Revista de la
Unión Matemática Argentina
Volumen 31, 1983.

INNER DERIVATIONS WITH CLOSED RANGE IN THE CALKIN ALGEBRA. II: THE NON-SEPARABLE CASE

Lawrence A. Fialkow and Domingo A. Herrero

1. INTRODUCTION.

In [1], C. Apostol characterized the Hilbert space operators which induce inner derivations having closed range. Let $L(H)$ denote the algebra of all (bounded linear) operators acting on a complex Hilbert space H of infinite dimension h. An operator T in $L(H)$ induces an inner derivation δ_T: $L(H) \longrightarrow L(H)$ defined by $\delta_T(X) = TX-XT$. Apostol's results give necessary and sufficient conditions on an operator T so that ran(δ_T), the range of δ_T, is norm closed in $L(H)$:

THEOREM 1 [1, Theorem 3.5]. *For* T *in* $L(H)$, *the following are equivalent:*

(i) ran(δ_T) *is closed in* $L(H)$.

(ii) $p(T) = 0$ *for some monic polynomial* p *and* ran q(T) *is closed in* H *for each polynomial* q *dividing* p.

(iii) T *is similar to a Jordan operator* J.

(By *Jordan operator*, we mean that $J = \bigoplus_{j=1}^{m} [\lambda_j + \bigoplus_{i=1}^{m_j} q_{k_{ij}}^{(\alpha_{ij})}]$, where $0 < m < \infty$, $1 \leq m_j < \infty$, for each j, $1 \leq j \leq m$, $\lambda_1,\lambda_2,\ldots,\lambda_m$ are distinct complex scalars, q_k denotes the Jordan nilpotent cell in C^k, $q_k^{(\alpha)}$ denotes the orthogonal direct sum of α copies of q_k acting in the usual fashion on $(C^k)^{(\alpha)}$, the orthogonal direct sum of α copies of C^k, and $\alpha_{ij} \geq 1$ for all i and j).

In [4], the authors proved the analogues of Apostol's results for the quotient Calkin algebra $A(H) = L(H)/K(H)$, where $K(H)$ denotes the ideal of all compact operators acting on H: If π: $L(H) \longrightarrow A(H)$ is the canonical projection, then ran $\delta_{\pi(T)}$ is closed in $A(H)$ if and only if T = = A+K, where $A \in L(H)$ has the property that ran δ_A is closed, and $K \in K(H)$.

This research was partially supported by Grants of the National Science Foundation.

The purpose of this note is to extend the results of [4] to the case when $\dim H = h > \aleph_o$ and $A(H)$ is replaced by the quotient C*-algebra $A_\alpha(H) = L(H)/J_\alpha$, where J_α denotes some closed bilateral ideal of $L(H)$, strictly larger than $K(H)$.

The (not too surprising) answer is the same as in the case of $A(H)$: if J_α is a closed bilateral ideal in $L(H)$ and π_α: $L(H) \longrightarrow A_\alpha(H)$ is the canonical projection, then ran $\delta_{\pi_\alpha(T)}$ is closed in $A_\alpha(H)$ if and only if ran δ_A is closed in $L(H)$ for some A in $\pi_\alpha^{-1}[\pi_\alpha(T)]$; i.e., the range of δ_A is closed for some A of the form T-K, with $K \in J_\alpha$. However, some subtleties concerning the two possible types of cardinals involved in the definition of J_α make it difficult to extrapolate the proofs given for the case when $J_\alpha = K(H)$. The necessary modifications will be explained in the next section.

2. INNER DERIVATIONS WITH CLOSED RANGE IN QUOTIENT ALGEBRAS OF $L(H)$ FOR A NON-SEPARABLE HILBERT SPACE H.

Throughout this article, H will be a complex Hilbert space of (topological) dimension $h > \aleph_o$. Let α be an infinite cardinal such that $\aleph_o \leqslant \alpha \leqslant h$. Then $I_\alpha = \{T \in L(H)$: $\dim(\text{ran } T)^- < \alpha\}$ is a bilateral ideal of $L(H)$ and $J_\alpha = (I_\alpha)^-$ is a closed bilateral ideal of $L(H)$. Moreover, it is well known that every non-trivial closed bilateral ideal of $L(H)$ is equal to J_α for some α, $\aleph_o \leqslant \alpha \leqslant h$ (see [2], [5], [9]). (In the sequel the term "ideal" always refers to a non-trivial closed bilateral ideal of $L(H)$). In particular, if $\alpha = \aleph_o$, then I_α is the (non-closed) bilateral ideal of all finite rank operators and $J_\alpha = K(H)$ is the ideal of all compact operators.

