Revista de la
Unión Matemática Argentina
Volumen 31 , 1983.

ON THE RELATION BETWEEN DARLINGTON REALIZATIONS OF CONTRACTIVE AND j-EXPANSIVE MATRIX-VALUED FUNCTIONS

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ABSTRACT. We obtain in this paper a relation between the matrices of coefficients of Darlinton realizations of a j-expansive matrix-valued function T(z) and the contractive matrix-valued function S(z), given, in terms of T(z) as a linear fractional transformation, over T(z), with constant coefficients.

1. We recall some known results on contractive and j-expansive matrix-valued functions [1,2,3].

A matrix S is called contractive if $I - S^*S \geqslant 0$, where I is a unit matrix and the symbol * denotes Hermitian conjugation.

Let J be a matrix for which $J^* = J$ and $J^2 = I$. A matrix A is called J-expansive if $A^*JA - J \geqslant 0$, and J-unitary if $A^*JA - J = 0$.

We set $P = \frac{1}{2}(I_{2n} + j)$ and $Q = \frac{1}{2}(I_{2n} - j)$, where $j = \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix}$.

For a j-expansive matrix T, of order 2n, the following matrix is defined

$$S = (QT+P)(PT+Q)^{-1}J_p \ , \text{ where } J_p = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}. \qquad (1;1)$$

Since $$I - S^*S = J_p(QT+P)^{*-1}(T^*jT-j)(QT+P)^{-1}J_p,$$

we can afirm that the matrix T is j-expansive if and only if the matrix S, defined by (1;1), is contractive. The matrix T is expressed, in terms of S, by the formula $T = (Q-SJ_pP)^{-1}(P-SJ_pQ)$.

A matrix-valued function $S(z)$ $(z=re^{it})$ defined on the unit circle $D = \{z;\ |z| < 1\}$, is called contractive if it is holomorphic and $\|S(z)\| \leqslant 1$ $(z \in D)$. We use $\mathcal{S}$ to design this class. $S(z) \in \mathcal{S}$ is inner if it satisfies a.e. $I-S^*(\xi)S(\xi) = 0$ $(\xi=e^{it})$.

A meromorphic matrix-valued function A(z) $(z \in D)$ is J-expansive if it assumes J-expansive values at each point of holomorphicity z, i.e.

$$A^*(z)JA(z) - J \geqslant 0,$$

and A(z) is J-inner if it is J-expansive and it satisfies

$$A^*(\xi)JA(\xi) - J = 0 \text{ a.e.}$$

Henceforth we will consider a j-expansive matrix-valued function T(z) $(z \in D)$ of order 2n.

In accordance with formula (1;1), T(z) is j-expansive if and only if the matrix-valued function

$$S(z) = [QT(z)+P][P+T(z)Q]^{-1}J_p \qquad (1;2)$$

belongs to the class $\mathcal{S}$. T(z) is given, in terms of S(z), by the formula

$$T(z) = [Q-S(z)J_pP]^{-1}[P-S(z)J_pQ] \qquad (1;3)$$

A paper by Efimov and Potapov [1], which studies the applications of J-expansive matrix-valued functions to passive electrical networks, shows that an arbitrary passive 2-port can be obtained by connecting a passive 2-port across the output of a lossless 4-port. A combination of this result and Arov's results [2] leads us to establish, in Sec.2, a relation between the linear fractional transformations of a j-expansive matrix-valued function T(z) and the corresponding contractive matrix-valued function S(z) given, in terms of T(z), by (1;2).

2. Darlington realization of a contractive matrix-valued function S(z) means the representation of S(z) as the linear fractional transformation [2]

$$S(z) = [\alpha(z)\varepsilon+\beta(z)][\gamma(z)\varepsilon+\delta(z)]^{-1} \qquad (2;1)$$

over a constant matrix $\varepsilon \in \mathcal{S}$ with a j-inner matrix of coefficients

$$G(z) = \begin{pmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & \delta(z) \end{pmatrix}$$

We shall consider also a Darlington realization of a j-expansive matrix-valued function T(z), of order 2n, i.e. the representation of T(z) as the linear fractional transformation [4]

$$T(z) = [A(z)t+B(z)][C(z)t+D(z)]^{-1} \qquad (2;2)$$

over a j-expansive constant matrix t, with a J'-inner matrix of coefficients

$$W(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}, \quad \text{where} \quad J' = \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}.$$

