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ON INVERSE PROBLEMS FOR SECOND-ORDER DIFFERENTIAL OPERATORS WITH BOUNDARY DEPENDENCE ON THE EIGENVALUE PARAMETER

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Dedicated to the Memory of Antonio A. Ribeiro Monteiro

1. INTRODUCTION.

In paper [5] Y.Li proved the following results:

THEOREM 1. *Assume that* $q(x)$ *is a real integrable function on* $[0,\pi]$ *and that* a *is a real non-null constant.*

i) *The spectrum of the boundary problem*

(1) $y'' + [\lambda - q(x)]y = 0 \quad , \quad \lambda = s^2 \quad , \quad 0 \leqslant x \leqslant \pi,$

(2) $y(0) = 0 \quad , \quad ay'(\pi) + sy(\pi) = 0 \quad ,$

uniquely determines $q(x)$ *almost everywhere.*

ii) *The spectrum of* (1) *with boundary conditions*

(3) $y(0) = 0 \quad , \quad y'(\pi) - isy(\pi) = 0 \quad ,$

uniquely determines $q(x)$ a.e..

The assumption $a \neq 0$ is essential. In [3], p.179, H.Hochstadt shows that Theorem 1, i) is logically equivalent to the following result:

THEOREM 2. *Assume* $q(x)$ *is a real function in* $L^1(0,\pi)$, *that* $f(s)$ *is an odd real entire function of* s *of order less than one and that* β *and* γ *are different numbers in* $[0,\pi)$. *Consider the boundary conditions:*

$$(4)\quad \begin{aligned} & y(0)\cos\alpha + y'(0)\sin\alpha = 0 \\ & [y(\pi)\cos\beta + y'(\pi)\sin\beta] + f(s)[y(\pi)\cos\gamma + y'(\pi)\sin\gamma] = 0. \end{aligned}$$

The eigenvalues of (1) *and* (4) *uniquely determine* $q(x)$ *a.e..*

The proof is based on the idea that the knowledge of the spectrum of (1)-(4) is equivalent to the knowledge of several spectra, in particu lar that of (1) with two sets of regular boundary conditions. Thus, the proof of Theorem 2 is reduced to the following theorem, mainly due to G.Borg but in its actual form due to N.Levinson.

THEOREM 3. *Let* $q(x) \in L^1(0,\pi)$ *be a real function and* $0 \leqslant \alpha,\beta,\gamma < \pi$, $\beta \neq \gamma$. *Assume given the two spectra corresponding to* (1) *and the boundary conditions*

$$(5) \qquad \begin{aligned} y(0)\cos\alpha + y'(0)\sin\alpha &= 0 ,\\ y(\pi)\cos\beta + y'(\pi)\sin\beta &= 0 ,\end{aligned}$$

$$(6) \qquad \begin{aligned} y(0)\cos\alpha + y'(0)\sin\alpha &= 0 ,\\ y(\pi)\cos\gamma + y'(\pi)\sin\gamma &= 0 .\end{aligned}$$

Then, α,β,γ *and the spectra determine* $q(x)$ *uniquely (a.e.).*

In this paper our aim is to generalize these results. Next Theorem 7 generalizes Li's theorem and the content of our Theorem 9 is a partial generalization of Hochstadt's theorem.

2. BASIC NOTATION.

The differential equation (1) will be denoted by (q) and the first boundary condition in (5) and (6) by $\{\alpha$, the second ones by $\beta\}$ and $\gamma\}$ respectively. Thus, (q), $\{\alpha;\beta\}$ will denote the boundary problem (1) (5). Instead of $\{\alpha$ we could have

$$(7) \qquad P(\lambda).y(0) + Q(\lambda).y'(0) = 0$$

a boundary condition that will be denoted by $\{P,Q$. The corresponding boundary condition with 0 replaced by π will denoted by $P,Q\}$. Here, P and Q are entire functions of λ and more precisely, polynomials in $\lambda = s^2$. If instead we had

$$(8) \qquad R(s).y(\pi) + S(s).y'(\pi) = 0$$

with R and S polynomials in s, we shall write $R,S\rangle$.

We shall assume that the polynomials $P(\lambda),Q(\lambda)$ that appear in (7), or in the corresponding pair at $x = \pi$, satisfy the following hypotheses:

I) if one of them is identically zero the other one is identically one.

II) $\text{g.c.d.}(P(\lambda),Q(\lambda)) = 1$.