Let J_α be an ideal of $L(H)$. If π_α: $L(H) \longrightarrow A_\alpha(H)$ is the canonical projection of $L(H)$ onto $A_\alpha(H) = L(H)/J_\alpha$, and $T \in L(H)$, then $\pi_\alpha(T)$ will also be denoted by t_α and $\sigma(t_\alpha) = \sigma_\alpha(T)$ will denote the spectrum of $t_\alpha \in A_\alpha(H)$, the α-*weighted spectrum* of T [3]. The reader is also referred to [2], [6], and [8] for the analysis of weighted spectra. The principal result of this article is the following analogue of Theoremen 1.2 of [4]:

THEOREM 2. *The following are equivalent for* $t_\alpha \in A_\alpha(H)$:

(i) $\text{ran}(\delta_{t_\alpha})$ *is closed;*

(ii) $\text{ran}(\delta_T) + J_\alpha$ *is closed in* $L(H)$*;*

(iv) $\operatorname{ran}(\delta_{T+K})$ *is closed in* $L(H)$ *for some* K *in* J_α*;*

(v) T *is similar to a* J_α*-perturbation of a Jordan operator;*

(vi) $t_\alpha \sim j_\alpha$ *for some Jordan operator* J*;*

(vii) $\rho(t_\alpha)$ *is similar to a Jordan operator for all unital *-representations* ρ *of the C*-algebra* $C^*(t_\alpha)$ *generated by* t_α *and* 1_α*;*

(viii) $\rho(t_\alpha)$ *is similar to a Jordan operator for some isometric unital *-representation* ρ *of* $C^*(t_\alpha)$*;*

(ix) $\rho(t_\alpha)$ *is similar to a Jordan operator for all unital *-representations* ρ *of* $A_\alpha(H)$*;*

(x) $p(T) \in J_\alpha$ *for some monic polynomial* p*, and* 0 *is an isolated point of* $\sigma[q(t_\alpha)^*q(t_\alpha)]$ *for all polynomials* q *dividing* p*;*

(xi) $p(T) \in J_\alpha$ *for some monic polynomial* p*, and* ran q(T) *is the algebraic direct sum of a (closed) subspace* H_q *and the range* R_q *of an operator* $R_q \in J_\alpha$ *for all polynomials* q *dividing* p.

Moreover, (i) *implies that*

(iii) $[\operatorname{ran}(\delta_T)]^- \subset \operatorname{ran}(\delta_T) + J_\alpha$, *and* (i) *is equivalent to* (iii) *for the case when* α *is a countably cofinal cardinal (in particular, for* $\alpha = \aleph_0$*).*

(An infinite cardinal α is *countably cofinal* if α is the supremum of a countable collection of cardinals less than α; e.g., $\alpha = \aleph_0$. If α is countably cofinal, then $I_\alpha \neq J_\alpha$; if α is not countably cofinal, then I_α is closed and therefore $I_\alpha = J_\alpha$ [3]).

The implications (v) $\Rightarrow$ (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) and (iv) $\Leftrightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (xi) $\Rightarrow$ (x) $\Leftrightarrow$ (vii) $\Leftrightarrow$ (viii) $\Leftrightarrow$ (ix) follow exactly as in the proof of [4, Theorem 1.2]. Thus, in order to complete the proof it only remains to show that (viii) $\Rightarrow$ (v), (i) $\Rightarrow$ (v) (if α *is not* countably cofinal) and (iii) $\Rightarrow$ (v) (if α *is* countably cofinal).