THEOREM 2.1. *Let* T(z) *be a j-expansive matrix-valued function* $(z \in D)$ *of order* 2n, *that admits a representation* (2;2) *and satisfies one of the following conditions*

i) $T^*(\xi)jT(\xi)-j > 0$ a.e., ii) $T^*(\xi)jT(\xi)-j = 0$ a.e.;

and S(z) *the contractive matrix-valued function given, in terms of*

T(z), *by formula* (1;2). *Then,* S(z) *has a Darlington realization* (2;1), *with a matrix of coefficients* G(z) *related to the matrix of coefficients* W(z), *of the representation* (2;2), *by the expression*

$$G(z) = R^*W(z)Rt_1 \tag{2;3}$$

where $R = \begin{pmatrix} Q & PJ_p \\ P & QJ_p \end{pmatrix}$ *and* t_1 *is a constant matrix.*

Proof. i) We shall prove the thesis when T(z) satisfies condition i) with

$$t = \frac{1}{2}(Q+5P+J_p) \tag{2;4}$$

and

$$t_1 = \frac{1}{2}\begin{pmatrix} \sqrt{5}\, I_{2n} & \frac{1}{\sqrt{5}}(Q-P+2J_p) \\ \frac{1}{\sqrt{5}}(Q-P+2J_p) & \sqrt{5}\, I_{2n} \end{pmatrix} \tag{2;5}$$

Let us consider the representation (2;2) of T(z) over the matrix t given by (2;4), and introduce the notation

$$A(z) = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix} \quad ; \quad B(z) = \begin{pmatrix} b_{11}(z) & b_{12}(z) \\ b_{21}(z) & b_{22}(z) \end{pmatrix} \quad ;$$

$$C(z) = \begin{pmatrix} c_{11}(z) & c_{12}(z) \\ c_{21}(z) & c_{22}(z) \end{pmatrix} \quad ; \quad D(z) = \begin{pmatrix} d_{11}(z) & d_{12}(z) \\ d_{21}(z) & d_{22}(z) \end{pmatrix}. \tag{2;6}$$

Therefore we can set

$$T(z) \overset{\text{def}}{=} \begin{pmatrix} t_{11}(z) & t_{12}(z) \\ t_{21}(z) & t_{22}(z) \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{2}[a_{11}(z)+a_{12}(z)] + b_{11}(z) & \frac{1}{2}[a_{11}(z)+5a_{12}(z)] + b_{12}(z) \\ \frac{1}{2}[a_{21}(z)+a_{22}(z)] + b_{21}(z) & \frac{1}{2}[a_{21}(z)+5a_{22}(z)] + b_{22}(z) \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} \frac{1}{2}[c_{11}(z)+c_{12}(z)] + d_{11}(z) & \frac{1}{2}[c_{11}(z)+5c_{12}(z)] + d_{12}(z) \\ \frac{1}{2}[c_{21}(z)+c_{22}(z)] + d_{21}(z) & \frac{1}{2}[c_{21}(z)+5c_{22}(z)] + d_{22}(z) \end{pmatrix}^{-1}. \tag{2;7}$$

It is useful to introduce now the expressions

$$a_1(z) = \frac{1}{2}[a_{11}(z)+a_{12}(z)] + b_{11}(z) \quad ; \quad a_2(z) = \frac{1}{2}[a_{11}(z)+5a_{12}(z)] + b_{12}(z);$$

$$a_3(z) = \frac{1}{2}[a_{21}(z)+a_{22}(z)] + b_{21}(z) \quad ; \quad a_4(z) = \frac{1}{2}[a_{21}(z)+5a_{22}(z)] + b_{22}(z);$$

$$c_1(z) = \frac{1}{2}[c_{11}(z)+c_{12}(z)] + d_{11}(z) \quad ; \quad c_2(z) = \frac{1}{2}[c_{11}(z)+5c_{12}(z)] + d_{12}(z);$$

$$c_3(z) = \frac{1}{2}[c_{21}(z)+c_{22}(z)] + d_{21}(z) \quad ; \quad c_4(z) = \frac{1}{2}[c_{21}(z)+5c_{22}(z)] + d_{22}(z). \tag{2;8}$$

