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CLIFFORD ISOMETRIES OF COMPACT HOMOGENEOUS RIEMANNIAN MANIFOLDS

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ABSTRACT. Let G be a compact connected simple Lie group and $H \subset G$ a closed connected Lie subgroup.

Consider the manifold $M = G/H$ with a G-invariant Riemannian metric.

Let T' be a maximal torus of H and let T be a maximal torus of G containing T'. Assume the Lie algebra of T has an orthonormal basis B that contains a basis for the Lie algebra of T' and such that the Weyl group W(G,T) contains every transposition of B.

In this situation we give a necessary condition for an element $g \in G$ to be a Clifford isometry of M which generalizes the eigenvalue condition for the spheres.

We exemplify and give preliminary applications.

KEY WORDS AND PHRASES.

Compact Lie group, maximal torus, Weyl group, homogeneous Riemannian manifold, Clifford isometry.

INTRODUCTION AND NOTATION.

Let G be a compact connected Lie group and $H \subset G$ a closed subgroup. $\mathcal{L}(G)$ and $\mathcal{L}(H) \subset \mathcal{L}(G)$ will denote the respective Lie algebras of G and H.

Consider in G a G-invariant Riemannian metric and induce from it a G-invariant Riemannian metric on the manifold $M = G/H$ associated to the decomposition $\mathcal{L}(G) = \mathcal{L}(H) \oplus \mathcal{L}(H)^{\perp}$.

A Clifford isometry of M is by definition an isometry $f: M \longrightarrow M$ such that if d is the distance function on M then the displacement function

$$d(x,f(x))$$

is a constant function for $x \in M$.

A group of Clifford isometries is a group all of whose elements are Clifford isometries of M.

For the relation between homogeneity and finite groups of Clifford isometries see for instance Wolf [1, page 230].

In the next section we give a necessary condition for an element $g \in G$ to be a Clifford isometry of M for pairs (G,H) satisfying certain conditions and in the last section we exemplify and we use the main result to prove that if (G,H) is a pair to which the theorem applies, then a finite subgroup Γ of G of Clifford isometries of M has the property that every abelian subgroup of Γ included in a torus of G is cyclic (c.f. Wolf [1, chapter 5] and also compare to Wolf [1, theorem 9.1.2. page 301]).

These conditions on the pair (G,H) include always the connectedness of H. In relation to the conjecture (see Wolf [1, page 230]) that if Γ is a finite group of Clifford isometries of M then M/Γ is homogeneous it would be interesting to look at the case H disconnected.

In this case if H_o is the identity component of M, the manifold $M_o = G/H_o$ is a covering manifold of M. In particular, when M is simply connected we must have H connected.

It is clear that the elements of G are not all the possible candidates to be Clifford isometries of M since in general the full group of isometries of M is bigger than G.

In fact, since the metric of G is also right invariant, every element of the normalizer of H in G acting on the right on M is an isometry of M and it is immediately verified that it is always a Clifford isometry of M.

In any case, in relation to the conjecture we mentioned these are the least interesting of the Clifford isometries of M since they clearly give a homogeneous quotient manifold.

When we have rk(G) = rk(H), H not necessarily connected, the theorem of conjugation of maximal tori of a compact connected Lie group implies that any $g \in G$ has a fixed point in M and therefore the only possible Clifford isometry G may contain is the identity. This is the case for the even dimensional spheres $S^{2n} = SO(2n+1)/SO(2n)$.

Here rk(G) denotes the dimension of any maximal torus of G.

Finally, we should say that the necessary conditions we find for an element $g \in G$ to be a Clifford isometry of $M = G/H$ are sufficient in the case of the spheres (c.f. Wolf [1, page 227]) but not in general (c.f. Wolf [2, page 95]).

The sufficiency in the case of the spheres seems strongly related to the property that equal chords imply equal arcs. In this sense it is then no surprise what we find for the remaining Stiefel manifolds, which are the subject of a coming paper. Neither it is a surprise what Wolf finds in [2] for the remaining symmetric spaces since the necessary condition it is always sufficient to obtain a Clifford isometry in the chordal distance and therefore in order to be a Clifford isometry in the intrinsic distance the displacement must be between points with the property that subtend equal "chords" and equal "arcs". This is a very strong restriction for the isometry.

THE MAIN RESULT.

Let G be a compact connected Lie group and $\{e\} \neq H \subset G$ a closed connected subgroup.

Let T' be a maximal torus of H and let T be a maximal torus of G such that $T' \subsetneq T$.