LEMMA 3. *If* $p(t_\alpha)$ *is similar to a Jordan operator for some isometric unital *-representation* ρ *of* $C^*(t_\alpha)$*, then* $T \sim J+K$*, where* J *is a Jordan operator and* $K \in J_\alpha$ *(i.e.,* (viii) $\Rightarrow$ (v) *in Theorem 2).*

Proof. We proceed exactly as in the proof of [4, Proposition 2.8] except that, in this case, we have to apply C.L. Olsen's theorem for the ideals J_α [10, Theorem 4.3] in order to conclude that S = T+J (a suitable J_α-perturbation of T) admits a matrix of the form

$$\begin{pmatrix} T_1 & & * \\ & T_2 & \\ & & \ddots & \\ 0 & & & T_n \end{pmatrix}.$$

Cont·nuing as in the proof of [4, Proposition 2.8], with s_α, J_α ... instead of s, $K(H)$ ..., we may reduce our problem to the case when $A = \rho(t_\alpha)$ satisfies $A^k = 0$, $A^{k-1} \neq 0$.

As in the proof of [4, Lemma 2.7], let $\eta > 0$ be such that $(0,\eta) \cap \cap\, \sigma(A^{*j}A^j) = \emptyset$, $j = 1,2,\ldots,k-1$. We claim that, after perhaps replacing η by a suitable number in the interval $[\eta/2,\eta]$, we may assume that $\eta \notin \bigcup_{j=0}^{k} \sigma(T^{*j}T^j)$. Observe that $\sigma_\alpha(T^{*j}T^j) = \sigma(t_\alpha{}^{*j}t_\alpha{}^j) = \sigma(A^{*j}A^j)$, so that if $E_j(.)$ denotes the spectral measure of $T^{*j}T^j$, then $\text{rank}[E_j([\eta/2,\eta])] = \beta_j < \alpha$ for all $j = 1,2,\ldots,k-1$. It follows from the analysis of weighted spectra [3], [8] that either 0 is an isolated point of $\sigma_{\aleph_o}(T^{*j}T^j) = \sigma(t^{*j}t^j)$ and $[\eta/2,\eta] \cap [\bigcup_{j=0}^{k} \sigma(T^{*j}T^j)]$ is finite (in which case the validity of the claim is clear), or there exists a subspace H_γ of dimension γ, $\beta_j < \gamma < \alpha$, such that H_γ reduces T and such that if P_γ is the orthogonal projection of H onto H_γ, then $\sigma_\alpha([(1 - P_\gamma)T^*]^j[(1 - P_\gamma)T]^j) \cap [\eta/2,\eta] = \emptyset$ for all $j = 1,2,\ldots,k-1$. Since $P_\gamma T \in J_\alpha$, by replacing (if necessary) T by $T-P_\gamma T$ and η by a suitable number in $[\eta/2,\eta]$, we can assume that $\eta \notin \bigcup_{j=0}^{k-1} \sigma(T^{*j}T^j)$.

Now the proof continues to follow that of [4, Lemma 2.7], with $K(H)$ replaced by J_α (namely, $(L_j - L_{j-1}) - R_j \in J_\alpha$, etc.) and π replaced by π_α, until the point where we show that $\rho\circ\pi_\alpha(1 \oplus [\bigoplus_{j=2}^{k} T_{j,j+1}{}^*T_{j,j+1}]) = 1 \oplus [\bigoplus_{j=2}^{k} A_{j,j+1}{}^*A_{j,j+1}]$ is invertible in $C^*(A)$. If $\alpha = \aleph_o$, the proof may be completed exactly as in the proof of [4, Lemma 2.7]. In the remaining case ($\alpha > \aleph_o$), it is still true that $T_{j,j+1}: H_{j+1} \longrightarrow \text{ran}(T_{j,j+1})$ has closed range in H_j and "small" nullity, i.e., $\text{nul}(T_{j,j+1}) = \dim\ker T_{j,j+1} < \alpha$ $(j = 1,2,\ldots,k-1)$. In this case, we can find a reducing subspace $H_{j+1,\beta}$ of $T_{j,j+1}{}^*T_{j,j+1}$ such that $\dim(H_{j+1,\beta}) = \beta = \aleph_o \text{nul}(T_{j,j+1}) < \alpha$ and such that the restriction of $T_{j,j+1}$ to $H_{j+1} \ominus H_{j+1,\beta}$ is bounded below. It is easy to check that $\dim[\text{ran}(T_{j,j+1}) \ominus T_{j,j+1}(H_{j+1,\beta})]$ is equal to h and $\dim[T_{j,j+1}(H_{j+1,\beta})]^- = \beta$; thus, we can find an operator $K_{j,j+1}: H_{j+1} \longrightarrow \text{ran}(T_{j,j+1})$ such that $H_{j+1,\beta}$ reduces $K_{j,j+1}{}^*K_{j,j+1}$,

