THE PLENARY HULL OF THE GENERALIZED JACOBIAN MATRIX AND THE INVERSE FUNCTION THEOREM IN SUBDIFFERENTIAL CALCULUS

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INTRODUCTION.

Some of the most important recent advances in optimization have come about as a result of a systematic replacement of smoothness assumptions by convexity. This is exemplified by the work of Rockafellar [2]. It is natural to ask whether analogous results can be proven without either smoothness or convexity. A general theory of necessary conditions for such problems has been obtained [3]. The conditions are expressed, in part, by means of generalized gradients.

The classical inverse function theorem gives conditions under which a $C^r$ function admits (locally) a $C^r$ inverse. The purpose of this article is to give conditions under which a lipschitzian (not necessarily differentiable) function admits (locally) a lipschitzian inverse by means the characterization of the plenary hull of the generalized jacobian matrix.

1. LOCALLY LIPSCHITZ FUNCTIONS.

Let $F: \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz on $B$ a bounded subset of $\mathbb{R}^n$. That is, for each bounded subset $B$ of $\mathbb{R}^n$ there exists a constant $K$ such that

$$|f(x_1) - f(x_2)| \leq K|x_1 - x_2|$$

for all points $x_1$ and $x_2$ of $B$.

It is known [1] that such function has at almost all points $x$ a derivative (gradient), which we denote $\nabla f(x)$. It is easily verified that the function $\nabla f$ is bounded on bounded subsets of its domain of definition.

Let now $F$ be locally Lipschitz on $0$ a nonempty open subset of $\mathbb{R}^n$ and taking values in $\mathbb{R}^m$. One could be tempted to define the generalized derivative of $F = (f_1, \ldots, f_m)^t$ at $x_0 \in 0$ by simply considering $[\partial f_1(x_0), \ldots, \partial f_m(x_0)]^t$ where this set consists of matrices whose $i$th row belongs to $\partial f_i(x_0)$.

The usual $m \times n$ jacobian matrix of partial derivatives, when it exists,
is denoted $J_F(x)$. We topologize the vector space of $m \times n$ matrices with the norm

$$|M| = \max |m_{ij}| \text{ where } M = (m_{ij}), \ 1 \leq i \leq m, \ 1 \leq j \leq n$$

A mathematical tool is what Clarke, F.H. called the generalized jacobian matrix defined in the following way:

**DEFINITION 1.1.** The generalized jacobian matrix of $F$ at $x_0 \in \mathbb{O}$, denoted by $\mathcal{J}_F(x_0)$, is the convex hull of all matrices $M$ of the form

$$M = \lim_{n \to \infty} J_F(x_n) \text{ where } x_n \text{ converges to } x_0 \text{ in } \text{dom} F'.$$

In this definition, $\text{dom} F'$ denotes the subset of full measure of $\mathbb{O}$ where $F$ is differentiable.

In doing so, $\mathcal{J}_F(x_0)$ is nonempty compact convex subset of the vector space of $m \times n$ matrices, which is reduced to $\{J_F(x_0)\}$ whenever $F$ is strictly differentiable at $x_0$.

**DEFINITION 1.2.** $\mathcal{J}_F(x_0)$ is said to be of maximal rank if every $M$ in $\mathcal{J}_F(x_0)$ is of maximal rank.

2. **PLENARY HULL OF THE GENERALIZED JACOBIAN MATRIX.**

Let us denote $\langle \cdot, \cdot \rangle$ the inner product on the vector space of $m \times n$ matrices defined by $\langle M, U \rangle = \text{Trace of } M^t U$; it comes from Definition 1.1.

that for all $U \in \mathbb{R}^{m \times n}$:

$$\max_{M \in \mathcal{J}_F(x_0)} \langle M, U \rangle = \lim_{x \to x_0} \sup_{x \in \text{dom} F'} \langle J_F(x), U \rangle \quad (2.1)$$