Hence from (2;7) and (2;8) it results

$$t_{11}(z) = [a_2(z)-a_1(z)c_3^{-1}(z)c_4(z)][c_2(z)-c_1(z)c_3^{-1}(z)c_4(z)]^{-1} \;;$$
$$t_{12}(z) = [a_2(z)-a_1(z)c_1^{-1}(z)c_2(z)][c_4(z)-c_3(z)c_1^{-1}(z)c_2(z)]^{-1} \;;$$
$$t_{21}(z) = [a_4(z)-a_3(z)c_3^{-1}(z)c_4(z)][c_2(z)-c_1(z)c_3^{-1}(z)c_4(z)]^{-1} \;;$$
$$t_{22}(z) = [a_4(z)-a_3(z)c_1^{-1}(z)c_2(z)][c_4(z)-c_3(z)c_1^{-1}(z)c_2(z)]^{-1} \;. \tag{2;9}$$

Replacing (2;5) and (2;6) in (2;3), we obtain the matrix-valued function

$$G(z) = \begin{pmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & \delta(z) \end{pmatrix} =$$

$$= \frac{1}{2}\begin{pmatrix} Q & P \\ QJ_p & PJ_p \end{pmatrix}\begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}\begin{pmatrix} Q & PJ_p \\ P & QJ_p \end{pmatrix}\cdot\begin{pmatrix} \sqrt{5}\, I_{2n} & \frac{1}{\sqrt{5}}(Q-P+2J_p) \\ \frac{1}{\sqrt{5}}(Q-P+2J_p) & \sqrt{5}\, I_{2n} \end{pmatrix},$$

whose blocks are

$$\alpha(z) = \{[QA(z)+PC(z)]Q + [QB(z)+PD(z)]P\}\frac{\sqrt{5}}{2} I_{2n} + $$
$$+ \{[QA(z)+PC(z)]PJ_p + [QB(z)+PD(z)]QJ_p\}\frac{1}{2\sqrt{5}}(Q-P+2J_p) \;;$$

$$\beta(z) = \{[QA(z)+PC(z)]Q + [QB(z)+PD(z)]P\}\frac{1}{2\sqrt{5}}(Q-P+2J_p) + $$
$$+ \{[QA(z)+PC(z)]PJ_p + [QB(z)+PD(z)]QJ_p\}\frac{\sqrt{5}}{2} I_{2n} \;;$$

$$\gamma(z) = \{[QJ_pA(z)+PJ_pC(z)]Q + [QJ_pB(z)+PJ_pD(z)]P\}\frac{\sqrt{5}}{2} I_{2n} + $$
$$+ \{[QJ_pA(z)+PJ_pC(z)]PJ_p + [QJ_pB(z)+PJ_pD(z)]QJ_p\}\frac{1}{2\sqrt{5}}\cdot(Q-P+2J_p);$$

$$\delta(z) = \{[QJ_pA(z)+PJ_pC(z)]Q + [QJ_pB(z)+PJ_pD(z)]P\}\frac{1}{2\sqrt{5}}(Q-P+2J_p) + $$
$$+ \{[QJ_pA(z)+PJ_pC(z)]PJ_p + [QJ_pB(z)+PJ_pD(z)]QJ_p\}\frac{\sqrt{5}}{2} I_{2n}. \tag{2;10}$$

Observe that, since $W(z)$ is J'-inner, $G(z)$ is, by its construction, j_{2n}-inner, where $j_{2n} = \begin{pmatrix} -I_{2n} & 0 \\ 0 & I_{2n} \end{pmatrix}$. Then, the linear fractional trans formation (2;1) over the matrix $\varepsilon = 0_{2n}$, with G(z) as matrix of coefficients, is a representation of some contractive matrix-valued function S(z) ($z \in D$), of order 2n. Let us calculate S(z) by replacing the expressions (2;10) in the formula

$$S(z) = \beta(z)\,\delta^{-1}(z).$$

We have

$$S(z) = \{Q[\tfrac{1}{2}A(z)(Q+5P+J_p) + B(z)] + P[\tfrac{1}{2}C(z)(Q+5P+J_p) + D(z)]\}\,.$$
$$.\{QJ_p[\tfrac{1}{2}A(z)(Q+5P+J_p) + B(z)] + PJ_p[\tfrac{1}{2}C(z)(Q+5P+J_p) + D(z)]\}^{-1}.$$

