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A TWO-STEPS INTERCHANGE MARKET MODEL

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ABSTRACT. In this paper we introduce a new model of interchange market among producers, traders and consumer in a two-steps context. We study general characterization of extremals of the convex polihedron of possible solutions for given prices. Particular cases with only two merchandises are studied.

1. INTRODUCTION.

If one wishes to study an interchange market in two-steps, that is to say the various merchandises in the market go through traders only one time, then it would be possible to consider the model of transportation in two-steps introduced and studied recently [2], where the ports are now transformed into the set of traders of the economy. Moreover if we consider many merchandises a price from the producers to the traders has to be to considered first. Next since we have only two-steps in the market a different price from the traders to the consumers must be introduced. Such prices under the assumption that the market is cleared, must be related by a type of Walras law. This new condition makes the mathematical model more interesting.

In the next section we consider the new model in its mathematical formulation. In the third section we consider a general characterization of extremals of the convex set of possible solutions of the problem for fixed prices. Finally in the next two sections we describe a general class of extremals for same particular cases with only two merchandises, showing that there appears a new subject, that of composing a special class of cycles which are related with those introduced by Jurkat-Ryser in [1], and by same of the authors in [2].

2. PROBLEM FORMULATION.

As we have mentioned in the Introduction in this section we are going to formulate precisely the interchange market.

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Thus, consider i: 1,...,n agents of the economy. They are allowed to interchange different marchandises ℓ: 1,...,m. Each agent i has an initial endowment $x_i(\ell) \geqslant 0$ of the ℓ-th merchandise. The market is conservative in the sense that the total amount of each merchandise is only interchanged but conserved. There are only two-steps of transactions: $x^1_{ij}(\ell) \geqslant 0$ is the amount of merchandise ℓ that is interchanged from agent i to the j-th agent in the first transaction; $x^2_{jk}(\ell) \geqslant 0$ has the same meaning between the j-th to the k-th agent in the second step of the operation. The final amount after the two-steps transaction for agent k of the merchandise ℓ is $y_k(\ell) \geqslant 0$ and in the arguments of this paper is considered to be given. On the other hand there are two prices, one for the first step transaction 'p' and the remaining for the second step transaction 'q'. As usual they are vectors normalized in the simplex

$$p,q \in S^m = \{z \in R^m: z(\ell) \geqslant 0 \quad \sum_{\ell=1}^{m} z(\ell) = 1\}$$

where R^m indicates the Euclidean space of m-dimensions.

From the previous conservation facts, we have that formally the equations to be considered are:

$$\left.\begin{aligned} &\sum_{j=1}^{n} x^1_{ij}(\ell) = x_i(\ell) && \text{for each } i,\ell \\ &\sum_{i=1}^{n} x^1_{ij}(\ell) = \sum_{k=1}^{n} x^2_{jk}(\ell) && \text{for each } j,\ell \\ &\sum_{j=1}^{n} x^2_{jk}(\ell) = y_k(\ell) && \text{for each } k,\ell \end{aligned}\right\} \qquad (1)$$

On the other hand, if the market mantains the value of the whole economy that is to say a type of Walras law, then

$$\sum_{\ell=1}^{m} p(\ell) \sum_{i=1}^{n} x^1_{ij}(\ell) = \sum_{\ell=1}^{m} q(\ell) \sum_{k=1}^{n} x^2_{jk}(\ell) \qquad \text{for each } j$$

or equivalently

$$\sum_{\ell=1}^{m} [p(\ell)-q(\ell)] \sum_{i=1}^{n} x^1_{ij}(\ell) = 0 \qquad \text{for each } j \qquad (2)$$

Adding on i the first equation of (1), on j the second and on k the last one of (1) we obtain:

$$\sum_{i=1}^{n} x_i(\ell) = \sum_{k=1}^{n} y_k(\ell) \qquad \text{for each } \ell$$

which tells us that indeed the amounts of merchandises are conserved through the two-steps transactions in the economy. It is clear that there is much interest to study the set of solutions of such interesting economic problem. This solution set is a convex polyhedron C and then it is of interest to know the extremals of such polyhedron. It is known that the study of the extremals of special polyhedrons, as for example the previous one, presents many difficulties when one wants to describe all the extremals.

