, p_{k)} = \frac{1}{2}(\delta^i_j\, p_k + \delta^i_k\, p_j)$.
The equivalence class of connections related to each other by eq. (3) is said to define a projective structure [15]. Eq. (2) says that freely falling particles are geodesics of this structure ($\mathcal{P}$-geodesics). Obviously they are timelike geodesics (with respect to the conformal structure) because a particle passing through x lies inside of $C^+(x) \cup C^-(x)$, which tends locally to the null cone at x, the tangent vector to the particle lying inside the cone.

6. AFFINE STRUCTURE.

A projective and a conformal structures defined on a manifold M are said to be compatible when there exists a unique affine connection Γ^i_{jk} on M, having the same geodesics as the projective structure, and mapping every null vector of the conformal structure, under parallel transfer, into another null vector. In other words: Γ^i_{jk} must be at the same time a projective transformation π^i_{jk} and a conformal one of $\{^{\ i}_{j\ k}\}$ (the connection derived from g_{ij} by means of the second kind Christoffel symbol). In such a case M is said to be a Weyl space, and if we construct a covariant derivative with Γ^i_{jk} we will obtain (cf. [11])

$$\nabla_k\, g_{ij} = \lambda_k\, g_{ij} \qquad (4)$$

λ_k being an arbitrary vector.
This compatibility condition is equivalent, as we shall see below, to the requirement that $\mathcal{C}$-null geodesics should be identical to $\mathcal{P}$-null geodesics, a property that can be derived from the causal structure introduced at section 2. In EPS this property is deduced from a specially formulated axiom C which states: "Each event e has a neighborhood U such that an event p ∈ U, p ≠ e, lies on a particle P through e if and only if p is contained in the interior of the light cone of e". We don't need to formulate such an axiom, having this property already introduced with the causal structure. So we can follow the line of reasoning of EPS (pp. 79-80) to demonstrate that projective and conformal null geodesics are identical, using the fact that every event over

$C^+(x)$ can be approximated arbitrarily closely by events situated along particles through x (as guaranteed by axiom C_4). Then the equations

$$g_{ij}\, \dot{x}^i \dot{x}^j = 0 \quad \text{and} \quad \ddot{x}^i + \{{}_j{}^i{}_k\}\, \dot{x}^j \dot{x}^k = \beta\, \dot{x}^i \qquad (5)$$

imply

$$\ddot{x}^i + \pi^i_{jk}\, \dot{x}^j \dot{x}^k = \alpha\, \dot{x}^i \qquad (6)$$

We then must have $\pi^i_{jk} - \{{}_j{}^i{}_k\} = \varphi^i g_{jk}$ and $\alpha = \beta$, where φ^i is an arbitrary vector. But we are able to do a projective transformation of π^i_{jk}, without changing the physical situation. We make $\Gamma^i_{jk} = \pi^i_{jk} - 2\delta^i_{(j}\varphi_{k)}$ and then

$$\Gamma^i_{jk} = \{{}_j{}^i{}_k\} + \tfrac{1}{2}(\delta^i_j\, \psi_k + \delta^i_k\, \psi_j - \psi^i\, g_{jk}) \qquad (7)$$

This is the expression for the affine connection of a Weyl space, i.e., if this connection is used for taking covariant derivatives, the identity (4) is obtained.

7. MEASUREMENT OF PROPER TIME.

In a Weyl space there is a unique affine connection Γ^i_{jk} defined, with which it is possible to construct a parallel transport process with Lévi-Civita method. The arc length along any curve is also unambiguously defined (except for a linear transformation). Both concepts are related: the arc length can be also defined as a parameter along the curve such that its tangent vector, evaluated with this parameter, is always congruent to an arbitrary vector, which has been fixed at a point and equipolently transported along the curve. The freedom in the choice of this vector and the point where it is fixed imply that a linear transformation of the parameter so defined is available. If the curve is a geodesic the arc length coincides with its affine parameter.

In General Relativity the arc length along timelike curves is commonly associated with proper time, as measured by standard clocks (e.g. Einstein founded his choice of a riemannian space-time in the sharpness of the spectral lines emmited by atoms with surely very different previous histories [16]). Indeed, many concepts such as parallelism between paths, simultaneity between events, rigidity, etc. are interpreted in terms of this "atomic time" (cf. chronometry in space-time [1]).