Hence from (2;8) it follows that

$$S(z) = \begin{pmatrix} \frac{1}{2}[a_{11}(z)+a_{12}(z)]+b_{11}(z) & \frac{1}{2}[a_{11}(z)+5a_{12}(z)]+b_{12}(z) \\ \frac{1}{2}[c_{21}(z)+c_{22}(z)]+d_{21}(z) & \frac{1}{2}[c_{21}(z)+5c_{22}(z)]+d_{22}(z) \end{pmatrix}\,.$$

$$.\begin{pmatrix} \frac{1}{2}[a_{21}(z)+a_{22}(z)]+b_{21}(z) & \frac{1}{2}[a_{21}(z)+5a_{22}(z)]+b_{22}(z) \\ \frac{1}{2}[c_{11}(z)+c_{12}(z)]+d_{11}(z) & \frac{1}{2}[c_{11}(z)+5c_{12}(z)]+d_{12}(z) \end{pmatrix}^{-1}$$

After elementary calculations we can obtain the blocks of S(z)

$$s_{11}(z) = [a_2(z)-a_1(z)c_1^{-1}(z)c_2(z)][a_4(z)-a_3(z)c_1^{-1}(z)c_2(z)]^{-1}\ ;$$
$$s_{12}(z) = [a_2(z)-a_1(z)a_3^{-1}(z)a_4(z)][c_2(z)-c_1(z)a_3^{-1}(z)a_4(z)]^{-1}\ ;$$
$$s_{21}(z) = [c_4(z)-c_3(z)c_1^{-1}(z)c_2(z)][a_4(z)-a_3(z)c_1^{-1}(z)c_2(z)]^{-1}\ ;$$
$$s_{22}(z) = [c_4(z)-c_3(z)a_3^{-1}(z)a_4(z)][c_2(z)-c_1(z)a_3^{-1}(z)a_4(z)]^{-1}\ . \quad (2;11)$$

The j-expansive matrix-valued function related to S(z) by (1;3) is

$$\begin{pmatrix} s_{12}(z)-s_{11}(z)s_{21}^{-1}(z)s_{22}(z) & s_{11}(z)s_{21}^{-1}(z) \\ -s_{21}^{-1}(z)s_{22}(z) & s_{21}^{-1}(z) \end{pmatrix}$$

To prove the thesis, we have to show that this matrix-valued function coincides with T(z). From (2;9) and (2;11) it is immediate to see that

$$t_{22}(z) = s_{21}^{-1}(z) \quad ; \quad t_{12}(z) = s_{11}(z)s_{21}^{-1}(z).$$

We will show now that $s_{21}^{-1}(z)s_{22}(z) = t_{21}(z)$. Using the expressions (2;11) we obtain

$$s_{21}^{-1}(z)s_{22}(z) = [a_4(z)-a_3(z)c_1^{-1}(z)c_2(z)][c_4(z)-c_3(z)c_1^{-1}(z)c_2(z)]^{-1}\ .$$
$$.[c_4(z)-c_3(z)a_3^{-1}(z)a_4(z)][c_2(z)-c_1(z)a_3^{-1}(z)a_4(z)]^{-1}$$

$$s_{21}^{-1}(z)s_{22}(z) = -\{[a_4(z)-a_3(z)c_3^{-1}(z)c_4(z)] + [a_3(z)c_3^{-1}(z)c_4(z) - $$
$$- a_3(z)c_1^{-1}(z)c_2(z)]\}[c_4(z)-c_3(z)c_1^{-1}(z)c_2(z)]^{-1} .$$
$$.\{[c_4(z)-c_3(z)c_1^{-1}(z)c_2(z)] + [c_3(z)c_1^{-1}(z)c_2(z) -$$
$$- c_3(z)a_3^{-1}(z)a_4(z)]\}[c_2(z)-c_1(z)a_3^{-1}(z)a_4(z)]^{-1} =$$
$$= [a_4(z)-a_3(z)c_3^{-1}(z)c_4(z)]\ \{I_n + [c_4(z) -$$
$$- c_3(z)c_1^{-1}(z)c_2(z)]c_3(z)c_1^{-1}(z)[c_2(z)-c_1(z)a_3^{-1}(z)a_4(z)]\}$$
$$[c_2(z)-c_1(z)a_3^{-1}(z)a_4(z)]^{-1} + a_3(z)c_3^{-1}(z)\ [c_4(z) -$$
$$- c_3(z)a_3^{-1}(z)a_4(z)][c_2(z)-c_1(z)a_3^{-1}(z)a_4(z)]^{-1} =$$
$$= [a_4(z)-a_3(z)c_3^{-1}(z)c_4(z)]\{[c_2(z)-c_1(z)a_3^{-1}(z)a_4(z)]^{-1} +$$
$$+ \{c_1(z)c_3^{-1}(z)[c_4(z)-c_3(z)c_1^{-1}(z)c_2(z)]\}^{-1}\}+a_3(z)c_3^{-1}(z).$$
$$. [c_4(z)-c_3(z)a_3^{-1}(z)a_4(z)][c_2(z)-c_1(z)a_3^{-1}(z)a_4(z)]^{-1} =$$
$$= [a_4(z)-a_3(z)c_3^{-1}(z)c_4(z)][c_2(z)-c_1(z)c_3^{-1}(z)c_4(z)]^{-1} +$$
$$+ [a_4(z)-a_3(z)c_3^{-1}(z)c_4(z) + a_3(z)c_3^{-1}(z)c_4(z)-a_4(z)] .$$
$$. [c_2(z)-c_1(z)a_3(z)a_4(z)]^{-1} =$$
$$= [a_4(z)-a_3(z)c_3^{-1}(z)c_4(z)][c_2(z)-c_1(z)c_3^{-1}(z)c_4(z)]^{-1} =$$
$$= t_{21}(z).$$

