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ON SCALAR CONCOMITANCE OPERATORS

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INTRODUCTION.

The theory of concomitants had a great development during the last decade, due to the systematic use of the invariance identitites [5]. Being its language the classical tensor analysis, sometimes there is a lack of precision both in the results and in the techniques of proof. Following the current trends about the use of the language of jets for every local problem in differential geometry [1], in this paper we study a possible setting of the concomitants in that context. We do not consider the most general possible case, i.e., a geometric object concomitant of certain geometric objects, but the most commonly used, i.e., scalars concomitants of arbitrary tensors and their derivatives up to a finite order.

In section 1, we give the preliminaries that we need in the following sections (for further information see [4]). In section 2, we propose the definition of the concomitance operators and the invariance identities they must satisfy, and we show how classical statements of the type "Let L be a concomitant of ..." can be translated into this language. Finally, in section 3, we show how the concepts developed in the previous section can be used to prove rigorously two particular results of the theory not previously known.

1. PRELIMINAIRES.

Let M and P be differentiable manifolds with dimensions m and m+n respectively, and let π: P $\longrightarrow$ M be a surjective submersion (i.e., its differential is everywhere surjective). A *local section* of π is a differentiable function s: U $\longrightarrow$ P, where U $\subset$ M is an open set, such that $\pi \circ s$ = id. We will denote $\Gamma(U,P)$ the set of all local sections whose domain is U.

For $s_1 \in \Gamma(U_1,P)$, $s_2 \in \Gamma(U_2,P)$ and $x_0 \in U_1 \cap U_2$, we say that s_1 and s_2 are equivalent if, in local coordinates, their partial derivatives up to k-th order are the same at x_0. Let $P^k(x_0)$ be the quotient and let P^k be the union of all $P^k(x_0)$ for $x_0 \in M$. We define the projection π^k: $P^k \longrightarrow M$ the natural way, $\pi^k(y) = x$ iff $y \in P^k(x)$.

For $s \in \Gamma(U,P)$ we define

$$J^k(s): U \longrightarrow P^k$$

as the mapping whose value at x_0, $J^k(s)(x_0)$, is the equivalence class of s for the relation defined above. We will say that $J^k(s)$ is the k-jet of s. We also define, if $k \geq h$, $\pi^k_h: P^k \longrightarrow P^h$ by

$$\pi^k_h(J^k(s)(x)) = J^h(s)(x)$$

It is easy to see that $P^o \simeq P$.

An *adapted chart* is a 3-tuple (U,ϕ_0,ϕ), where $U \subset P$ is an open set and $\phi_0: \pi(U) \longrightarrow R^m$, $\phi: U \longrightarrow R^m \times R^n$ are diffeormisms commuting the diagram

$$\begin{array}{ccc} U & \xrightarrow{\phi} & R^m \times R^n \\ \pi\downarrow & & \downarrow P_1 \\ \pi(U) & \xrightarrow{\phi_0} & R^m \end{array}$$

(P_1: projection onto the first factor).

For an adapted chart (U,ϕ_0,ϕ), let

$$V_U = (\pi^k_0)^{-1}(U)$$

$$h = m + n + n \sum_{i=1}^{k} c'_{m,i} \qquad (1)$$

where $c'_{m,i}$ is the number of combinations with repetition of m elements taken i times.

If $y \in V_U$, then it will be $y = J^k(s)\ (\pi^k(y))$ for some $s \in \Gamma(W,U)$. We define:

$$s^i = \phi^i \circ s \circ \phi_0^{-1} \qquad (1 \leq i \leq n)$$

$$s^i_{\alpha_1 \ldots \alpha_t} = D_{\alpha_1 \ldots \alpha_t}(s^i) \qquad (1 \leq t \leq k\ ,\ 1 \leq \alpha_1 \leq \ldots \leq \alpha_t \leq n)$$

and $\phi^k: V_U \longrightarrow R^h$ by

$$\phi^k(y) = (\phi_0(x), s^i(\phi_0(x)), \ldots, s^i_{\alpha_1 \ldots \alpha_k}(\phi_0(x))) ,$$

where $x = \pi^k(y)$. The family of all (V_U,ϕ^k) is an atlas for P^k, and so P^k is a differentiable manifold with dimension h given by (1).