Nevertheless we have renounced to use standard clocks in the foundation of space-time geometry, for the reasons explained at the introduction. For instance, Marzke and Wheeler have shown that proper time can be measured in a Riemann space in terms of light rays and freely falling particles, the only elements from which we want to derive all the concepts involved in this theory. The method, called "geodesic clock" [5], consists essentially in a light ray bouncing back and forth between two particles with parallel world lines. The same idea has been developed by one of us (M.C. [3]) in a much more general space than a riemannian one. Assuming natural conditions about free fall and light propagation, a method for constructing parallel paths and a parallel transport process was deviced, based in physical considerations. Then this method was used to construct the geodesic clock, and natural assumptions about its readings lead finally to the riemannian character of space-time.

Another interesting construction of the geodesic clock in terms of the conformal and projective structures has been proposed by Ehlers [17]. The idea is the following: conformal structure allows the definition of the orthogonality of a line element with a geodesic (the physic notion involved is that of Einstein simultaneity). Using the projective structure it is possible to construct a "plane strip" containing the given geodesic (a zig-zag of geodesics crossing the original one). Then a curve of the strip is said to be parallel to the given geodesic when its points can be joined by geodesic segments intersecting both orthogonally. A geodesic clock can then be constructed. Köhler [18] improved later the limiting processes involved in this construction, and proved that in a Weyl space such clock's readings coincide with affine parameters along timelike geodesics. For more details we refer to the literature ([17,18,19]).

In this work we prefer to adopt the arguments developed in [3], because in this way parallel transport in Weyl space can be given a physical interpretation. Moreover, we think that in this way the construction of parallel paths is more clearly related to light and particles propagation properties, although the foundation of Ehler's construction is very similar.

a) Constructing parallel paths in Minkowski space.

The Desargues theorem of affine geometry allows, given two pairs of parallel lines, to obtain another pair with such a property (fig.4). In the flat space-time of Special Relativity we have naturally such pairs: the light rays emitted at different points by the same particle and lying on a bidimensional plane. Based on this property of light propagation, a method for constructing parallel paths of particles has been sugested [12]. Suppose that A and B at figure 5 are the world lines of two particles, and we want them to be parallel. We shoot simultaneously from the event e a set of particles with different velocities, being C the faster and D the slower ones. When C arrives at A this last one emmits a light ray L which is reflected by D (L') and then returns to A. We allow particles C and D to pass through A, and also the particle E that arrives at A at the same time of the arrival of

L'. We then repeat the operation with particle B, and if the reflected ray L''' arrives at B simultaneously with E, then the figure of Desargues theorem is obtained, and so A is parallel to B. If not, we change the position of B, moving it either to the left or to the right, if L''' arrives at B before or after E respectively.

b) Parallel transport in curved space-time.

The previous construction can be extended to curved space-time: in any sufficiently small neighborhood of every event the above described situation must be reproducible, and if we then make the neighborhood tend to a point, we can obtain a differential equation for the parallel transport of a vector along a curve (fig. 6). This analytic development has been made in a space with a projective and a conformal structures [3]. It was also shown [20] that the so obtained parallel transport process, called Desargues transfer, coincides with ordinary Lévi-Civita transfer in a Weyl space, which has then an obvious physical interpretation.

Although the mentioned works assume that vectors are transported along geodesic curves, the Desargues transfer coincides with Lévi-Civita transfer also along non-geodesic curves. In effect, the equation of parallel transfer of a vector v^i along a curve $x^i = x^i(s)$ is, to first order in s

$$\Delta v^i + \Gamma^i_{jk} v^j v^k \Delta s = 0 \qquad (8)$$

and, up to the same order, it doesn't matter along which curve we go between the two points separated by Δs. So we can do it along a geodesic. Eq. (8) is that of parallel transport when $\Delta s \to 0$, so the approximation is good. It is equivalent to consider a non geodesic curve as the limit of a polygonal of geodesics, when its number of sides tendsto infinity.

We can now define, in a natural way, when two particles will have parallel world lines. That will be the case if the tangent vector to one of them, when parallely displaced along a geodesic orthogonal to it and crossing also the other, gives the tangent vector to the second curve. This definition makes sense in a small neighborhood of an event and for nearby particles, because it means that the above described construction repeated at two neighboring events becomes a square (fig. 7). Let us develop this analytically.