For boundary conditions of type (8) we shall also assume that R(s) and S(s) satisfy I), II).

$q(x)$ will always denote a function in $L^1(0,\pi)$.

3. AUXILIARY RESULTS.

Let us denote by $U_s(x)$ the solution of (q) $y'' + (\lambda - q)y = 0$ such that $U_s(0) = -Q(\lambda)$, $U'_s(0) = P(\lambda)$, and by $\widetilde{U}_s(x)$ the solution that $\widetilde{U}_s(\pi) =$

$= -S(s)$, $\tilde{U}'_s(\pi) = R(s)$.

$W(P,Q ; R,S)(s)$ is the characteristic function of the problem (q), $\{P(\lambda),Q(\lambda) ; R,S\rangle$ defined by the wronskian $W(U_s,\tilde{U}_s) = W(s)$:

$$(9) \qquad W(P,Q ; R,S)(s) = U_s(x)\tilde{U}'_s(x) - U'_s(x)\tilde{U}_s(x) = W(s).$$

The zeroes of the entire function W(s) are, by definition, the eigenvalues, and the spectrum is this set of zeroes, each zero counted as many times as its multiplicity. By the multiplicity of an eigenvalue we shall understand its multiplicity as a zero of W, so, a simple eigenvalue is a simple zero of W(s).

THEOREM 4. *Consider the problem* (q) $\{P,Q ; R,S\rangle$ *where* q *is complex-valued and* P,Q,R *and* S *are complex polynomials. Assume that* R *and* S *are not simultaneously even polynomials in* s. *Then,*

(i) *the spectrum and the boundary conditions determine completely the characteristic function* $W(P,Q ; R,S)(s)$,

(ii) *for any pair of polynomials* $X(s),Y(s)$, *the characteristic function* $W(P,Q;X,Y)$ *is completely determined by* $W(P,Q;R,S)(s)$.

Proof. (i) is nothing but Theorem 1, [6] , §2. (Observe that hypothesis iv) in p.7 is verified in our situation).

ii) In the proof that follows we make use of the following

PROPOSITION 1. R(s) *and* S(s) *are not simultaneously even if and only if* $R(s).S(-s)$ *is not even.*

(In fact, assume R(s) not even. Then there exists s_0, root of $R(s) = 0$, such that $R(-s_0) \neq 0$. If $R(s).S(-s)$ were even, s_0 and $-s_0$ would be roots of this polynomial and therefore $S(s_0) = 0$. But then g.c.d.(R,S) $\neq 1$, a contradiction).

From proposition 1 it follows that $\Delta = R(-s).S(s) - R(s).S(-s) \not\equiv 0$ and therefore for s such that $\Delta(s) \neq 0$, $\tilde{U}_s(x)$ and $\tilde{U}_{-s}(x)$ are linearly independent. Let us call $\tilde{V}_s(x)$ the solution of (q) that verifies the initial conditions:

$$(10) \qquad \tilde{V}_s(\pi) = -Y(s) \quad , \quad \tilde{V}'_s(\pi) = X(s).$$

Then, if s is not a zero of the polynomial $\Delta(s)$ and

$$(11) \qquad \begin{aligned} A &= (Y(s).R(-s) - X(s).S(-s))\Delta(s)^{-1}, \\ B &= (X(s).S(s) - Y(s).R(s))\Delta(s)^{-1} \end{aligned}$$

we have

$$(12) \qquad \tilde{V}_s(x) = A.\tilde{U}_s(x) + B.\tilde{U}_{-s}(x).$$

Therefore, for s such that $\Delta(s) \neq 0$:

$$(13)\quad W(P,Q\ ;\ X,Y)(s) = -P(\lambda).\widetilde{V}_s(0) - Q(\lambda).\widetilde{V}'_s(0) =$$
$$= -P(s^2).(A.\widetilde{U}_s(0) + B.\widetilde{U}_{-s}(0)) - Q(s^2).(A.\widetilde{U}'_s(0) + B.\widetilde{U}'_{-s}(0)) =$$
$$= A(s).W(P,Q\ ;\ R,S)(s) + B(s).W(P,Q\ ;\ R,S)(-s),\quad \text{QED}.$$

THEOREM 5. *Consider the boundary problem* (q), $\{P,Q;\beta\}$, q(x) *a complex-valued function. Except for a denumerable set of* β*'s in* $[0,\pi)$ *the spectrum of the problem is simple (i.e. all the eigenvalues are simple).*