3. AN EXTREMAL CHARACTERIZATION.

As we have mentioned above, the characterization of the extremals of the convex polyhedron given by the equalities (1) and (2) is important. Here we are going to give this characterization. In order to do this, we define for our problem a global cycle as a $z = (z^1,z^2) \neq 0$ for given prices p and q such that:

$$\left.\begin{aligned} &\sum_{j=1}^{n} z^1_{ij}(\ell) = 0 && \text{for each } i,\ell \\ &\sum_{i=1}^{n} z^1_{ij}(\ell) = \sum_{k=1}^{n} z^2_{jk}(\ell) && \text{for each } j,\ell \\ &\sum_{j=1}^{n} z^2_{jk}(\ell) = 0 && \text{for each } k,\ell \end{aligned}\right\} \quad (3)$$

and finally

$$\sum_{\ell=1}^{m} [p(\ell)-q(\ell)] \sum_{i=1}^{m} z^1_{ij}(\ell) = 0 \quad \text{for each } j \qquad (4)$$

We emphasize that in general such a global cycle has positive and negative componentes.

We say that a solution or simply a "matrix" $x = (x^1,x^2)$ of our problem contains a global cycle $z = (z^1,z^2)$ if

$$\text{supp }(z) \subset \text{supp }(x)$$

where

$$\text{supp }(x) = \{(x^1_{ij},x^2_{jk}) : x^1_{ij} \neq 0 \quad x^2_{jk} \neq 0\}$$

As a first result we have the following characterization:

THEOREM 1. *For given prices* p *and* q, *a solution* $x = (x^1,x^2)$ *of our problem is extremal if and only if it does not contain a global cycle.*

Proof. We first will prove the necessity. Consider a solution $x = (x^1,x^2)$ which contains a global cycle $z = (z^1,z^2)$. Let us define:

$$\lambda = \frac{\min \{ x_{ij}^1 , x_{jk}^2 : (i,j);(j,k)\ x_{ij}^1 > 0 \quad x_{jk}^2 > 0\}}{\max \{|z_{ij}^1|, |z_{jk}^2| : (i,j);(j,k)\ z_{ij}^1 \neq 0 \quad z_{jk}^2 \neq 0\}}$$

This is well defined and $\lambda > 0$ which results from the fact that the global cycle is contained in x. On the other hand we can easily see that:

$$x + \lambda z \qquad \text{and} \qquad x - \lambda z$$

belong to the set of points C. Moreover

$$x = \frac{1}{2}(x + \lambda z) + \frac{1}{2}(x - \lambda z)$$

which implies that x is not an extremal. Thus the necessity is proven. Now we consider the sufficiency. Consider a point $x = \frac{1}{2} x_1 + \frac{1}{2} x_2$ where $x_1 \neq x_2$ and are points in C. Then it is clear that

$$T = x_1 - x_2$$

is a global cycle. Now we have immediately that:

$$\text{supp } T = \text{supp } (x_1 - x_2) \subset \text{supp } (x_1) \cup \text{supp } (x_2) = \text{supp } (x)$$

and therefore x contains a global cycle T. (q.e.d.)

At this point we need the following definition. We say that for a given ℓ : $x(\ell) = (x_{ij}^1(\ell), x_{kj}^2(\ell))$ is a two-steps extremal for ℓ if $x(\ell)$ is an extremal of the problem (1) independently of the condition (2). We refer the reader to [2].

We now have the next result regarding the parts of an extremal $x = (x^1, x^2)$ with respect to a component ℓ with $p(\ell) = q(\ell)$.

THEOREM 2. *Given an extremal* $x = (x^1, x^2)$ *of* C, *if for some* ℓ, $p(\ell) = q(\ell)$, *then* $x(\ell) = (x_{ij}^1(\ell), x_{kj}^2(\ell))$ *is a two-steps extremal for* ℓ.

Proof. Suppose that for an ℓ such that $p(\ell) = q(\ell)$, $x(\ell)$ were not a two-steps extremal for ℓ. Then in the convex set $C(\ell)$ of solutions of the problem given by (1) for ℓ, we have that there are two different points $\bar{\bar{x}}(\ell)$ and $\bar{x}(\ell)$ such that:

$$x(\ell) = \frac{1}{2}\bar{x}(\ell) + \frac{1}{2}\bar{\bar{x}}(\ell)$$

Now construct the points

$$\bar{x}(\bar{\ell}) = \begin{cases} x(\bar{\ell}) & \bar{\ell} \neq \ell \\ \bar{x}(\bar{\ell}) & \bar{\ell} = \ell \end{cases}$$

and

$$\bar{\bar{x}}(\bar{\ell}) = \begin{cases} x(\bar{\ell}) & \bar{\ell} \neq \ell \\ \bar{\bar{x}}(\bar{\ell}) & \bar{\ell} = \ell \end{cases}$$

We finally have that:

$$x(\ell) = \frac{1}{2}\,\bar{x}(\ell) + \frac{1}{2}\,\bar{\bar{x}}(\ell)$$

for the two new points $\bar{x}$ and $\bar{\bar{x}}$ of C wich are different by construction. Therefore x would not be extremal, contrary to the assumption.