To finish the proof of i), it only remains to show that

$$t_{11}(z) = s_{12}(z)-s_{11}(z)s_{21}^{-1}(z)s_{22}(z). \qquad (2;12)$$

Replacing expressions (2;8) in the preceding relation it results

$$s_{12}(z)-s_{11}(z)s_{21}^{-1}(z)s_{22}(z) = \{[a_2(z)-a_1(z)a_3^{-1}(z)a_4(z)] .$$
$$. [c_2(z)-c_1(z)a_3^{-1}(z)a_4(z)]^{-1}\ [c_2(z)-c_1(z)c_3^{-1}(z)c_4(z)] +$$
$$+ [a_2(z)-a_1(z)c_1^{-1}(z)c_2(z)][a_4(z)-a_3(z)c_1^{-1}(z)c_2(z)]^{-1} .$$
$$. [a_4(z)-a_3(z)c_3^{-1}(z)c_4(z)]\}[c_2(z)-c_1(z)c_3^{-1}(z)c_4(z)]^{-1}. \qquad (2;13)$$

After long but elementary calculations we obtain the following form for the factor inside braces:

$$[a_2(z)-a_1(z)a_3^{-1}(z)a_4(z)][c_2(z)-c_1(z)a_3^{-1}(z)a_4(z)]^{-1} .$$
$$.[c_2(z)-c_1(z)c_3^{-1}(z)c_4(z)] + [a_2(z)-a_1(z)c_1^{-1}(z)c_2(z)] .$$
$$.[a_4(z)-a_3(z)c_1^{-1}(z)c_2(z)]^{-1}\ [a_4(z)-a_3(z)c_3^{-1}(z)c_4(z)] =$$

$$= [a_2(z)-a_1(z)c_3^{-1}(z)c_4(z)] + a_1(z)c_3^{-1}(z)c_4(z)-a_1(z) \cdot$$
$$\cdot \{[a_4^{-1}(z)a_3(z)-c_2^{-1}(z)c_1(z)]^{-1} + [c_2^{-1}(z)c_1(z)-a_4^{-1}(z)a_3(z)]^{-1} -$$
$$- c_3^{-1}(z)c_4(z) = a_2(z)-a_1(z)c_3^{-1}(z)c_4(z). \qquad (2;14)$$

Replacing (2;14) in (2;13) it results

$$s_{12}(z)-s_{11}(z)s_{21}^{-1}(z)s_{22}(z) = [a_2(z)-a_1(z)c_3^{-1}(z)c_4(z)][c_2(z) - -c_1(z)c_3^{-1}(z)c_4(z)]^{-1}.$$

From the preceding relation and (2;9), we conclude that (2;12) holds. This completes the proof of part i).