2. CONCOMITANCE OPERATORS.

Let M be a differentiable manifold with dimension m and, for a natural number c and non-negative integers $r_1,s_1,\ldots,r_c,s_c$, let

$$P = \bigcup_{p \in M} (T^{r_1}_{s_1}(M_p) \times \ldots \times T^{r_c}_{s_c}(M_p)) , \qquad (2)$$

where $T^r_s(M_p)$ is the tensor product of M_p (r factors) and M^*_p (s factors). As in the case of tensor bundles, it is easy to see that P has a natural differential structure with dimension:

$$m + n = \sum_{i=1}^{c} m^{r_i+t_i} + m \tag{3}$$

The natural projection $\pi: P \longrightarrow M$ is clearly a surjective submersion. To have a section $s \in \Gamma(U,P)$ is to have c sections $s_j \in \Gamma(U,T^{r_j}_{s_j}(M))$ such that:

$$s(p) = (s_1(p),\ldots,s_c(p)) \quad \text{for all } p \in U$$

If $A = \{(x,U)\}$ is an atlas for M with $x(U) = R^m$, then we have an atlas $A' = \{(t_U,\pi^{-1}(U))\}$ for P, where $t_U(\pi^{-1}(U)) = R^m \times R^n$. Using the results outlined at the previous section, we obtain an atlas

$A^k = \{(\phi^k,(\pi^k_0)^{-1}(\pi^{-1}(U))\}$ for P^k such that $\phi^k((\pi^k_0)^{-1}(\pi^{-1}(U))) = R^h$,

where h is given by (1) and n is given by (3).

Let (x,U) be an arbitrary but fixed chart for M with $x(U) = R^m$, and for each chart $(\bar{x},U)$ with $\bar{x}(U) = R^m$ let

$$\psi^k_p(\bar{x}) : R^{h-m} \longrightarrow R^h$$

be the map given by

$$\psi^k_p(x)\ (b^i,b^i_{\alpha_1},\ldots,b^i_{\alpha_1,\ldots,\alpha_k}) = \bar{\phi}^k \circ (\phi^k)^{-1}\ (x(p),b^i,\ldots,b^i_{\alpha_1,\ldots,\alpha_k})$$

where b^i are the last coordinates on P given by the adapted chart (and so $1 \leqslant i \leqslant n$, where n is given by (3)).

Let A be the set of all such functions $\psi^k_p(\bar{x})$ for $(\bar{x},U)$ a chart on M with $\bar{x}(U) = R^m$.

LEMMA. $A \simeq GL(m,R)\times R^q$, *where* $q = \dim \bar{P}^{k+1} - 2m - m^2$, *and* $\bar{P} = M\times M$ *with the natural projection onto the first factor.*

Proof. We define $f: A \longrightarrow GL(m,R)\times R^q$ by

$$f(\psi^k_p(\bar{x})) = (B^i_j\ (p),B^i_{j_1j_2}(p),\ldots,B^i_{j_1j_2\ldots j_{k+1}}(p))\ ,$$

where

$$B^i_{j_1\ldots j_s} = \frac{\partial^s x^i}{\partial\bar{x}^{j_1}\ldots\partial\bar{x}^{j_s}}$$

and $1 \leqslant j_1 \leqslant j_2 \leqslant \ldots \leqslant j_s \leqslant m$ for all $1 \leqslant s \leqslant k+1$.

Being $\bar{x}\circ x^{-1}$ an invertible mapping, then it is clear that

$(B^i_j(p)) \in GL(m,R)$, and so $f(A) \subset GL(m,R) \times R^q$. Let

$$g: GL(m,R) \times R^q \longrightarrow A$$

be the mapping defined in the following way. For $(c^i_j, c^i_{j_1 j_2}, \dots, c^i_{j_1 \dots j_{k+1}}) \in GL(m,R) \times R^q$, let $(\bar{x},U)$ a chart on M with $\bar{x}(U) = R^m$ such that:

$$\begin{aligned} B^i_j(p) &= c^i_j \\ &\vdots \\ B^i_{j_1 \dots j_{k+1}}(p) &= c^i_{j_1 \dots j_{k+1}} \end{aligned}$$

(there is such a chart; just compose x with the polynomial of degree $\leqslant k+1$ whose coefficients are all $c^i_{j_1 \dots j_s}$). We define then:

$$g(c^i_j, \dots, c^i_{j_1 \dots j_{k+1}}) = \psi^k_p(\bar{x})$$

and it follows at once that $f \circ g = id$, $g \circ f = id$. ///

Each element of A is a (differentiable) function from R^{h-m} to R^h, then the lemma allows us to define an action of $GL(m,R) \times R^q$ over R^{h-m}. Let

$$H^k: GL(m,R) \times R^q \times R^{h-m} \longrightarrow R^{h-m}$$

be defined as

$$H^k(B,b) = \pi'(f^{-1}(B)(b)) \ ,$$

where $\pi': R^h \longrightarrow R^{h-m}$ is the projection onto the last h-m coordinates. From the transformation law for tensors and their derivatives, it is clear that H^k is a differentiable function.

We will say that an open set $V \subset R^{h-m}$ is *invariant* by H^k if $H^k(GL(m,R) \times R^q \times V) \subset V$.