C: $x^i = x^i(s)$ is the original curve, and v^i is a vector orthogonal to C and parallely displaced along C, i.e.

$$\frac{D}{ds} v^i = \alpha(s)\, v^i \qquad (9)$$

(Lévi-Civita transfer conserves orthogonality). Now consider all the geodesics crossing C with tangent vector v^i at the intersecting point, and parametrize them as $x^i = x^i(v)$. They form a bidimensional surface $x^i = x^i(s,v)$, and they are the s = const. lines. We have, from the above definition, that the v = const. lines (C is one of them) are parallel to each other if their tangent vectors can be obtained from A^i (the tangent vector to C) by parallel transfer along the s = const. lines. In such a case we have over the surface $x^i = x^i(s,v)$ that

$$g_{ij}\frac{\partial x^i}{\partial s}\frac{\partial x^j}{\partial v} = 0 \tag{10}$$

Deriving this equation twice, once with respect to v and another to s, and combining them, having (9) into account, it immediately results $\alpha(s) = 0$, i.e.

$$\frac{D}{ds}V^i = 0 \tag{11}$$

In other words: a vector V^i defines parallel curves to C if and only if it satisfies eq. (11), named the equation of equipolent transfer. Its name derives from the fact that in a riemannian manifold eq. (11) implies that V^i is not only parallel transferred, but it also has always the same norm. In a Weyl space this has no sense, but it means that the above construction gives an orthogonal net. The vector V^i determines the surface in which the parallel curves of the net are contained.

c) Geodesic clock.

We proceed to the construction of the geodesic clock. Let C be the curve where we want to define proper time, and V^i an orthogonal vector to C equipolently transported along C, and so defining parallel paths with respect to C. Suppose we have a light pulse bouncing back and forth between C and a parallel particle (fig. 8). The definition arises naturally: proper time will be a parameter along C which suffers identical increments for each oscillation of the light pulse in every region of the world line of the particle.

In first approximation the tangent vector to the light ray at C, L^i, is given by

$$L^i = V^i + \frac{\Delta t}{2}\frac{dx^i}{dt}$$

It is a null vector, so $g_{ij}L^iL^j = 0$, and then, taking into account that V^i is orthogonal to C, it implies that

$$\frac{g_{ij}V^iV^j}{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}} = -\left(\frac{\Delta t}{2}\right)^2 \tag{12}$$

The parameter t will be, according to the previous definition, the pro per time along C, only if Δt is constant over C. This means, from eq. (12), that all along C the vector V^i, equipolently transported, is congruent to the tangent vector of the curve, evaluated with respect to t. Remembering the introduction to this section, this result implies that t is the arc length along C. The ambiguity by a linear transformation physically represents the freedom in the choice of the measuring scale and on the origin of time.

Then proper time, as defined by the geodesic clock constructed with a physically well motivated concept of parallelism, coincides with the arc length along timelike curves in a Weyl space (and with the affine parameters when the curves are geodesics).

8. RIEMANNIAN STRUCTURE.

In a Weyl space the arc length is a well defined concept. This fact is equivalent to the property that two vectors, if congruent at a point of a curve, remain congruent when they are equipolently transported along it. Nevertheless if each vector is transported to the same point over different paths, they will not be congruent in general.

We saw that the arc length coincides with the measure of a geodesic clock, and so we can ask about was does the above geometric property imply in terms of physical concepts. It means that if we define at a point a unit for measuring time, and we construct two geodesic clocks ticking once each unit of time at that event, and then we let them be carried along different world lines, and we make them cross again, the time unit transported to the intersection event will be different for each clock. In other words: two different geodesic clocks at the same event makes different measures depending on its previous histories. This seems unacceptable for actual space-time, so we postulate our last axiom

R: If two geodesic clocks intersect twice, they have the same relative rates at the crossing events.

In view of the above discussed relation between proper time and arc length, this axiom implies that if we have two vectors which are congruent at a given event, and we transport them to another event along different curves, they will be again congruents. This property characterices a riemannian manifold. In fact, it is easy to see that for every vector T^i of the tangent space at every point of M, the function $g_{ij}\, T^iT^j$ is then a scalar one.