Let $\{X_1,\dots,X_r\}$ be an orthonormal basis of $\mathcal{L}(T')$ and $\{X_1,\dots,X_n\}$ an orthonormal basis for $\mathcal{L}(T)$ (therefore $1 \leqslant r < n$).

Assume that the Weyl group $W(G,T)$ of G with respect to T contains every transposition of the set $\{X_1,\dots,X_n\}$.

By changing the metric in G by an appropriate positive constant we may assume the length of the geodesics $R \ni t \to \exp(t\,X_i)$ is 2π for $i = 1,2,\dots,n$.

THEOREM. *If* $g \in G$ *is a Clifford isometry of* M *then* g *can be conjugated in* G *to an element of the form*

$$\exp(\theta(\varepsilon_1 X_1 + \dots + \varepsilon_n X_n)), \quad 0 \leqslant \theta \leqslant \pi ,$$

$\varepsilon_i = \pm 1$ *for* $i = 1,2,\dots,n$.

Proof. Let $g \in G$ be a Clifford isometry of M.

Since conjugation by an element of G preserves this property and also the conclusion we want to reach, we may assume $g \in T$.

For $\theta_1,\theta_2,\dots,\theta_n \in R$ we write

$$t(\theta_1,\dots,\theta_n) = \exp(\theta_1 X_1 + \dots + \theta_n X_n).$$

Say then that $g = t(\theta_1^o,\dots,\theta_n^o)$, $-\pi < \theta_i^o \leqslant \pi$, $i = 1,2,\dots,n$.

Let $T_1 = \{t(0,\dots,0,\theta_{r+1},\dots,\theta_n)/\theta_{r+1},\dots,\theta_n \in R\}$.

Then T_1 is a closed torus of G of dimension $n-r$.

Since T' is a maximal torus of H and H is connected, it follows that T' is maximal abelian in H. Therefore $T_1 \cap H = \{e\}$.

$\mathcal{L}(T_1)$ is orthogonal to $\mathcal{L}(T')$ and since T' is a maximal torus of H, this implies that $\mathcal{L}(T_1)$ is orthogonal to $\mathcal{L}(H)$.

From this it follows that T_1 is isometric, via the canonical projection, to a closed submanifold of M (injective application with isometric derivative).

Let $g_1 = t(\theta_1^o,0,\dots,0)$, $g_2 = t(0,\theta_2^o,0,\dots,0),\dots,$ $g_n = t(0,\dots,0,\theta_n^o)$.

Since $g_1,\dots,g_r \in H$, we have that

$$d(H,g\cdot H) = d(H,g_{r+1}\dots g_n\cdot H)$$

and since T_1 is isometric to a closed submanifold of M containing H and $g_{r+1}\dots g_n\cdot H$, it follows that this last distance can be measured, inside T_1, from the identity to $g_{r+1}\dots g_n$.

By the election of $\theta^o_{r+1},\dots,\theta^o_n$, this last distance is $(\theta^{o2}_{r+1} + \dots + \theta^{o2}_n)^{1/2}$,i.e.,

(1) $$d(H,g\cdot H)^2 = (\theta^o_{r+1})^2 + \dots + (\theta^o_n)^2.$$

Say now that $x \in G$ is such that $Ad(x)X_r = X_{r+1}$ and $Ad(x)X_j = X_j$ for all $j \neq r,r+1$, (possible by the hypothesis on the Weyl group).

Then $xgx^{-1} = t(\theta^o_1,\dots,\theta^o_{r-1},\theta^o_{r+1},\theta^o_r,\theta^o_{r+2},\dots,\theta^o_n)$.

We use now the same procedure as before to compute $d(x^{-1}\cdot H,g\cdot x^{-1}\cdot H) =$ $= d(H,x\cdot g\cdot x^{-1}\cdot H)$ and we find that

(2) $$d(H,g\cdot H)^2 = d(H,xgx^{-1}H)^2 = (\theta^o_r)^2 + (\theta^o_{r+2})^2 + \dots + (\theta^o_n)^2$$

From (1) and (2) it follows that $|\theta_r| = |\theta_{r+1}|$.

Repeating the argument with different elements of the Weyl group the theorem follows. q.e.d.

EXAMPLES AND AN APPLICATION.

Let $G = SO(2n)$.

Let $X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. For $i = 1,2,\dots,n$ we call X_i to the $2n \times 2n$ real matrix that has all entries equal to zero except for a 2×2 block equal to X along the diagonal in the $(2i-1,2i)$-th place.

