INNER DERIVATIONS WITH CLOSED RANGE IN THE CALKIN ALGEBRA. II:
THE NON-SEPARABLE CASE

Lawrence A. Fialkow and Domingo A. Herrero

1. INTRODUCTION.

In [1], C. Apostol characterized the Hilbert space operators which induce inner derivations having closed range. Let $L(H)$ denote the algebra of all (bounded linear) operators acting on a complex Hilbert space $H$ of infinite dimension $h$. An operator $T$ in $L(H)$ induces an inner derivation $\delta_T: L(H) \to L(H)$ defined by $\delta_T(X) = TX - XT$. Apostol's results give necessary and sufficient conditions on an operator $T$ so that $\text{ran}(\delta_T)$, the range of $\delta_T$, is norm closed in $L(H)$:

**Theorem 1** [1, Theorem 3.5]. For $T$ in $L(H)$, the following are equivalent:

(i) $\text{ran}(\delta_T)$ is closed in $L(H)$.

(ii) $p(T) = 0$ for some monic polynomial $p$ and $\text{ran}(q(T))$ is closed in $H$ for each polynomial $q$ dividing $p$.

(iii) $T$ is similar to a Jordan operator $J$.

(By Jordan operator, we mean that $J = \bigoplus_{j=1}^{m} \bigoplus_{i=1}^{m_j} (\alpha_{ij})$, where $0 < m < \infty$, $1 < m_j < \infty$, for each $j$, $1 \leq j \leq m$, $\lambda_1, \lambda_2, \ldots, \lambda_m$ are distinct complex scalars, $q_k$ denotes the Jordan nilpotent cell in $C_k$, $q_k(a)$ denotes the orthogonal direct sum of $a$ copies of $q_k$ acting in the usual fashion on $(C_k)^{(a)}$, the orthogonal direct sum of $a$ copies of $C_k$, and $\alpha_{ij} > 1$ for all $i$ and $j$).

In [4], the authors proved the analogues of Apostol's results for the quotient Calkin algebra $A(H) = L(H)/K(H)$, where $K(H)$ denotes the ideal of all compact operators acting on $H$: If $\pi: L(H) \to A(H)$ is the canonical projection, then $\text{ran}(\delta_{\pi(T)})$ is closed in $A(H)$ if and only if $T = A \cdot K$, where $A \in L(H)$ has the property that $\text{ran}(\delta_A)$ is closed, and $K \in K(H)$.

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The purpose of this note is to extend the results of [4] to the case when \( \dim H = h > \aleph_0 \) and \( A(H) \) is replaced by the quotient C*-algebra \( A_\alpha(H) = L(H)/J_\alpha \), where \( J_\alpha \) denotes some closed bilateral ideal of \( L(H) \), strictly larger than \( K[H] \).

The (not too surprising) answer is the same as in the case of \( A[H] \): if \( J_\alpha \) is a closed bilateral ideal in \( L[H] \) and \( \pi_\alpha : L[H] \rightarrow A_\alpha[H] \) is the canonical projection, then \( \operatorname{ran} \delta_{\pi_\alpha(T)} \) is closed in \( A_\alpha[H] \) if and only if \( \operatorname{ran} \delta_A \) is closed in \( L[H] \) for some \( A \in \pi_\alpha^{-1}[\pi_\alpha(T)] \); i.e., the range of \( \delta_A \) is closed for some \( A \) of the form \( T-K \), with \( K \in J_\alpha \).

However, some subtleties concerning the two possible types of cardinals involved in the definition of \( J_\alpha \) make it difficult to extrapolate the proofs given for the case when \( J_\alpha = K[H] \). The necessary modifications will be explained in the next section.

2. INNER DERIVATIONS WITH CLOSED RANGE IN QUOTIENT ALGEBRAS OF \( L[H] \) FOR A NON-SEPARABLE HILBERT SPACE \( H \).

Throughout this article, \( H \) will be a complex Hilbert space of (topological) dimension \( h > \aleph_0 \). Let \( \alpha \) be an infinite cardinal such that

\( \aleph_0 < \alpha < h \).

Then \( I_\alpha = \{ T \in L[H] : \dim(\operatorname{ran} T)^- < \alpha \} \) is a bilateral ideal of \( L[H] \) and \( J_\alpha = (I_\alpha)^- \) is a closed bilateral ideal of \( L[H] \). Moreover, it is well known that every non-trivial closed bilateral ideal of \( L[H] \) is equal to \( J_\alpha \) for some \( \alpha, \aleph_0 < \alpha < h \) (see [2], [5], [9]). (In the sequel the term "ideal" always refers to a non-trivial closed bilateral ideal of \( L[H] \)). In particular, if \( \alpha = \aleph_0 \), then \( I_\alpha \) is the (non-closed) bilateral ideal of all finite rank operators and \( J_\alpha = K[H] \) is the ideal of all compact operators.