Proof. Next proposition will be used in the proof:

PROPOSITION 2. $U_s(\pi)$ *and* $U'_s(\pi)$ *are linearly independent.*

(In fact, if $F_1 = P - i.s.Q$, $F_2 = P + i.s.Q$ then these polynomials have the same degree m and

$$(14)\quad \begin{cases} U_s(\pi) = \dfrac{F_1(s)\ e^{is\pi} - F_2(s)\ e^{-is\pi}}{2is} + O(e^{|\mathrm{Im}\ s|\pi}.|s|^{m-2}) \\[2ex] U'_s(\pi) = \dfrac{F_1(s)\ e^{is\pi} + F_2(s)\ e^{-is\pi}}{2} + O(e^{|\mathrm{Im}\ s|\pi}.|s|^{m-1}) \end{cases}$$

as it follows from (5) and (8), [6], §2. For $s = -it$, $t > 0$, we have: $U_s(\pi) \sim c.t^{m-1}.e^{\pi t}$ and $U'_s(\pi) \sim c'.t^m.e^{\pi t}$, and the proposition follows).

In consequence,

$$(15)\qquad D(s) = \begin{vmatrix} U_s(\pi) & U'_s(\pi) \\ \dfrac{d\ U_s(\pi)}{ds} & \dfrac{d\ U'_s(\pi)}{ds} \end{vmatrix} \neq 0.$$

Let us call z(x,s) the solution such that $z(\pi,s) = -\sin\beta$, $z'(\pi,s) = \cos\beta$. Then $W(P,Q\ ;\ \beta)$ is equal to:

$$(16)\qquad W(U_s(x),z(x,s)) = U_s(\pi).\cos\beta + U'_s(\pi).\sin\beta\ .$$

If s were a non-simple eigenvalue, we would have

$$(17)\qquad \begin{cases} U_s.\cos\beta + U'_s.\sin\beta = 0\ , \\[1ex] \dfrac{d\ U_s}{ds}.\cos\beta + \dfrac{d\ U'_s}{ds}.\sin\beta = 0. \end{cases}$$

In consequence, $W(s) = 0$, $D(s) = 0$. Since always $|U_s| + |U'_s| \neq 0$, for each zero s_0 of D(s) it will exist exactly a unique β_0 such that $W(s_0) = 0$. QED.

EXAMPLE. Next we show that it is necessary to exclude certain β's for the preceding theorem to hold. Let us assume that u and v are two (real) consecutive normalized eigenfunctions of the problem (0), $\{\alpha;\beta\}$, $0 < \alpha,\beta < \pi$, and λ and μ the corresponding eigenvalues. Then $u+iv \neq 0$ on $[0,\pi]$. Besides, the function $y = u+iv$ satisfies the equation:

$$y'' + q(x).y = 0 ,$$

$$q(x) = \frac{\lambda u(x) + i\mu v(x)}{u(x) + iv(x)}$$

and the relation: $\int_0^\pi y^2(x)\,dx = 0$. From the corollary to next theorem 6 we know then that the spectrum of (q), $\{\alpha;\beta\}$ is not simple.

4. MAIN RESULTS.

The following result is a complement to Theorem 3.

THEOREM 6. *Consider the boundary value problems* (q) $\{\alpha;\beta\}$ *and* (q) $\{\alpha;\gamma\}$ *where* $0 \leqslant \alpha,\beta,\gamma < \pi$, $\gamma \neq \beta$, *and* q *is complex-valued. If the corresponding characteristic functions* $W(\alpha,\beta)(\lambda)$ *and* $W(\alpha,\gamma)(\lambda)$ *have simple zeroes then they uniquely determine* q(x) *a.e..*

Proof. The proof of Theorem 1 in Levinson [4] can be repeated verbatim to show that

$$\text{(18)} \qquad f(x) = \sum_{n=0}^{\infty} \frac{u_2(x,\lambda_n) . \int_0^\pi u_1(\xi,\lambda_n)\, f(\xi)\, d\xi}{c_n . W'(\alpha,\beta)(\lambda_n)}$$

where c_n is a non-null constant and the function $u_i(x,\lambda_n)$ is an eigenfunction corresponding to the eigenvalue λ_n for the problem (q_i), $\{\alpha;\beta\}$, $i = 1,2$.

q_1 and q_2 are the two complex-valued L^1-functions that are assumed to have the same characteristic functions $W(\alpha,\beta)$ and $W(\alpha,\gamma)$. The series converge uniformly on compact sets of $(0,\pi)$ and boundedly on $[0,\pi]$ for $f \in C^1([0,\pi])$, $f(0) = f(\pi) = 0$, f real.