So, in the expression $\nabla_i\, g_{jk} = \lambda_i\, g_{jk}$ the vector λ_i will verify $\lambda_i = \frac{\partial}{\partial x^i} \ln(g_{ij}\, T^iT^j)$. Being the gradient of a scalar function, a gauge transformation of λ_i can be made, obtaining $\nabla_i\, g_{jk} = 0$ [11], expres sion that defines a riemannian geometry.

Although axiom R seems to be a very reasonable one, it is not really well founded, because a geodesic clock is actually an ideal experience. It seems natural because there exists a large amount of experimental evidence about the integrability of the time units defined by atomic clocks [21]. So R would be well founded only if we additionally accept the equality of atomic time and gravitational standard time. As said in EPS, it would be better to test this equality experimentally, or to deduce it from a theory embracing both kind of phenomena.

Instead we shall show now, as an argument of plausibility for axiom R, that the most natural generalization for the equation of motion of a photon to a Weyl space is, in fact, only possible if the space is actually a riemannian one.

Suppose we want to make local measurements in space-time assuming it is a Weyl space. Then we must fix in an arbitrary way a particular g_{ij} among all possible ones. One way of doing such a thing would be to use a physically admisible system of coordinates S [22]. It consists of a fluid of parallel freely falling particles, each one carrying a geodesic clock. Then, at a spacelike surface orthogonal to the particle fluid, we fix a time unit (and so doing we fix the g_{ij} at the surface). Each particle carries this unit (and then the g_{ij}) by means of its clock. In this way we have selected a g_{ij} in the region of space-time where we want to measure. For example we can define the relative standard time between events x^i and $x^i + dx^i$ as the projection of its separation over the respective particle of the fluid

$$dT = \frac{1}{c} (g_{ij} \gamma^i dx^j)^{1/2} \tag{13}$$

where γ^i is the tangent vector to the particle of the fluid.

Suppose now that a particle P: $x^i = x^i(s)$, with tangent vector T^i parallel to γ^i, emits a light pulse whose wave fronts are parametrized as $x^i = x^i(1)$, with tangent vector L^i. Each wave front corresponds to a different value of s over P. Assume that s increases a unit from each wave front to the following one. Then the relative standard time between a wave front and the next one would be given by

$$\Delta T = \frac{1}{c} (g_{ij} V^i V^j)^{1/2}$$

and the frequency would be

$$\nu = \frac{c}{(g_{ij} V^i V^j)^{1/2}} \tag{14}$$

An electromagnetic wave can also be associated with a set of photons with four-momentum given, in special relativity, by the expression

$$P^i = \frac{h\nu}{c^2}\frac{dx^i}{dt} \tag{15}$$

and satisfying the following equation of motion

$$\frac{dP^i}{dt} = 0 \tag{16}$$

The definition of the four-momentum of a photon is usually generalized to curved space using covariant derivatives with respect to the relative standard time [23], because it is the most natural expression that coincides in a freely falling system with (15). So we define

$$P^i = \frac{h\nu}{c^2}\frac{Dx^i}{dT} \tag{17}$$

and we evaluate, using (14), its covariant derivative

$$\frac{DP^i}{dT} = \frac{h}{c^2}\frac{d\nu}{dT}\frac{dx^i}{dT} + \frac{h\nu}{c^2}\frac{D}{dT}\left(\frac{dx^i}{dl}\cdot\frac{dl}{dT}\right) = P^i\,\frac{d\nu/dT}{\nu} - \frac{\frac{d}{dT}\left(\frac{dT}{dl}\right)}{\frac{dT}{dl}} = -P^i\,\frac{d}{dT}\ln\frac{\frac{dT}{dl}}{\nu} \tag{18}$$

where l is an affine parameter along the photon path. Using (13) and (14)

$$\frac{DP^i}{dT} = -\frac{P^i}{g_{ij}\;T^i\;L^j}\frac{d}{dT}\left(g_{ij}\;T^i\;L^j\right)$$

$$\frac{DT^i}{dl} = \frac{DL^i}{ds} \text{ and } \frac{d}{ds}\left(g_{ij}\;L^i\;L^j\right) = 0, \text{ so } g_{ij}\frac{DL^i}{ds}L^j = 0.$$

l is an affine parameter along the photon world line, so $\frac{DL^i}{dl} = 0$. Recalling also that in a Weyl space $\nabla_i\;g_{jk} = \lambda_i\;g_{jk}$

$$\frac{d}{dT}\left(g_{ij}\;T^i\;L^j\right) = \frac{D}{dl}\left(g_{ij}\;T^i\;L^j\right)\frac{dl}{dT} = \lambda_k\frac{dx^k}{dT}$$