Let \( J_\alpha \) be an ideal of \( L[H] \). If \( \pi_\alpha : L[H] \rightarrow A_\alpha[H] \) is the canonical projection of \( L[H] \) onto \( A_\alpha[H] = L[H]/J_\alpha \), and \( T \in L[H] \), then \( \pi_\alpha(T) \) will also be denoted by \( t_\alpha \) and \( \sigma(t_\alpha) = \sigma_\alpha(T) \) will denote the spectrum of \( t_\alpha \in A_\alpha[H] \), the \( \alpha \)-weighted spectrum of \( T \) [3]. The reader is also referred to [2], [6], and [8] for the analysis of weighted spectra. The principal result of this article is the following analogue of Theoremen 1.2 of [4]:

**THEOREM 2.** The following are equivalent for \( t_\alpha \in A_\alpha[H] \):

(i) \( \operatorname{ran}(\delta_{t_\alpha}) \) is closed;

(ii) \( \operatorname{ran}(\delta_T) + J_\alpha \) is closed in \( L[H] \);
(iv) \( \text{ran}(\delta_{T+K}) \) is closed in \( L(H) \) for some \( K \) in \( J_a \);
(v) \( T \) is similar to a \( J_a \)-perturbation of a Jordan operator;
(vi) \( t_a \sim J_a \) for some Jordan operator \( J \);
(vii) \( \rho(t_a) \) is similar to a Jordan operator for all unital \( * \)-representations \( \rho \) of the C*-algebra \( C^*(t_a) \) generated by \( t_a \) and \( l_a \);
(viii) \( \rho(t_a) \) is similar to a Jordan operator for some isometric unital \( * \)-representation \( \rho \) of \( C^*(t_a) \);
(ix) \( \rho(t_a) \) is similar to a Jordan operator for all unital \( * \)-representations \( \rho \) of \( \mathcal{A}_a \{ H \} \);
(x) \( p(T) \in J_a \) for some monic polynomial \( p \), and \( 0 \) is an isolated point of \( \sigma(q(t_a)q(t_a)) \) for all polynomials \( q \) dividing \( p \);
(xi) \( p(T) \in J_a \) for some monic polynomial \( p \), and \( \text{ran} q(T) \) is the algebraic direct sum of a (closed) subspace \( H_q \) and the range \( R_q \) of an operator \( R_q \in J_a \) for all polynomials \( q \) dividing \( p \).

Moreover, (i) implies that

(iii) \( \text{ran}(\delta_T) \cap \text{ran}(\delta_T) + J_a \), and (i) is equivalent to (iii) for the case when \( a \) is a countably cofinal cardinal (in particular, for \( a = \aleph_0 \)).

An infinite cardinal \( a \) is countably cofinal if \( a \) is the supremum of a countable collection of cardinals less than \( a \); e.g., \( a = \aleph_0 \). If \( a \) is countably cofinal, then \( I_a \neq J_a \); if \( a \) is not countably cofinal, then \( I_a \) is closed and therefore \( I_a = J_a \) [3])

The implications (v) \( \Rightarrow \) (i) \( \iff \) (ii) \( \iff \) (iii) and (iv) \( \iff \) (v) \( \Rightarrow \) (vi) \( \Rightarrow \) (x) \( \iff \) (vi) \( \iff \) (vii) \( \iff \) (ix) follow exactly as in the proof of [4, Theorem 1.2]. Thus, in order to complete the proof it only remains to show that (viii) \( \Rightarrow \) (v), (i) \( \Rightarrow \) (v) (if \( a \) is not countably cofinal) and (iii) \( \Rightarrow \) (v) (if \( a \) is countably cofinal).

**Lemma 3.** If \( p(t_a) \) is similar to a Jordan operator for some isometric unital \( * \)-representation \( \rho \) of \( C^*(t_a) \), then \( T \sim J+K \), where \( J \) is a Jordan operator and \( K \in J_a \) (i.e., (viii) \( \Rightarrow \) (v) in Theorem 2).

**Proof.** We proceed exactly as in the proof of [4, Proposition 2.8] except that, in this case, we have to apply C.L. Olsen's theorem for the ideals \( J_a \) [10, Theorem 4.3] in order to conclude that \( S = T+J \) (a suitable \( J_a \)-perturbation of \( T \)) admits a matrix of the form
Continuing as in the proof of [4, Proposition 2.8], with \( s_\alpha \), \( J_\alpha \) ... instead of \( s \), \( K(H) \) ..., we may reduce our problem to the case when 
\[ A = \rho(T_\alpha) \] satisfies \( A^k = 0 \), \( A^{k-1} \neq 0 \).