ii) We will show that, when T(z) satisfies condition ii), the relation (2;3) holds with $t = I_{2n}$ and

$$t_1 = \begin{pmatrix} J_p & 0 \\ 0 & I_{2n} \end{pmatrix} \qquad (2;15)$$

Using the notation introduced in part i) for the blocks of W(z), i.e. the expressions (2;6), and replacing t and (2;6) in (2;2), we have

$$T(z) = \begin{pmatrix} a_1(z) & a_2(z) \\ a_3(z) & a_4(z) \end{pmatrix} \begin{pmatrix} c_1(z) & c_2(z) \\ c_3(z) & c_4(z) \end{pmatrix}^{-1}, \qquad (2;16)$$

where

$$a_1(z) = a_{11}(z)+b_{11}(z) \;;\; a_2(z) = a_{12}(z)+b_{12}(z) \;;\; a_3(z) = a_{21}(z)+b_{21}(z);$$
$$a_4(z) = a_{22}(z)+b_{22}(z) \;;\; c_1(z) = c_{11}(z)+d_{11}(z) \;;\; c_2(z) = c_{12}(z)+d_{12}(z);$$
$$c_3(z) = c_{21}(z)+d_{21}(z) \;;\; c_4(z) = c_{22}(z) + d_{22}(z). \qquad (2;17)$$

By virtue of (2;16) we can observe that the expressions (2;9) hold for the blocks of T(z), but in this case with $a_i(z)$ and $c_i(z)$ (i=1,4) given by (2;17)

Let us write now the blocks of $G(z) = \begin{pmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & \delta(z) \end{pmatrix}$, applying (2;3), (2;9) and (2;15).

$$\alpha(z) = [QA(z)+PC(z)]QJ_p + [QB(z)+PD(z)] PJ_p \;;$$
$$\beta(z) = [QA(z)+PC(z)]PJ_p + [QB(z)+PD(z)] QJ_p \;;$$
$$\gamma(z) = [QJ_pA(z)+PJ_pC(z)] QJ_p + [QJ_pB(z)+PJ_pD(z)] PJ_p \;;$$
$$\delta(z) = [QJ_pA(z)+PJ_pC(z)] PJ_p + [QJ_pB(z)+PJ_pD(z)] QJ_p . \qquad (2;18)$$

It can be easily checked that G(z) is, by its construction, j_{2n}-inner. Then, the linear fractional transformation

$$[\alpha(z)\varepsilon+\beta(z)][\gamma(z)\varepsilon+\delta(z)]^{-1} \tag{2;19}$$

over the constant matrix $\varepsilon = I_{2n}$, with $G(z)$ as matrix of coefficients, is a representation of a contractive matrix-valued function $S(z)$. Moreover, $S(z)$ is, in this case, an inner matrix-valued function.

It follows from (2;18) and (2;19)

$$S(z) = \{Q[A(z)+B(z)] + P[C(z)+D(z)]\}J_p.\{QJ_p[A(z)+B(z)]J_p + PJ_p[C(z) + D(z)]J_p\}^{-1}.$$

Hence using (2;15) we have

$$S(z) = \begin{pmatrix} a_1(z) & a_2(z) \\ c_3(z) & c_4(z) \end{pmatrix} \begin{pmatrix} a_3(z) & a_4(z) \\ c_1(z) & c_2(z) \end{pmatrix}^{-1} \tag{2;20}$$

From the preceding relation it turns out that the blocks of $S(z)$ are given by expressions that take the form (2;11), with $a_i(z)$ and $c_i(z)$ $(i=1,4)$ defined by (2;15).

To complete the proof it only remains to show that we can obtain $T(z)$ replacing $S(z)$, given by (2;20), in (1;3). Note that, since the expressions defining $T(z)$ and $S(z)$ are, in this case, formally the same than those of part i), the calculations we have to perform to conclude the proof are also the same we have done in part i), and the results are essentially similar.

This completes the proof of theorem 2.1.

REFERENCES

[1] A.V.EFIMOV and V.P.POTAPOV, *J-expanding matrix-valued functions and their role in the analytical theory of circuits*, Uspehi Mat. Nauk. 28 (1973), N°1, 65-130. English Transl. Russian Math. Surveys. 28 (1973), N°1, 69-140.

[2] D.Z.AROV, *Darlington realization of matrix-valued functions*, Izv. Akad. Nauk SSSR, Ser. Mat. Tom 37, 6 (1973). English Transl.Math USSR, Izvestija, 7 (1973), 1295-1326.

[3] M.G.KREIN and Ju.L.SMULJIAN, *On the linear fractional transformations with operators coefficients*, Mat. Issled., 2 (1967),N°3, 64-96. English Transl., Amer. Math. Soc. Transl., (2) 103 (1973), 125-152.

[4] E.A.CORTINA, *j-expansive matrix-valued functions and Darlington realization of transfer scattering matrices*, to be published in Jour. of Math. Anal. and Appl.

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Recibido en noviembre de 1982.