(q.e.d)

It is interesting to note that the extremality of a two-steps "matrix" $x(\ell)$ can be easily characterized by means of two-steps cycles. These are matrices such that they fulfill (3) for the corresponding merchandise. On the other hand for a two-steps problem we have already introduced in [2], the concept of cycles which now we define as two-steps path cycle. This is obtained as two-steps partial cycles. Indeed a partial cycle is either a set of indices:

$$i_0j_0\ ,\ i_0j_1\ ,\ldots\ldots,i_ej_e\ ,\ i_ej_{e+1}$$

or

$$k_0j_0\ ,\ k_0j_1\ ,\ldots\ldots,k_ej_e\ ,\ k_ej_{e+1}$$

wich we will write:

$$i_0j_0\ ,\ S\ ,\ i_ej_{e+1} \quad \text{and} \quad k_0j_0\ ,\ S\ ,\ k_ej_{e+1}$$

respectively. Now a connected two-steps path cycle is a set of partial cycles as follows:

$$i_oi_o, S_1, i_ej_{e_1+1}, k_0j_{e_1+1}, S_2, k_1j_{e_2+1}, \ldots\ldots, k_fj_{2f+1}, S_{2f+1}, k_{f+1}, j_0\ .$$

A two-steps path cycle is an arbitrary set of connected two-steps path cycles.

It is natural to give the name of two-steps path cycle to a matrix such that its support is a two-step path cycle such that it satisfies (3) having only two values λ and $-\lambda$ in its support for the corresponding merchandise. We will call an upper or lower two-step path cycle a two-step path cycle such that it contains only elements of the upper or lower part respectively.

We have the following result:

PROPOSITION 3. *Any two-step cycle is formed as sum of two-step path cycles.*

Proof. Given a two-step cycle A then there is a component which is either $x_{i_0j_0}(\ell) \neq 0$ or $x_{j_0k_0}(\ell) \neq 0$. Now it is clear as expressed in [2] that there exists a two-step path cycle C such that

$$\text{supp } C \subset \text{supp } A$$

Now consider

$$A - \lambda C$$

with λ such that $A - \lambda C$ has more zeros than A. Again $A - \lambda C$ is a two-step cycle and we repeat the procedure obtaining finally the result.

(q.e.d.)

4. EXTREMALITY: FOR A SIMPLE CASE.

Having the previous results, we now will study a particular but important case, namely when we have two merchandises with the coefficients $x_i(\ell) = y_k(\ell) = 1$ for $\ell = 1,2,\ldots,m$ and all i,k. As a first simple consequence we have that the number of i has to be equal to the number of k. We call it two-step stochastic.

In order to characterize all the extremals of this simple but important problem, we first present a simpler result which is concerned with the case of only one merchandise.

THEOREM 4. *Any extremal of the problem* (1) *in the two-step stochastic, with only one merchandise takes values zero and one and reciprocally.*

Proof. It is clear that if a matrix has only zeros and ones is naturally an extremal. On the other hand, if a matrix is extremal, suppose that it is not formed by only zeros and ones. Therefore as we have seen in [2] it is clear that it contains in its support a two-step path cycle, which implies by a result analogous to Theorem 1 for the particular situation, that such a matrix would be not extremal. This is impossible. (q.e.d.)

With this result we now will consider the case of two marchandises in the two-steps stochastic case. Moreover we study this in the case when $p \neq q$. We have a general condition that:

$$r\left(\sum_{i=1}^{n} x_{ij}^{1}(1) - \sum_{i=1}^{n} x_{ij}^{1}(2)\right) = 0 \qquad (5)$$

where

$$r = p(1) - q(1) = -(p(2) - q(2)) \neq 0$$

From here we have that (5) can be written as:

$$\varepsilon(1) = \varepsilon(2)$$

where the ε's are the respective vectors which take into account the projection sum.

As a first result we have:

THEOREM 5. *A matrix of the two-steps stochastic case with only zeros and ones is an extremal.*

Proof. It is clear that such a matrix can not have in its support a global cycle, therefore by Theorem 1, it follows immediately. (q.e.d.).

At this point it is natural to expect the existence of matrices having two-step cycles in each of the merchandises but they do not contitute a global cycle. This is due to the fact of the "transversality condition" or condition (4). This fact is shown in the following example with two i's and four j's:

1/2	1/2		
		1/2	1/2

	1/2	1/2	
1/2			1/2

for the first merchandise and for the second one:

1/2	1/2		
		1/2	1/2

	1/2		1/2
1/2		1/2	

It is clear that in each merchandise they form a two-step path cycle. However it does not contain a global cycle. The reader will realize the fact that the transversality condition cannot be satisfied. In other words, this is due to the fact that both two-step path cycles cannot be "ensembled" in a way according with (5). Since it does not contain a global cycle such that a matrix is an extremal.