We turn our attention to those $U \in \mathbb{R}^{m \times n}$ of the form $u \circ v: x \to (u, x)v$, where $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$. In such a way, $\langle M, u \circ v \rangle$ reduces to $\langle Mu, v \rangle$ and $(2.1)$ can be rephrased as:

$$\max_{M \in \mathcal{J}_F(x_0)} \langle Mu, v \rangle = \lim_{x \to x_0} \sup_{x \in \text{dom} F'} \langle J_F(x)u, v \rangle \quad (2.2)$$

We can use results on chain rules so that the left-hand of $(2.2)$ appears as the generalized gradient of a particular real-valued function. Given $v \in \mathbb{R}^m$, the generalized gradient of $F_v: x \to (F(x), v)$ at $x_0$ can be exactly described as: $\mathcal{G}_v(x_0) = \mathcal{J}^T F(x_0)v$, ([6]).

Therefore, for all $u \in \mathbb{R}^n$, we have that:
Although $\tilde{J}F(x_0)$ is convex and compact, one generally cannot separate an $M_0$ from $\tilde{J}F(x_0)$ by using only linear mappings (on $R^{m \times n}$) of the form $u \mapsto v$, $u \in R^n$, $v \in R^m$. This state of affairs led Sweetser [5] to introduce the following definition:

- a subset $A \subset R^{m \times n}$ is plenary if and only if it includes every $M$ in $R^{m \times n}$ satisfying $Mu \in Au$ for all $u \in R^n$.

Since the intersection of plenary sets is plenary, Sweetser defined the plenary hull of $A$, denoted $\text{plen } A$, as

- a smallest plenary set containing $A$.

Namely, when $\min(m,n) > 1$, $\text{plen } \tilde{J}F(x_0)$ is a convex compact (plenary) set of matrices containing $\tilde{J}F(x_0)$.

Since $\tilde{J}F(x_0)u = [\text{plen } \tilde{J}F(x_0)]u$ for all $u \in R^n$, Hiriart-Urruty [8], formulated the following theorem:

THEOREM 2.1. Let $u \in R^n$ and $v \in R^m$ then,

$$\max_{M \in \text{plen } \tilde{J}F(x_0)} \langle Mu, v \rangle = F^0(x_0; u, v)$$

(2.4)

In another setting, $M \in \text{plen } \tilde{J}F(x_0)$ if and only if

$$\langle Mu, v \rangle \leq F^0(x_0; u, v) \text{ for all } (u, v) \in R^n \times R^m.$$

To summarize, let's say that $\text{plen } \tilde{J}F(x_0)$ is the convex compact (plenary) set of matrices satisfying

$[\text{plen } \tilde{J}F(x_0)]u = \tilde{J}F(x_0)u$ for all $u \in R^n$. When $F = (f_1, \ldots, f_m)^t$ we have that:

$$\tilde{J}F(x_0) \subset \text{plen } \tilde{J}F(x_0) \subset [\partial f_1(x_0), \ldots, \partial f_m(x_0)]^t$$

The set $[\partial f_1(x_0), \ldots, \partial f_m(x_0)]^t$ is obviously convex, compact and plenary. It actually yields the same image set as $\tilde{J}F(x_0)$ does when the considered vectors $u$ are the elements $e_i$ of the canonical basis in $R^n$. In other words,

$$\{x_1^*, [x_1^*, \ldots, x_i^*, \ldots, x_m^*]^t \in \tilde{J}F(x_0) \} = [\partial f_i(x_0)] [6].$$

3. THE PLENARY HULL OF $\tilde{J}F(x_0)$ AND THE INVERSE FUNCTION THEOREM.

THEOREM 3.1. If every matrix $M$ in $\text{plen } \tilde{J}F(x_0)$ is of maximal rank, then exist neighborhoods $U$ and $V$ of $x_0$ and $F(x_0)$ respectively, and a lipschitzian function $G: V \rightarrow R^n$ such that:
for all \( u \in U \) is \( F(u) = v \) if and only if for all \( v \in V \) is \( G(v) = u \).