$\ker(K_{j,j+1}) \supset H_{j+1} \ominus H_{j+1,\beta}$, and $T'_{j,j+1} = T_{j,j+1} + K_{j,j+1}$ is an invertible mapping from H_{j+1} onto $\mathrm{ran}(T_{j,j+1})$ (= $\mathrm{ran}(T'_{j,j+1})$). It is apparent that $\mathrm{rank}(K_{j,j+1}) \leqslant \beta$ and therefore $K_{j,j+1} \in J_\alpha$. Thus, $T'_{j,j+1}: H_{j+1} \longrightarrow \mathrm{ran}(T'_{j,j+1})$ is an invertible J_α-perturbation of $T_{j,j+1}$. It now follows from Apostol's criterion [1, Lemma 3.2, Corollary 3.4 and Theorem 3.5] that some J_α-perturbation of T is similar to a Jordan operator, and the result follows.

COROLLARY 4. *Suppose* $\dim(H_0) > \aleph_0$, H_1 *is separable,* $\alpha > \aleph_0$ *is a countably cofinal cardinal, and* $H = H_0 \oplus H_1^{(\alpha)}$. *Let* $A \in L(H_0)$, $T \in L(H_1)$, $K \in J_\alpha(H)$, *and* $L = A \oplus T^{(\alpha)} + K$. *If* $\mathrm{ran}\,\delta_T$ *is not closed, then* $[\mathrm{ran}(\delta_L)]^-$ *is not contained in* $\mathrm{ran}(\delta_L) + J_\alpha$.

Proof. The proof is based on suitable modifications of the proof of [4, Lemma 2.10], which we now outline. As in that proof, we can find $Y \in L(H_1)$ and $\{X_n\}_{n=1}^\infty \subset L(H_1)$ such that $\|(TX_n - X_nT) - Y\| \longrightarrow 0$, but $Y \notin \mathrm{ran}(\delta_T)$ and $f(n) = \|X_n\| \uparrow \infty$ "very slowly". For each β, $1 \leqslant \beta \leqslant \alpha$, clearly,

$$\|(T^{(\beta)}X_n^{(\beta)} - X_n^{(\beta)}T^{(\beta)}) - Y^{(\beta)}\| \longrightarrow 0 \ (n \to \infty) \text{ and } Y^{(\beta)} \notin \mathrm{ran}(\delta_{T^{(\beta)}}).$$

Modifications of the proof of [4, Lemma 2.10] permit us to construct an increasing sequence $\{\alpha_n\}_{n=1}^\infty$ of infinite cardinals such that $\alpha = \sum_{n=1}^\infty \alpha_n = \sup_n \alpha_n$ and $H = H_0 \oplus H_1^{(\alpha)} = H_0 \oplus [\oplus_{n=1}^\infty H_1^{(\alpha_n)}]$. If P_n denotes the orthogonal projection of H onto $H_0 \oplus [\oplus_{j=1} H_1^{(\alpha_j)}]$, then we may also assume that $\|K(L - P_n)\| + \|(1 - P_n)K\| < 2^{-n}/([1 + f(n+1)]$, $n \geqslant 1$.