It is proved as usual for $\lambda_n \neq \lambda_m$ that

$$\text{(19)} \qquad \int u_2(x,\lambda_n) . u_2(x,\lambda_m)\, dx = 0$$

Assume that (19) holds for m=n. Then, we would have $\bar{u}_2(x,\lambda_n) \perp$ $\perp u_2(x,\lambda_j)$, $\forall$ j, a contradiction since (18) shows that $\{u_2(x,\lambda_j)\}$ is a complete set in L^2.

Then, $\{u_2(x,\lambda_j)\}$ and $\{\bar{u}_2(x,\lambda_k)\}$ form a complete biorthogonal set in L^2. This implies that, after a suitable normalization, $u_2(x,\lambda_n) =$ $= u_1(x,\lambda_n)$ holds for every n, and therefore that $q_1(x) = q_2(x)$ a.e.
QED.

COROLLARY. *If the problem* (q), $\{\alpha:\beta\}$, q(x) *a complex-valued function, has a simple spectrum, then any eigenfunction* u *verifies:*

$$\int_0^\pi u^2\, dx \neq 0.$$

THEOREM 7. *Consider the boundary value problem* (q), $\{\alpha;\ R,S\rangle$, *with* $0 \leq \alpha < \pi$, R *and* S *not simultaneously even. Then, the boundary conditions and the spectrum determine completely the potential* q(x) *even for* q(x) *a complex-valued integrable function.*

Proof. For any β, $W(\alpha,\beta)(s)$ is uniquely determined by $W(\alpha\ ;\ R,S)$ (Theorem 4) and for an infinite number of β's, $W(\alpha,\beta)$ has simple zeroes (Theorem 5). Theorem 7 follows now from Theorem 6. QED.

The following result is proved in [1] following the same line of proof as in Levinson's paper.

THEOREM 8. *Assume that* q(x) *is a real function and the boundary condition:*

$$[\beta:\ -(\beta_1 y(0) + \beta_2 y'(0)) = \lambda(\beta_1' y(0) + \beta_2' y'(0))$$

verifies $\beta_1'\beta_2 - \beta_1\beta_2' > 0$. *For* $0 \leq \gamma,\delta < \pi$, $\gamma\neq\delta$, *the spectra of the boundary value problems* (q) $[\beta;\gamma\}$ *and* (q)$[\beta;\delta\}$ *and the boundary conditions determine* q(x) *uniquely a.e..*

THEOREM 9. *Consider the boundary value problem* (q)$[\beta;\ R,S\rangle$ *with* q(x) *real-valued, and* R,S *not simultaneously even. Then, the spectrum and the boundary conditions uniquely determine* q(x) *a.e..*

Proof . The same as in Theorem 7 but using Theorem 8 instead of Theorem 6. In fact, the characteristic functions of problems (q) $[\beta;\gamma\}$ and (q)$[\beta;\delta\}$ are determined in particular for some pair γ,δ such that $\sin(\gamma-\delta) \neq 0$. This unequality implies that these functions have no zero in common. QED.

REFERENCES

[1] BENEDEK,A.I. and PANZONE,R., *On inverse eigenvalue problems for a second-order differential equation with parameter contained in the boundary conditions*, Notas de Algebra y Análisis N°9, Instituto de Matemática, Universidad Nacional del Sur, Argentina, (1980), 1-13.

[2] BORG,G., *Eine Umkehrung der Sturm-Liouvilleschen Aufgabe*, Acta Math. 78, (1945), 1-96.

[3] HOCHSTADT,H., *On inverse problems associated with second-order differential operators*, Acta Mathematica. 119, (1967), 173-192.

[4] LEVINSON,N., *The inverse Sturm-Liouville problem*, Mat. Tidsskr., B (1949), 25-30.

[5] LI,YI-SHEN, *On an inverse eigenvalue problem for a second-order differential equation with boundary dependence on the parameter*, Acta Mathematica Sinica, 15, (1965), 375-381.

[6] BENEDEK,A.I. and PANZONE,R., *On Sturm-Liouville problems with the square-root of the eigenvalue parameter contained in the boundary conditions*, Notas de Algebra y Análisis N°10, Instituto de Matemática, Universidad Nacional del Sur, Argentina, (1981), 1-62.

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