As in the proof of [4, Lemma 2.7], let \( \eta > 0 \) be such that \( (0, \eta) \cap \sigma(A^{j}A^j) = \emptyset, j = 1, \ldots, k-1 \). We claim that, after perhaps replacing \( \eta \) by a suitable number in the interval \([\eta/2, \eta]\), we may assume that \( \eta \notin \bigcup_{j=0}^{k} \sigma(T^{j}T^{j}) \). Observe that \( \sigma(T^{j}T^{j}) = \sigma(t^{j}T^{j}) = \sigma(A^{j}A^j) \), so that if \( E_j(.) \) denotes the spectral measure of \( T^{j}T^{j} \), then 
\[ \text{rank}[E_j([\eta/2, \eta])] = \beta_j < \alpha \text{ for all } j = 1, \ldots, k-1. \]
It follows from the analysis of weighted spectra [3], [8] that either \( 0 \) is an isolated point of \( \sigma_0(T^{j}T^{j}) = \sigma(t^{j}T^{j}) \) and \([\eta/2, \eta] \cap \bigcup_{j=0}^{k} \sigma(T^{j}T^{j}) \) is finite (in which case the validity of the claim is clear), or there exists a subspace \( H_\gamma \) of dimension \( \gamma \), \( \beta_j < \gamma < \alpha \), such that \( H_\gamma \) reduces \( T \) and such that if \( P_\gamma \) is the orthogonal projection of \( H \) onto \( H_\gamma \), then 
\[ \sigma_\alpha([1 - P_\gamma]T^{j}) \cap [\eta/2, \eta] = \emptyset \text{ for all } j = 1, \ldots, k-1. \]

Since \( P_\gamma T \in J_\alpha \), by replacing (if necessary) \( T \) by \( T - P_\gamma T \) and \( \eta \) by a suitable number in \([\eta/2, \eta]\), we can assume that \( \eta \notin \bigcup_{j=0}^{k} \sigma(T^{j}T^{j}) \).

Now the proof continues to follow that of [4, Lemma 2.7], with \( K(H) \) replaced by \( J_\alpha \) (namely, \( (L_j - L_{j-1}) - R_j \in J_\alpha \), etc.) and \( \pi \) replaced by \( T_\alpha \), until the point where we show that \( \rho = \pi_\alpha([1 \oplus \bigoplus_{j=2}^{k} T_{j, j+1}^* T_{j, j+1}]) = 1 \oplus \bigoplus_{j=2}^{k} A_{j, j+1}^* A_{j, j+1} \) is invertible in \( C^*(A) \). If \( \alpha = \mathcal{N}_0 \), the proof may be completed exactly as in the proof of [4, Lemma 2.7]. In the remaining case (\( \alpha > \mathcal{N}_0 \)), it is still true that \( T_{j, j+1} : H_j \to \text{ran}(T_{j, j+1}) \) has closed range in \( H_j \) and "small" nullity, i.e., \( \text{null}(T_{j, j+1}) = \dim \ker T_{j, j+1} < \alpha \) (\( j = 1, \ldots, k-1 \)). In this case, we can find a reducing subspace \( H_{j+1, \beta} \) of \( T_{j, j+1}^* T_{j, j+1} \) such that \( \dim(H_{j+1, \beta}) = \beta = \mathcal{N}_0 \text{null}(T_{j, j+1}) < \alpha \) and such that the restriction of \( T_{j, j+1} \) to \( H_{j+1, \beta} \) is bounded below. It is easy to check that 
\[ \dim[\text{ran}(T_{j, j+1}) \Theta T_{j, j+1}(H_{j+1, \beta})] \] is equal to \( h \) and 
\[ \dim[\text{ran}(T_{j, j+1}(H_{j+1, \beta})] = \beta; \] thus, we can find an operator \( K_{j, j+1} : H_{j+1} \to \text{ran}(T_{j, j+1}) \) such that \( H_{j+1, \beta} \) reduces \( K_{j, j+1} \).
ker(K_{j,j+1}) \supset H_{j+1} \oplus H_{j+1, \mathcal{S}}$, and $T_{j,j+1} = T_{j,j+1} + K_{j,j+1}$ is an invertible mapping from $H_{j+1}$ onto $\text{ran}(T_{j,j+1}) (= \text{ran}(T_{j,j+1}'))$. It is apparent that $\text{rank}(K_{j,j+1}) \leq \mathcal{S}$ and therefore $K_{j,j+1} \in J_{\mathcal{A}}$. Thus,

$$T_{j,j+1}: H_{j+1} \rightarrow \text{ran}(T_{j,j+1})$$

is an invertible $J_{\mathcal{A}}$-perturbation of $T_{j,j+1}$. It now follows from Apostol's criterion [1, Lemma 3.2, Corollary 3.4 and Theorem 3.5] that some $J_{\mathcal{A}}$-perturbation of $T$ is similar to a Jordan operator, and the result follows.