When \( F \) is \( C^1 \), \( JF(x_0) \) reduces to \( JF(x_0) \) and the function \( G \) above is necessarily \( C^1 \) as well. Thus we recover the classical theorem.

**REMARK 1.** This theorem remains true (without modifications in proof) if we impose the maximality of a rank for all \( M \in J^*_A F(x_0) \) where \( A \subseteq \text{dom} F' \) be such that its complementary set in \( O \) is of null measure and let \( J^*_A F(x_0) \) be defined as in (1.1) except that in this definition the points \( x_n \) are constrained in \( A \).

**REMARK 2.** Due to the definitions themselves, we have that

\[
\partial^*_A F_v(x_0) = J^*_A F(x_0)v \quad \text{(6)}
\]

It is known that the generalized gradient of real-valued functions is blind to sets of null measure [3, Proposition 1.11]. The desire to make the generalized derivative blind to sets of null measure led Porciau, R.H. [7] to alter Clarke's original definition by considering the Lebesgue set the \( \text{Leb} F' \) of \( F' \), instead of \( \text{dom} F' \), in the definition of \( JF(x_0) \) but, since \( F' \) is locally in \( L^m(0, R^m) \), almost every \( x \) in \( \text{dom} F' \) belongs to \( \text{Leb} F' \).

**REMARK 3.** Let \( A \) and \( J^*_A F(x_0) \) be as in Remark 1. Then

\[
\text{plen} J^*_A F(x_0) = \text{plen} JF(x_0)
\]

So, \( \text{plen} JF(x_0) \) is blind to sets of measure zero.

**Proof of the theorem 3.1.**

**LEMMA 1.** An exact chain rule in finite-dimensional case [6].

Let \( F \) be a locally Lipschitz function and \( g \) be continuously differentiable. Then

\[
\partial(g \circ F) = J^* F(x_0) \nabla g(F(x_0)) \quad \text{where} \quad F : R^n \rightarrow R^m \quad \text{and} \quad g : R^m \rightarrow R.
\]

**LEMMA 2.** Let \( \beta \) be a positive number. Then for all \( x \) sufficiently near \( x_0 \), \( \text{plen} JF(x) \subseteq \{ \text{plen} JF(x_0) + \beta \text{M}(0, 1) \} \) where \( \text{M}(0, 1) \) denotes the unit ball in the vector space of \( m \times n \) matrices.

This is a direct consequence of definition of the plenary hull of the generalized jacobian matrix.
Lemma 3. There are positive numbers $r$ and $\lambda$ with the following property: given any unit vector $v$ in $\mathbb{R}^n$, there is a unit vector $u$ in $\mathbb{R}^n$ such that, whenever $x$ lies in $(x_0 + rB)$ and $M \in \text{plen } \hat{J}F(x_0)$, then $\langle Mv, u \rangle \geq \lambda$ for all $M$, where $B$ denotes the open unit ball in $\mathbb{R}^n$.

Proof. Let $\Sigma_1$ denote the unit sphere in $\mathbb{R}^n$. The subset $\text{plen } \hat{J}F(x_0) \Sigma_1$ of $\mathbb{R}^n$ is compact and does not contain $0$ since $\text{plen } \hat{J}F(x_0)$ is of maximal rank.

Hence for some $\lambda > 0$, $\text{plen } \hat{J}F(x_0) \Sigma_1$ is distance at least $2\lambda$ from $0$. For positive $\beta$ sufficiently small, $[\text{plen } \hat{J}F(x_0) + \beta M(0,1)] \Sigma_1$ is distance at least $\lambda$ from $0$.

By Lemma 2, it follows that for some positive $r$,

$$x \in x_0 + rB \Rightarrow \text{plen } \hat{J}F(x_0) \subset [\text{plen } \hat{J}F(x_0) + \beta M(0,1)]$$

We may suppose $r$ chosen so that $F$ satisfies Lipschitz condition on $(x_0 + rB)$.