Now we are ready to give the general definition of the concomitance operators:

DEFINITION 1. An $(r_1,s_1,\dots,r_k,s_k)$-*scalar concomitance* k-*operator* is a differentiable function

$$F: V \subset R^{h-m} \longrightarrow R \ ,$$

where V is an invariant open set, such that

$$F(b) = F \circ H^k(B,b) \tag{4}$$

for all $b \in R^{h-m}$ and $B \in GL(m,R) \times R^q$.

DEFINITION 2. If $F: V \subset R^{h-m} \longrightarrow R$ is an $(r_1,s_1,\dots,r_k,s_k)$-scalar concomitance k-operator, then, from (4) it follows that

$$X\,(F\circ H^k) = 0 \qquad (5)$$

for all vector X tangent to $GL(m,R)\times R^q \subset GL(m,R)\times R^q\times R^{h-m}$. The identities (5) are denoted *invariance identities* for F.

It is clear that for a differentiable mapping $F\colon V\subset R^{h-m}\longrightarrow R$, the identities (5) are necessary and sufficient for F to be a concomitance operator.

We are now in position, in each particular case, to say what it means for a scalar to be a concomitant of arbitrary tensors. For instance, if G is a 2-covariant tensor and X is a vector field, then we say that a scalar L (i.e., an element of $C^\infty(M)$) is a concomitant of G and X up to k-th order if there is a (0,2,1,0)-scalar concomitance k-operator $F\colon R^{h-m}\longrightarrow R$ such that, for every chart (x,U) on M:

$$L(p) = F(g_{ij}(p);g_{ij,h_1}(p);\dots;g_{ij,h_k}(p);a^i(p);\dots;a^i_{h_1\dots h_k}(p)) ,$$

where $G = g_{ij}\,dx^i\otimes dx^j$ and $X = a^i\,\dfrac{\partial}{\partial x^i}$ and a dot means partial derivation. It is here $n = m^2 + 2m$ and so $h\text{-}m = n+n\sum_{i=1}^{k} c'_{m^2+2m,i}$.

3. APPLICATIONS.

We are going now to study two cases of scalar concomitance. Let L be a scalar and let $w\in D_1(M)$, $X\in D^1(M)$ be an 1-form and a vector field respectively such that w(X) is nowhere null. We say that L is a concomitant of w and X if there is a (0,1,1,0)-scalar concomitance 0-operator $F\colon R^{2m}\longrightarrow R$ such that, for every chart (x,U):

$$L(p) = F(\psi_1(p),\dots,\psi_m(p),\phi^1(p),\dots,\phi^m(p)) ,$$

where $w = \psi_i\,dx^i$ and $X = \phi^i\,\dfrac{\partial}{\partial x^i}$.

THEOREM 1. *If* L *is a scalar concomitant of the* 1*-form* w *and the vector field* X*(with* w(X) *nowhere null), then there exists a function* $g\colon R_{\neq 0}\longrightarrow R$ *such that* $L = g\circ w(X)$.

Proof. Now it is k=0. If $H = H^0\colon GL(m,R)\times R^{2m}\longrightarrow R^{2m}$ is the action previously defined, then it follows at once that

$$H(B,a,b) = (B^i_j\,a_i;(B^{-1})^i_j.b^j)$$

where $B = (B^i_j)\in GL(m,R)$ and $(a,b)\in R^{2m}$.

Being q=0, the invariance identities (5) are equivalents to the m^2 identities

$$\frac{\partial}{\partial B^s_t}(F \circ H) = 0$$

It is easy to see that these identities, evaluated at B = I, are

$$F^i(a,b)a_j - F_j(a,b)b^i = 0 \quad \text{for all } (a,b) \in R^{2m}$$

where $F^i = D_i F$, $F_j = D_{m+j} F$.

If $(x_1,\dots,x_m,y^1,\dots,y^m)$ is the usual coordinate system en R^{2m}, then we may write the invariance identities as

$$F^i x_j = F_j y^i \qquad (1 \leqslant i , j \leqslant m)$$

Then, out of the coordinate axes, it is $F^i/y^i = F_j/x^j$. Then there is a function $c: R^{2m} \longrightarrow R$ such that

$$F^i = c\, y^i \quad , \quad F_j = c\, x_j \tag{6}$$

(everything being continuous, (6) is valid everywhere).