Proceeding as in [4, Lemma 2.10], we define

$$Z_m = 0_{H_0} \oplus [X_1^{(\alpha_1)} \oplus X_2^{(\alpha_2)} \oplus \dots \oplus X_m^{(\alpha_m)}] \oplus [\oplus_{j=m+1}^\infty X_m^{(\alpha_j)}] ;$$

then

$$LZ_m - Z_mL = (0_{H_0} \oplus [\oplus_{j=1}^m (TX_j - X_jT)^{(\alpha_j)}] \oplus [\oplus_{j=m+1}^\infty (TX_m - X_mT)^{(\alpha_j)}]) + \\ + KZ_m - Z_mK.$$

Exactly as before, both the terms in parentheses and $\{KZ_m - Z_mK\}_{m=1}^\infty$ are Cauchy sequences, and the Cauchy sequence $\{\delta_L(Z_m)\}_{m=1}^\infty$ converges to $B = 0_{H_0} \oplus [\oplus_{j=1}^\infty Y^{(\alpha_j)} - \oplus_{j=1}^\infty A_j] + C$, where $C = (C_{ij})_{i,j=0}^\infty \in J_\alpha$. Similarly, the assumption that $B = LZ - ZL + R$ for some Z in $L(H)$ and R in

J_α leads to the contradiction that Y is $\mathrm{ran}(\delta_T)$. Hence $B \in [\mathrm{ran}(\delta_L)]^-$ but $B \notin \mathrm{ran}(\delta_L) + J_\alpha$.

COROLLARY 5. *If* α *is countably cofinal and* $L \in L(H)$ *is not similar to a* J_α*-perturbation of a Jordan operator, then* $[\mathrm{ran}(\delta_L)]^-$ *is not contained in* $\mathrm{ran}(\delta_L) + J_\alpha$.

Proof. If $\alpha = \aleph_0$, this is the result of [4, Corollary 2.11]. If $\alpha > \aleph_0$, we proceed exactly as in the proof of that result, except that in the present case, we have to use the results of [6] and [7] instead of [11, Theorem 1.3] in order to show that $L \cong L \oplus T^{(\alpha)} + K$ for some $K \in J_\alpha$ and some separably acting operator T such that $\mathrm{ran}(\delta_T)$ is not closed. Now the result follows from Corollary 4.

LEMMA 6. *If* $A \in L(H_0)$, T *acts on a separable space* H_1, $\mathrm{ran}(\delta_T)$ *is not closed,* α *is not countably cofinal, and* $L = A \oplus T^{(\alpha)} + K \in L(H)$ *(where* $H = H_0 \oplus H_1^{(\alpha)}$ *and* $K \in J_\alpha$*), then* $[\mathrm{ran}(\delta_L)]^-$ *is not contained in* $\mathrm{ran}(\delta_L) + J_\alpha$.

Proof. Assume that $Y \in [\mathrm{ran}(\delta_T)]^- \setminus \mathrm{ran}(\delta_T)$; then as in the proof of [4, Lemma 2.10], $Y^{(\aleph_0)} \in [\mathrm{ran}(\delta_{T^{(\aleph_0)}})]^- \setminus \mathrm{ran}(\delta_{T^{(\aleph_0)}})$. Since $\dim[\mathrm{ran}(K)]^- = \beta < \alpha$, it easily follows that $L = B \oplus T^{(\alpha)}$ with respect to a decomposition $H = H_B \oplus H_\gamma$, where $\dim(H_B) = \dim(H_0) + \beta$ and $H_\gamma \cong H_1^{(\alpha)}$.

Clearly, $[\mathrm{ran}(\delta_L)]^-$ contains an operator of the form $0 \oplus N$, where $N \in L(H_\gamma)$ is unitarily equivalent to $Y^{(\alpha)}$. Assume that $0 \oplus N = LZ - ZL + R$ for some $Z \in L(H)$ and some $R \in J_\alpha$; then $\dim[\mathrm{ran}(R)]^- = \beta' < \alpha$ and H_γ contains a separable subspace H' reducing L, Z, and R such that $R|H' = 0$, $L|H' \cong T^{(\aleph_0)}$, and $N|H' \cong Y^{(\aleph_0)}$. Therefore $T^{(\aleph_0)}Z' - Z'T^{(\aleph_0)} = Y^{(\aleph_0)}$ for a suitable operator $Z' \cong Z|H'$, whence we conclude that $Y^{(\aleph_0)} \in \mathrm{ran}(\delta_{T^{(\aleph_0)}})$, a contradiction.