Now we will consider the class of all the extremals. In order to find it we will introduce some useful results in order to characterize such a class.

We need the following definition. We say that two-step cycles one for each merchandise are ensembled if together they satisfy the transversality condition.

It is worth noting that for two-step path cycles this means just the fact that the "directions" of passing through from the upper to the lower matrices or viceversa are just the same.

We present now a result relevant to our purposes:

THEOREM 6. *Given a global cycle then there exist a sum of two-step path cycles, respectively for each merchandises, such that they satisfy the transversality condition.*

Proof. The global cycle gives rise immediately to the existence of two-step cycles each one for the corresponding merchandise. On the other hand each one of the two-step cycles for each merchandise is a sum of two-step path cycles satisfying the transversality condition.

(q.e.d.)

As an immediate consequence of this result we have:

COROLLARY 7. *If a matrix does not contain a sum of two-step path cycles satisfying the transversal condition in its support then it is an extremal.*

At this point it is worth noting that the previous two results are valid in the general case that is to say when the entries are not necessarily ones.

We remark that the sum of two-step path cycles not ensembled, with only one commun "column" can be also an extremal as is shown with the following example:

1				1/2			1/2
1					1/2	1/2	
1	1/2		1/2				
1		1/2		1/2			

	1/2	1/2					1
1/2			1/2				1
			1/2		1/2		1
				1/2		1/2	1

1				1/2			1/2
1					1/2	1/2	
1	1/2			1/2			
1		1/2	1/2				

		1/2	1/2				1
1/2	1/2						1
			1/2	1/2			1
					1/2	1/2	1

The reader surely realizes that with this example we have shown that the class of extremals are not just two-step path cycles not ensembled.

5. A FURTHER EXAMPLE:

Now we will consider a further example which will show that also numbers different from 1/2 can admit an extremal. In order to avoid lengthy calculations we will take into account the following example with matrices:

1	1/3	1/3	1/3			
1				1/3	1/3	1/3

1/3			1/3			2/3
	1/3			1/3		2/3
		1/3			1/3	2/3

in the first merchandise and

1	1/3	1/3	1/3			
1				1/3	1/3	1/3

1/3				1/3		2/3
	1/3				1/3	2/3
		1/3	1/3			2/3

for the second one. The amounts on the lateral places indicate the corresponding x_i and y_k. We remark that they are not all ones. It would be possible obtain ones with a further row in the superior matrix with three additional lines. However for simplicity we do not consider such a matrix in this paper.

Firstly, it is clear from the matrix that simple two-step path cycles ensembled do not exist.

On the other hand a detailed examination shows that the sum of two-step path cycle also cannot ensemble. Therefore in order to prove that it is an extremal by virtue of Corollary 7, it is sufficient to show that the sum of three two-step path cycles cannot exist. In order to prove it suppose that the sum of three elementary two-step path cycle exists

ε_1	$-\varepsilon_1$				
			$-\varepsilon_1$	ε_1	

ε_1			$-\varepsilon_1$		
	$-\varepsilon_1$			ε_1	

for the first merchandise and

ε_2	$-\varepsilon_2$				
				$-\varepsilon_2$	ε_2

ε_2				$-\varepsilon_2$	
	$-\varepsilon_2$				ε_2

for the second merchandise.

This was obtained by considering the elementary cycle between the first two elements of j. Similarly the elementary cycle between the second and third elements for j gives rise to an elementary cycle with amounts μ_1 and μ_2. Finally the only remaining possibility is the consideration of an elementary two-step cycle between the first and third j, with λ_1 and λ_2 for both merchandises. Thus we conclude that by the transversality condition we have the homogeneous system of equations:

$$\begin{pmatrix} 1 & 1 & 0 & -1 & 0 & -1 \\ -1 & 0 & 1 & 1 & -1 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 & -1 & -1 \\ 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \lambda_1 \\ \mu_1 \\ \varepsilon_2 \\ \mu_2 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

From here, it is possible to see that the determinant of the system is different from zero, therefore the only solution is the trivial which indicates the not existence of sum of three elementary two-step path cycles. Even though that this is only a case of sum of three elementary two-step path cycles it is clear that any other one is homomorphic to it. Therefore such a sum cannot exist and consequently the matrix shown is an extremal with values different from 1/2.

The previous examples prove the great variety of possible extremals that appear in such model. All the problem of extremals is then related with the structure of the sum of the two-step path cycles which is a new subject for further research.

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