**Corollary 4.** Suppose $\dim(H_0) > \aleph_0$, $H_1$ is separable, $\alpha > \aleph_0$ is a countably cofinal cardinal, and $H = H_0 \oplus H_1(\alpha)$. Let $A \in \mathcal{L}(H_0)$, $T \in \mathcal{L}(H_1)$, $K \in J_\alpha(H)$, and $\Lambda = A \oplus T(\alpha) + K$. If $\text{ran} \delta_T$ is not closed, then $[\text{ran}(\delta_\Lambda)]^{-1}$ is not contained in $\text{ran}(\delta_\Lambda) + J_\alpha$.

**Proof.** The proof is based on suitable modifications of the proof of [4, Lemma 2.10], which we now outline. As in that proof, we can find $Y \in \mathcal{L}(H_1)$ and $(X_n)_{n=1}^\infty \subseteq \mathcal{L}(H_1)$ such that $\|TX_n - X_nT - Y\| \rightarrow 0$, but $Y \notin \text{ran}(\delta_T)$ and $f(n) = \|X_n\| \rightarrow \mathcal{Y}$ "very slowly". For each $\beta$, $1 \leq \beta < \alpha$,

$$\|T(\beta)X_n(\beta) - X_n(\beta)T(\beta) - Y(\beta)\| \rightarrow 0 \quad (n \rightarrow \infty) \quad \text{and} \quad Y(\beta) \notin \text{ran}(\delta_T(\beta)).$$

Modifications of the proof of [4, Lemma 2.10] permit us to construct an increasing sequence $(\alpha_n)_{n=1}^\infty$ of infinite cardinals such that

$$\alpha = \sum_{n=1}^\infty \alpha_n = \sup_n \alpha_n \quad \text{and} \quad H = H_0 \oplus H_1(\alpha) = H_0 \oplus \bigoplus_{n=1}^\infty H_1(\alpha_n).$$

If $P_n$ denotes the orthogonal projection of $H$ onto $H_0 \oplus \bigoplus_{j=1}^n H_1(\alpha_j)$, then we may also assume that $\|K(L - P_n)\| + \|(1 - P_n)K\| < 2^{-n}/(1 + f(n+1))$, $n > 1$.

Proceeding as in [4, Lemma 2.10], we define

$$Z_m = 0_{H_0} \oplus (X_1 \oplus X_2 \oplus \cdots \oplus X_m)^{\alpha_j} \oplus [\bigoplus_{j=m+1}^\infty X_j^{\alpha_j}]$$

then

$$LZ_m - Z_mL = (0_{H_0} \oplus (TX_j - X_jT)^{\alpha_j}) \oplus [\bigoplus_{j=m+1}^\infty (TX_j - X_jT)^{\alpha_j}] + KZ_m - Z_mK.$$  

Exactly as before, both the terms in parentheses and $\{KZ_m - Z_mK\}_{m=1}^\infty$ are Cauchy sequences, and the Cauchy sequence $\{\delta_\Lambda(Z_m)\}_{m=1}^\infty$ converges to $B = 0_{H_0} \oplus (Y_j \oplus A_j) + C$, where $C = (C_{i,j})_{i,j=0}^\infty \in J_\alpha$. Similarly, the assumption that $B = LZ - ZL + R$ for some $Z$ in $\mathcal{L}(H)$ and $R$ in
$J_\alpha$ leads to the contradiction that $Y$ is $\text{ran}(\delta_T)$. Hence $B \in [\text{ran}(\delta_L)]^-$ but $B \not\in \text{ran}(\delta_L) + J_\alpha$.

COROLLARY 5. If $\alpha$ is countably cofinal and $L \in L[H]$ is not similar to a $J_\alpha$-perturbation of a Jordan operator, then $[\text{ran}(\delta_L)]^-$ is not contained in $\text{ran}(\delta_L) + J_\alpha$.

