Revista de la
Unión Matemática Argentina
Volumen 31, 1983.

WHAT F_σ SETS CAN BE NUMERICAL RANGES OF OPERATORS?

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The classical Toeplitz-Hausdorff theorem asserts that if T is a bounded linear operator acting on a Hilbert space H, then its *numerical range* W(T) (defined by

$$W(T) = \{\langle Tx,x\rangle : x \in H,\ \|x\| = 1\})$$

is a nonempty bounded convex subset of the complex plane C. It is also known and easy to prove that the closure $W(T)^-$ of W(T) contains the spectrum of T and that

$$\frac{1}{2} \leqslant \frac{1}{\|T\|} \sup \{|\lambda| : \lambda \in W(T)\} \leqslant 1$$

(see [2, Chapter 17, p.114]).

In a recent article, J. Agler proved that if T acts on a *separable* Hilbert space H and the boundary W(T) of its numerical range does not contain denumerably many linear segments, then W(T) is an F_σ subset of C [1].

Agler's result suggests the obvious problem: For which nonempty bounded convex F_σ subsets F of $\mathbb{C}$ does there exist T = T(F) such that W(T) = F?.

A large family of examples will be exhibited; it will show that the boundary of W(T) can be very pathological and, in particular, that for each F as above there exists T such that W(T) is homeomorphic to F and "approximately equal" to F.

In addition to the above observations, we shall only need two results about numerical ranges:

(1) If $\{T_\nu\}_{\nu\in\Gamma}$ (T_ν acting on H_ν for each ν in Γ) is a uniformly bounded family of Hilbert space operators and $T = \oplus_{\nu\in\Gamma} T_\nu$ denotes the direct sum of the T_ν's acting in the usual fashion on the *orthogonal direct sum* $H = \oplus_{\nu\in\Gamma} H_\nu$ of the underlying Hilbert spaces, then W(T) coincides with the *convex hull*, $co[\cup_{\nu\in\Gamma} W(T_\nu)]$ of the union of the numerical ranges of the T_ν's ($\nu \in \Gamma$).

(*) This research has been supported by a National Science Foundation Grant. AMS CLASSIFICATION NUMBER: 47A12.

(ii) (M. Radjabalipour and H. Radjavi [3]) If γ is a nonempty bounded open arc of a conic curve, then there exists an operator T_γ such that $W(T_\gamma) = co(\gamma)$.

The author wishes to thank Professor Jim Agler for several helpful discussions.

THE CONSTRUCTION. Let $F = \bigcup_{n=1}^{\infty} F_n$, where F_n $(n = 1,2,\ldots)$ is a nonempty closed subset of ∂D, and D denotes the open unit disk, and let $\varepsilon > 0$ be given.

Define $T_0 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ (acting on C^2). A straightforward computation shows that $W(T_0) = D^-$. In what follows, rD $(r > 0)$ denotes the open unit disk of radius r.

Assume that $\partial D \setminus F_1 = \bigcup_{n\in N_1} \gamma_{1n}$, where $\{\gamma_{1n}\}_{n\in N_1}$ is an enumeration of the (pairwise disjoint) arcs complementary to F_1 in ∂D. If γ_{1n} is an arc in this family with extreme points α_{1n} and β_{1n} (we shall always assume that γ is "positively oriented", so that $\gamma \subset \{\lambda: \arg \alpha < \arg \lambda < \arg \beta\}$), then we replace γ_{1n} by an arc of ellipse δ_{1n} tangent to ∂D at α_{1n} and at β_{1n} and $\delta_{1n} \subset (1+\varepsilon/2)D \setminus D^-$. As observed above, there exists an operator T_{1n} such that $W(T_{1n}) = co(\delta_{1n})$. Define $T_1 = \bigoplus_{n\in N_1} T_{1n}$; then $M_1 = W(T_0 \oplus T_1) = co[W(T_0) \cup W(T_1)]$ is the *closed* convex set with boundary equal to $(\partial D \setminus \bigcup_{n\in N_1} \gamma_{1n}) \cup (\bigcup_{n\in N_1} \delta_{1n})$.

Assume that we have already defined $T_0, T_1, \ldots, T_m$, so that $W(\bigoplus_{n=0}^{m} T_n) = co[\bigcup_{n=0}^{m} W(T_n)]$ is a closed convex set M_m satisfying the conditions

$(i)_m$ $D^- \subset M_m \subset (1 + \sum_{n=1}^{m} \varepsilon/2^n)D$;

$(ii)_m$ ∂M_m has a continuous tangent;

$(iii)_m$ $\partial M_m = (\bigcup_{n=1}^{m} F'_n) \cup (\bigcup_{n\in N_m} \delta_{mn})$, where $\bigcup_{n=1}^{m} F'_n = \partial M_m \cap (\bigcup\{re^{i\theta}: r > 0,\ e^{i\theta} \in \bigcup_{n=1}^{m} F_n\})$ and $\{\delta_{mn}\}_{n\in N_m}$ is an enumeration of the (pairwise disjoint) open arcs complementary to $\bigcup_{n=1}^{m} F'_n$ in ∂M_m;

$(iv)_m$ For each $n \in N_m$, δ_{mn} is an arc of ellipse.