For each $t \neq 0$, let

$$S_t = \{(x,y) \in R^{2m}: x_i\, y^i = t\}$$

Then S_t is an hypersurface of R^{2m}. Let $i: S_t \longrightarrow R^{2m}$ be the inclusion.
Then:

$$\begin{aligned} d(F \circ i) &= i^*(dF) = i^*(F^i\, dx_i + F_i\, dy^i) = \quad (\text{using } (6)) \\ &= i^*(c\, y^i\, dx_i + c\, x_i\, dy^i) = i^*(c d(y^i\, x_i)) = \\ &= c \circ i\,(d(y^i\, x_i \circ i)) = 0 \end{aligned}$$

since $y^i\, x_i \circ i$ is the function constantly t. Then $F|_{S_t}$ = const. = = g(t), and so:

$$L(p) = F(\psi_i(p),\phi^j(p)) = g(\psi_i(p)\ \phi^i(p)) = g(w_p(X_p)) = g(w(X)(p)),$$

and so the theorem is proved. ///

We now consider $G \in D^o_2(M)$, a 2-covariant non-singular symmetric tensor, and $X \in D^1(M)$, a vector field nowhere null. Let $V = GL(m,R)\times R^m_{\neq 0}$. We say that a scalar L is a concomitant of G and X if there is a (0,2,1,0)-scalar concomitance 0-operator $F: V \longrightarrow R$ such that

i) $F|_{GL(m,R)\times\{a\}}$ is symmetric for every $a \in R^m_{\neq 0}$.

ii) if (x,U) is a chart on M, then $L(p) = F(g_{ij}(p), \phi^i(p))$,

where $G = g_{ij}\, dx^i \otimes dx^j$ and $X = \phi^i \dfrac{\partial}{\partial x^i}$.

THEOREM 2. *Let* L *be a scalar concomitant of the symmetric, non-singu-*

lar 2-covariant tensor G *and a vector field* X *nowhere null. Then there is a function* $h\colon R_{\neq 0} \longrightarrow R$ *such that*

$$L = h \circ G(X,X)$$

Proof. Once again it is k=0 and $H = H^o\colon GL(m,R) \times GL(m,R) \times R^m \longrightarrow$ $\longrightarrow GL(m,R) \times R^m$ is the function:

$$H(B,a,b) = (B_i^r\, B_j^\ell\, a_{r\ell}\ ,\ (B^{-1})_j^i\, b^j)$$

The invariance identities (5) are equivalent to the m^2 identities:

$$\frac{\partial}{\partial B_t^s}\,(F \circ H) = 0$$

and, evaluated at $B = I$, they are $F^{tj}(a,b)(a_{sj}+a_{js}) = F_s(a,b)\, b^t$, where $F^{tj} = \dfrac{\partial F}{\partial x_{tj}}$, $F_s = \dfrac{\partial F}{\partial y^s}$, and (x_{ij}, y^s) is the usual coordinate system on $GL(m,R) \times R^m$. If a is symmetric, then it is:

$$2\, F^{tj}\,(a,b)\, a_{sj} = F_s(a,b)\, b^t$$

or else:

$$F^{tj} = \frac{1}{2}\, F_s\, y^t\, x^{sj} \tag{7}$$

Since F is symmetric, we get from (7):

$$F_\ell\, y^j\, x_{js} = F_s\, y^j\, x_{j\ell}$$

and so we deduce, as in the previous theorem, that:

$$F_\ell = c\, x_{j\ell}\, y^j \tag{8}$$

Now, for each $t \neq 0$, let

$$S_t = \{(x,y) \in SGL(m,R) \times R^m\colon\ x_{ij}\, y^i\, y^j = t\}$$

where SGL(m,R) stands for the symmetric matrices of GL(m,R). If $i\colon S_t \longrightarrow SGL(m,R) \times R^m$ stands for the inclusion map, then:

$$\begin{aligned} d(F \circ i) = i^*\,(dF) &= i^*(F^{ij}\, dx_{ij} + F_s\, dy^s) = \text{(Using (7))}\\ &= i^*\,(\tfrac{1}{2}\, F_s\, y^i\, x^{sj}\, dx_{ij} + F_s\, dy^s) = \text{(Using (8))}\\ &= i^*\,(\tfrac{1}{2}\, c\, x_{hs}\, y^h\, y^i\, x^{sj}\, dx_{ij} + F_s\, dy^s) =\\ &= i^*\,(\tfrac{1}{2}\, c\, y^j\, y^i\, dx_{ij} + c\, x_{ij}\, y^i\, dy^j) =\\ &= i^*\,(\tfrac{1}{2}\, c\, d(y^i\, y^j\, x_{ij})) =\\ &= \tfrac{1}{2}\, c \circ i\,\,(d(y^i y^j . x_{ij} \circ i)) = 0 \end{aligned}$$

since $x_{ij}\, y^i\, y^j \circ i$ is the function constantly t. Then $F\big|_{S_t} = \text{const.} =$ $= h(t)$, and so:

$$L(p) = F(g_{ij}(p),\Phi^i(p)) = h(g_{ij}(p)\Phi^i(p)\Phi^j(p)) =$$
$$= h\ (G_p(X_p,X_p)) = h(G(X,X)\ (p))$$

and so the theorem is proved. ///

Finally, we remark that theorem A.1 from [2] and theorem 1 from [3] can be equally translated into this language.

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