Proof. If $\alpha = \mathcal{H}_0$, this is the result of [4, Corollary 2.11]. If $\alpha > \mathcal{H}_0$, we proceed exactly as in the proof of that result, except that in the present case, we have to use the results of [6] and [7] instead of [11, Theorem 1.3] in order to show that $L \cong L \circ T(\alpha) + K$ for some $K \in J_\alpha$ and some separably acting operator $T$ such that $\text{ran}(\delta_T)$ is not closed. Now the result follows from Corollary 4.

LEMMA 6. If $A \in L[H_0]$, $T$ acts on a separable space $H$, $\text{ran}(\delta_T)$ is not closed, $\alpha$ is not countably cofinal, and $L = A \circ T(\alpha) + K \in L[H]$ (where $H = H_0 \oplus H(\alpha)$ and $K \in J_\alpha$), then $[\text{ran}(\delta_L)]^-$ is not contained in $\text{ran}(\delta_L) + J_\alpha$.

Proof. Assume that $Y \in [\text{ran}(\delta_T)]^- \setminus [\text{ran}(\delta_T)]^-$; then as in the proof of [4, Lemma 2.10], $Y^o \in [\text{ran}(\delta_T)(N_0)]^- \setminus [\text{ran}(\delta_T)(N_0)]^-$. Since $\dim(\text{ran}(K))^- = \beta < \alpha$, it easily follows that $L = B \circ T(\alpha)$ with respect to a decomposition $H = H_B \oplus H(\gamma)$, where $\dim(H_B) = \dim(H_0) + \beta$ and $H(\gamma) \cong H(\alpha)$.

Clearly, $[\text{ran}(\delta_L)]^-$ contains an operator of the form $0 \circ N$, where $N \in L[H(\gamma)]$ is unitarily equivalent to $Y(\alpha)$. Assume that $0 \circ N = LZ - ZL + R$ for some $Z \in L[H]$ and some $T \in J_\alpha$; then $\dim(\text{ran}(R))^- = \beta < \alpha$ and $H(\gamma)$ contains a separable subspace $H'$ reducing $L$, $Z$, and $R$ such that $R|H' = 0$, $L|H' \cong T^{N_0}$, and $N|H' \cong Y^{N_0}$. Therefore $T^{N_0}Z - Z'T^{N_0} = Y^{N_0}$ for a suitable operator $Z' \cong Z|H'$, whence $Y^{N_0} \in \text{ran}(\delta_T(N_0))$, a contradiction.

COROLLARY 7. If $\alpha$ is not countably cofinal and $L \in L[H]$ is not similar to a $J_\alpha$-perturbation of a Jordan operator, then $\text{ran}(\delta_L)$ is not closed.

Proof. If $L$ is not of the form $W^{-1}JW + K$, where $W$ is invertible, $J$ is a Jordan operator, and $K \in J_\alpha$, then (by Lemma 3) there exists $T \in L[H_1]$ (where $H_1$ is a separable Hilbert space), not similar to a
Jordan operator, such that \( \rho(\ell_a) = T \) for some unital *-representation of \( C^*(\ell_a) \). Now it follows from [6] and [7] that the closure of the unitary orbit \( U(L) = \{U^*LU: U \text{ is unitary} \} \) of \( L \) contains an operator \( M \equiv L \circ T^a \).

Since \( \text{ran}(\delta_T) \) is not closed [1], it follows from Lemma 6 that \( \text{ran}(\delta_T) \) cannot be contained in \( \text{ran}(\delta_T) + J_a \), and thus \( \text{ran}(\delta_T) \) is not closed.

Now, if \( \text{ran}(\delta_T) \) is closed, then we can proceed exactly as in the proof of [1, Proposition 2.1] in order to show that \( U(\ell_a) \subset S(\ell_a) = (w_a^{-1}\ell_a w_a: w_a \text{ is invertible in } A_a(H)) \); thus \( m_a \sim \ell_a \), whence we conclude that \( \text{ran}(\delta_T) \) is closed too, a contradiction.

Now we are in a position to complete the proof of Theorem 2:
(viii) \( \Rightarrow \) (v) is the content of Lemma 3. If \( \alpha \) is countably cofinal, then it follows from Corollary 5 that (iii) \( \Rightarrow \) (v), completing the proof in this case. Finally, if \( \alpha \) is not countably cofinal, then it follows from Corollary 7 that (i) \( \Rightarrow \) (v).
REFERENCES


Lawrence A. Fialkow  
Western Michigan University  
Kalamazoo, MI 49104, USA

and

Domingo A. Herrero  
Arizona State University  
Tempe, AZ 85287, USA

Adelphi University  
Garden City, NY 11530, USA

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