COROLLARY 7. *If* α *is not countably cofinal and* $L \in L(H)$ *is not similar to a* J_α*-perturbation of a Jordan operator, then* $\mathrm{ran}(\delta_{\ell_\alpha})$ *is not closed.*

Proof. If L is not of the form $W^{-1}JW + K$, where W is invertible, J is a Jordan operator, and $K \in J_\alpha$, then (by Lemma 3) there exists $T \in L(H_1)$ (where H_1 is a separable Hilbert space), not similar to a

Jordan operator, such that $\rho(\ell_\alpha) = T$ for some unital *-representation of $C^*(\ell_\alpha)$. Now it follows from [6] and [7] that the closure of the unitary orbit $\mathcal{U}(L) = \{U^*LU\colon U \text{ is unitary}\}$ of L contains an operator $M \cong L \oplus T^{(\alpha)}$.

Since $\mathrm{ran}(\delta_T)$ is not closed [1], it follows from Lemma 6 that $[\mathrm{ran}(\delta_M)]^-$ cannot be contained in $\mathrm{ran}(\delta_M) + J_\alpha$, and thus $\mathrm{ran}(\delta_{m_\alpha})$ is not closed.

Now, if $\mathrm{ran}(\delta_{\ell_\alpha})$ is closed, then we can proceed exactly as in the proof of [1, Proposition 2.1] in order to show that $\mathcal{U}(\ell_\alpha)^- \subset \mathcal{S}(\ell_\alpha) = \{w_\alpha^{-1}\ell_\alpha w_\alpha\colon w_\alpha \text{ is invertible in } A_\alpha(H)\}$; thus $m_\alpha \sim \ell_\alpha$, whence we conclude that $\mathrm{ran}(\delta_{m_\alpha})$ is closed too, a contradiction.

Now we are in a position to complete the proof of Theorem 2: (viii) ⇒ (v) is the content of Lemma 3. If α is countably cofinal, then it follows from Corollary 5 that (iii) ⇒ (v), completing the proof in this case. Finally, if α is not countably cofinal, then it follows from Corollary 7 that (i) ⇒ (v).

REFERENCES

[1] C. APOSTOL, *Inner derivations with closed range*, Rev. Roum. Math. Pures et Appl. 21(1976), 249-265.

[2] L.A. COBURN and A. LEBOW, *Components of invertible elements in quotient algebras of operators*, Trans. Amer. Math. Soc. 130(1966), 359-366.

[3] G. EDGAR, J. ERNEST and S.-G. LEE, *Weighing operator spectra*, Indiana Univ. Math., J. 21(1971), 61-80.

[4] L.A. FIALKOW and D.A. HERRERO, *Inner derivations with closed range in the Calkin algebra*, Indiana Univ. Math. J. (To appear).

[5] B. GRAMSCH, *Eine Idealstruktur Banachscher Operatoralgebren*, J. Reine Angew. Math. (Crelle's Journal) 225(1967), 97-115.

[6] D.W. HADWIN, *Nonseparable approximate equivalence*, Trans. Amer. Math. Soc. 266(1981), 203-231.

[7] D.W. HADWIN, *Approximate equivalence and completely positive maps*, (Preprint).

[8] D.A. HERRERO, *Norm limits of nilpotent operators and weighted spectra in non-separable Hilbert spaces*, Rev. Un. Mat. Argentina 27(1975), 83-105.

[9] E. LUFT, *The two-sided closed ideals of the algebra of bounded linear operators on a Hilbert space*, Czekoslovak Math. J. 18(1968), 595-605.

[10] C.L. OLSEN, *A structure theorem for polynomially compact operators*, Amer. J. Math. 93(1971), 688-698.

[11] D. VOICULESCU, *A non-commutative Weyl-von Neumann theorem*, Rev. Roum. Math. Pures et Appl. 21(1976), 97-113.

Lawrence A. Fialkow
Western Michigan University
Kalamazoo, MI 49104, USA

and

Adelphi University
Garden City, NY 11530, USA

Domingo A. Herrero
Arizona State University
Tempe, AZ 85287, USA

Recibido en marzo de 1983.