Now let any unit vector $v$ be given. It follows from above that the convex set $[\text{plen } \hat{J}F(x_0) + \beta Mv] \Sigma_1$ for all $v$ in $\mathbb{R}^n$, is distance at least $\lambda$ from $0$. By the usual separation theorem for convex sets, there is a unit vector $u$ such that: $\langle u, Mv \rangle \geq \lambda$ for all $M \in \text{plen } \hat{J}F(x_0)$.

Lemma 4. If $x_1$ and $x_2$ lie in $(x_0 + rB)$, then

$$|F(x_1) - F(x_2)| \geq \lambda \|x_1 - x_2\|$$

Proof. We may suppose $x_1 \neq x_2$ and by the continuity of $F$ that $x_1, x_2 \in (x_0 + rB)$.

Set $v = \frac{x_2 - x_1}{\|x_2 - x_1\|}$, $a = \|x_2 - x_1\|$ so that $x_2 = x_1 + av$.

Let $\pi$ be the plane perpendicular to $v$ and passing through $x_1$. The set $P$ of points $x$ in $(x_0 + rB)$ where $F'$ fails to exist is of measure zero, and hence by Fubini's theorem, for almost every $x$ in $\pi$, the ray $x + tv$, $t \geq 0$ meets $P$ in a set of null one-dimensional measure. Choose an $x$ with the above property and sufficiently close to $x_1$ so that $x + tv$ lies in $(x_0 + rB)$ for every $t$ in $[0,a]$. Then the function $t \rightarrow F(x + tv)$ is Lipschitzian for $t$ in $[0,a]$ and has a.e. on this interval the derivative $JF(x + tv)v$. Thus

$$F(x + av) - F(x) = \int_0^a JF(x + tv)v dt$$

Let $u$ be as in Lemma 3. We deduce,

$$\langle u, (F(x + av) - F(x)) \rangle = \langle u, \int_0^a JF(x + tv)v dt \rangle \geq \int_0^a \lambda \ dt = \lambda a$$
Recalling the definition of $a$, we arrive at:

$$|F(x+av) - F(x)| \geq \lambda \|x_2 - x_1\|$$

This may be done for $x$ arbitrarily close to $x_1$. Since $F$ is continuous, the lemma ensues.

**LEMMA 5.** $F(x_0 + rB)$ contains $F(x_0) + (r \lambda/2)B$.

**Proof.** Let $y$ be any point in $F(x_0) + (r \lambda/2)B$, and let the minimum of $\|y - F(x)\|^2$ over $(x_0 + rB)$ be attained at $x$. We claim $x$ belongs to $(x_0 + rB)$.

$$r \lambda/2 > \|y - F(x_0)\| \geq \|F(x) - F(x_0)\| - \|y - F(x)\| \geq \lambda \|x - x_0\| - \|y - F(x)\| \geq \lambda r - \|y - F(x_0)\| > \lambda r - r \lambda/2 = r \lambda/2$$

which is a contradiction. Thus $x$ yields a local minimum for the function $\|y - F(x)\|^2$, and consequently [3, Corollary 1.1],

$$0 \in \delta \|y - F(x)\|^2$$

We now use Lemma 1 to conclude that $0$ belongs to the set

$$\mathcal{F}^e F(x)(y - F(x)) \quad ([6])$$

that coincides with $\{\text{plen } \mathcal{F}^e F(x)(y - F(x))\}$ for all vector in $\mathbb{R}^m$ by Theorem 2.1. But Lemma 3 implies that every matrix in $\text{plen } \mathcal{F}^e F(x)$ is non singular, hence the above is possible only if $F(x) = y$.

We now set $V = F(x_0) + (r \lambda/2)B$, and we define $G$ on $V$ as follows:

$G(y)$ is the unique $x$ in $(x_0 + rB)$ such that $F(x) = y$. We choose $U$ as any neighborhood of $x_0$ satisfying $F(U) \supset V$. The theorem is now seen to follow, since Lemma 4 implies that $G$ is Lipschitz with constant $\lambda^{-1}$. 
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