Then $\{X_1,X_2,\dots,X_n\}$ is an orthogonal basis for the Lie algebra of the standard maximal torus T of G.

The Weyl group W(G,T) contains every transposition of the set $\{X_1,X_2,\dots,X_n\}$.

We take a closed connected subgroup H' of G such that $\mathcal{L}(H')$ has a maximal torus generated by a subset of $\{X_1,X_2,\dots,X_n\}$ and we take as H any subgroup conjugated in G to H'.

A similar construction gives examples when $G = SO(2n+1)$.

This gives, in particular, all the (real) Stiefel manifolds.

In general, for any compact connected simple Lie group G we take T to be a maximal torus.

Next we consider an orthogonal basis $\{X_1,X_2,\dots,X_n\}$ for $\mathcal{L}(T)$ such that the Weyl group $W(G,T)$ contains every transposition of the set $\{X_1,\dots,X_n\}$.

When $G = SU(n)$ one such basis for the standard maximal torus of diagonal elements is given by

$$X_j = \mathrm{diag}(\sqrt{-1}\,(1-\sqrt{n}), \sqrt{-1}, \dots, \sqrt{-1}, -\sqrt{-1}(n-1-\sqrt{n}), \sqrt{-1}, \dots, \sqrt{-1})$$

for $j = 1,2,\dots,n-1$. The entry $-\sqrt{-1}(n-1-\sqrt{n})$ in X_j appears in the $j+1$-st place.

Similarly, for $G = Sp(n)$ we may take the corresponding basis through the inclusion $SU(n) \subset Sp(n)$ given by

$$\mathcal{L}(SU(n)) \ni A \longrightarrow \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix} \in \mathcal{L}(Sp(n))$$

Then we consider T' to be the connected subgroup of G, whose Lie algebra is generated by a subset of $\{X_1,\dots,X_n\}$.

Let H' be any closed connected subgroup of G having T' as a maximal torus, and let H be any conjugate subgroup of H' in G.

Then the result applies to the pair (G,H).

As for an application of the result, consider (G,H) to be a pair where the theorem applies (in particular, $1 \leqslant rk(H) < rk(G)$).

Let $\Gamma \subset G$ be a finite group of Clifford isometries of $M = G/H$.

THEOREM. *An abelian subgroup of Γ included in a torus of G is cyclic.*

Proof. We show that a finite subgroup of the maximal torus of G generated by two elements g_1, g_2 that generate a group of Clifford isometries of M, is cyclic. We may take $\mathrm{ord}(g_1) = p^r$, $\mathrm{ord}(g_2) = p^s$, p prime, $r \leqslant s$.

Using the notation and the conclusion of the main theorem we may take the two elements to be

$$g_1 = \exp\left(\frac{\pi}{p^r}\left(\sum_{i=1}^{n} \varepsilon_1(i)\, X_i\right)\right)$$

$$g_2 = \exp\left(\frac{\pi}{p^s}\left(\sum_{i=1}^{n} \varepsilon_2(i)\, X_i\right)\right)$$

where $\varepsilon_1(i), \varepsilon_2(i) = \pm 1$, $i = 1,2,\dots,n$.

Then

$$g_1 g_2 = \exp\left(\frac{\pi}{p^s} \sum_{i=1}^{n} (p^{s-r}\varepsilon_1(i) + \varepsilon_2(i))\, X_i\right)$$

is a Clifford isometry of M.

Applying again the main theorem, it follows that

$$|p^{s-r}\varepsilon_1(i) + \varepsilon_2(i)| = |p^{s-r}\varepsilon_1(j) + \varepsilon_2(j)| \quad , \quad \forall\ i,j.$$

Therefore we have that if $\varepsilon_1(i) = \varepsilon_2(i)$, then $\varepsilon_1(j) = \varepsilon_2(j)$, $\forall$ j.
Taking g_2^{-1} in place of g_2 if necessary, we may assume $\varepsilon_1(1) = \varepsilon_2(1)$.
Then $\varepsilon_1(i) = \varepsilon_2(i)$, $\forall$ $i = 1,2,\ldots,n$.

Therefore $g_2^{p^{s-r}} = g_1$ and the subgroup generated by $\{g_1,g_2\}$ has to be cyclic. q.e.d.

REFERENCES

[1] J.A. WOLF, *Spaces of constant curvature*, Fourth edition, Publish or Perish, Inc.

[2] J.A. WOLF, *Locally symmetric homogeneous spaces*, Comm. Math.Helv. 37, 1962-63, p.65.

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