So, finally

$$\frac{DP^i}{dT} = -\,P^i\,\lambda_k\,\frac{dx^k}{dT} \tag{19}$$

This equation is valid for every $\frac{dx^k}{dT}$ timelike, so the unique way in which it can coincide with equation (16) for a freely falling observer, and so express the four-momentum conservation, is when $\lambda_k = 0$. This means that the space-time must be in fact a riemannian one.

FIGURES

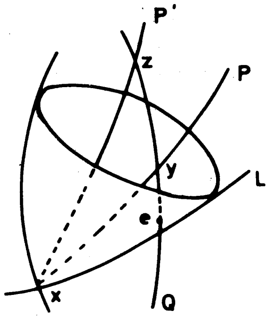

Figure 1

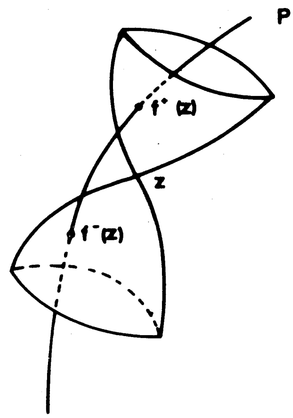

Figure 2: Message functions

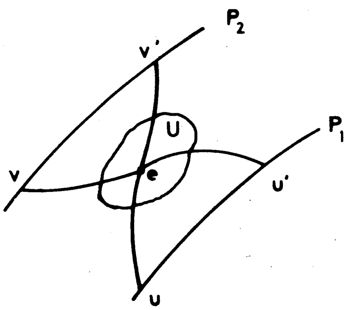

Figure 3: Radar coordinates

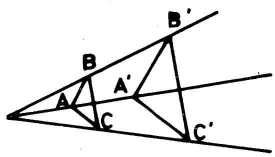

Figure 4: Desargues Theorem

AB//A' B', AC//A' C' $\Rightarrow$ BC//B' C'

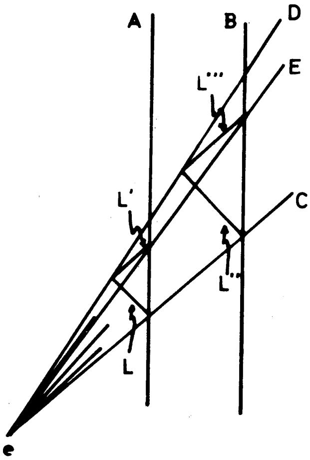

Figure 5

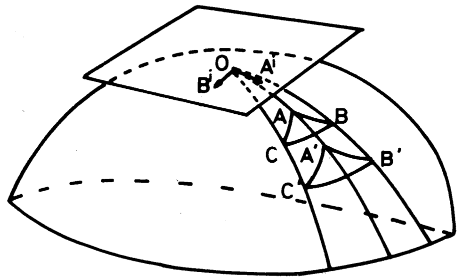

Figure 6

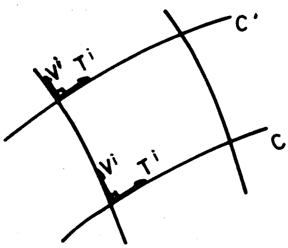

Figure 7: Parallel paths

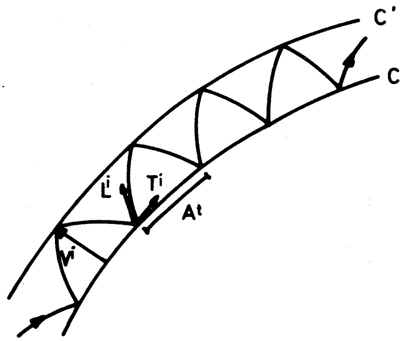

Figure 8: Geodesic clock

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