Let $F'_{m+1} = (\bigcup_{n\in N_m} \delta_{mn}) \cap (\bigcup\{re^{i\theta}: r > 0,\ e^{i\theta} \in F_{m+1}\})$ and let $\{\gamma_{m+1,n}\}_{n\in N_{m+1}}$ be an enumeration of the (pairwise disjoint) open arcs of ellipse complementary to F'_{m+1} in $\bigcup_{n\in N_m} \delta_{mn}$. Replace each of the arcs

$\gamma_{m+1,n}$ by an arc of ellipse $\delta_{m+1,n}$ lying in the same angular sector as $\gamma_{m+1,n}$ and with the same extreme points, such that $\delta_{m+1,n}$ is tangent to ∂M_m at those extreme points and $\delta_{m+1,n} \subset (1 + \sum_{n=1}^{m+1} \varepsilon/2^n)D \setminus M_m$;

then there exists an operator $T_{m+1,n}$ such that $W(T_{m+1,n}) = co(\delta_{m+1,n})$.

Define $T_{m+1} = \bigoplus_{n \in N_{m+1}} T_{m+1,n}$; then $W(\bigoplus_{n=0}^{m+1} T_n) = co[\bigcup_{n=0}^{m+1} W(T_n)]$ is a closed convex set M_{m+1} satisfying the conditions $(i)_{m+1}$, $(ii)_{m+1}$, $(iii)_{m+1}$ and $(iv)_{m+1}$ such that $\partial M_{m+1} \setminus \partial M_m = \bigcup_{n=1}^{m+1} F'_n$ is homeomorphic (via projection through the origin) to $\bigcup_{n=1}^{m} F_n \subset \partial D$.

Finally, we define $T = \bigoplus_{n=0}^{\infty} T_n$.

Clearly, $W(T) = co[\bigcup_{n=0}^{\infty} W(T_n)]$ satisfies:

(i) $D^- \subset W(T) \subset (1+\varepsilon)D$;

(ii) $\partial W(T)$ has a continuous tangent;

(iii) $W(T) \setminus \partial W(T) = \bigcup_{n=1}^{\infty} F'_n$ (F'_1 is defined equal to F_1) is homeomorphic (via projection through the origin) to $\bigcup_{n=1}^{\infty} F_n = F$; and

(iv) Each arc $\gamma \subset \partial W(T) \setminus W(T)$ is an arc of ellipse.

Let $\rho = \rho(\theta)$ be the polar equation of $\partial W(T)$; property (i) implies that $1 \leqslant \rho(\theta) < 1+\varepsilon$ for all $\theta \in [0,2\pi)$ and property (ii) says that ρ' is continuous. On the other hand, it is completely apparent that, by a clever choice of the arcs $\{\delta_{mn}\}_{n \in N_m}$ $(m = 1,2,\dots)$, we can also obtain

(v) $|\rho'(\theta)| < \varepsilon$, $0 \leqslant \theta < 2\pi$, and

(vi) $|D^{\pm}[\rho'(\theta)]| < \varepsilon$ and $|D_{\pm}[\rho'(\theta)]| < \varepsilon$, where D^+ and D^-, and D_+ and D_- denote the right and, respectively, the left Darboux derivative numbers (of the continuous function $\rho'(\theta)$).

THE GENERAL EXAMPLE. Ad hoc modifications of the previous construction show that, given an arbitrary bounded convex F_σ subset H of C with nonempty interior and $\varepsilon > 0$, there exists an operator $T = T(H,\varepsilon)$ acting on a separable Hilbert space $\mathcal{H}$ such that W(T) satisfies the following conditions·

(i) $H \subset W(T) \subset \{\lambda: \text{dist}[\lambda,H] < \varepsilon\}$,

(ii) Furthermore, given $\alpha \in$ interior H, T can be chosen that there exists a continuous function $\rho = r(\theta)$ $(0 \leqslant \theta \leqslant 2\pi)$ satisfying $1 \leqslant r(\theta) < 1+\varepsilon$, such that $W(T) = \{\alpha\} \cup \{\alpha + (\lambda-\alpha)\, r(\arg[\lambda-\alpha]) : \lambda \in H \setminus \{\alpha\}\}$. In particular, W(T) is homeomorphic to H, and $\partial W(T)$ is homeomorphic to ∂H via projection through the point α.

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Recibido en marzo de 1983.