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DISCONTINUITY OF MAPPINGS IN BITOPOLOGICAL SPACES

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ABSTRACT. In this paper we introduce the definition of removable discontinuity of mappings in a bitopological space and enquire when such mappings are continuous.

INTRODUCTION.

In this paper, we introduce the definition of removable discontinuity of mappings from one bitopological space (X,P,Q) [2] into another such space and find out conditions when a mapping having at worst a removable discontinuity at a point becomes continuous at that point. A continuous mapping has clearly at worst a removable discontinuity at each point, but we show by an example that the converse is not true. We introduce the definition of removable discontinuity in (X,P,Q) in such a way that when the two topologies P and Q coincide, our definition becomes the same as that of Halfer [1] who establishes various results on discontinuity of mappings in a single topological space. For our investigations in (X,P,Q) we require the concepts of local connectedness and connected mappings in a bitopological space, which also we introduce here. Connectedness in a bitopological space has been widely investigated by Pervin [4].

1. KNOWN DEFINITIONS.

DEFINITION 1.1 [2]. A space X where two (arbitrary) topologies P and Q are defined is called a bitopological space and is denoted by (X,P,Q).

DEFINITION 1.2 [4]. A bitopological space (X,P,Q) is called *connected* if and only if X cannot be expressed as the union of two non-empty disjoint sets A and B such that

$$[A \cap cl_P(B)] \cap [cl_Q(A) \cap B] = \emptyset$$

where cl_P and cl_Q denote the closures with respect to P and Q topologies respectively and $\emptyset$ denotes the empty set. If X can be so expressed, then A and B are called separated sets.

NOTE 1.1. If X can be so expressed, we say that A and B are (P,Q)-separated. Throughout the paper we shall follow this convention.

DEFINITION 1.3 [4]. A subset E of (X,P,Q) is called connected if and only if the space (E,P/E,Q/E) is connected.

DEFINITION 1.4 [4]. A function f mapping a bitopological space (X,P,Q) into a bitopological space (X*,P*,Q*) is said to be *continuous* if and only if the induced mappings f_1: (X,P) $\longrightarrow$ (X*,P*) and f_2: (X,Q) $\longrightarrow$ (X*,Q*) are continuous.

DEFINITION 1.5 [2]. In a bitopological space (X,P,Q), P is said to be *regular* with respect to Q if, for each point $x \in X$ and each P-closed set C such that $x \notin C$, there is a P-open set U and a Q-open set V such that $x \in U$, $C \subset V$ and $U \cap V = \emptyset$. (X,P,Q) is, or P and Q are, pairwise regular if P is regular with respect to Q and vice-versa.

DEFINITION 1.6 [2]. A bitopological space (X,P,Q) is said to be *pairwise Hausdorff* if, for each two distinct points x and y of X, there are a P-open neighbourhood U of x and a Q-open neighbourhood V of y such that $U \cap V = \emptyset$.

2. NEW DEFINITIONS.

DEFINITION 2.1. A bitopological space (X,P,Q) is said to be *locally connected* at a point $x \in X$ if and only if for every pair of P-open set U and Q-open set V each containing x, there exist connected Q-open set C and connected P-open set D such that $x \in C \subset U$ and $x \in D \subset V$. (X,P,Q) is said to be locally connected if and only if it is locally connected at every point of X.

DEFINITION 2.2. A function f mapping a bitopological space (X,P,Q) into a bitopological space (X*,P*,Q*) is said to be *connected* if and only if the image of every connected subset of (X,P,Q) is a connected subset of (X*,P*,Q*).

DEFINITION 2.3. A function f mapping a bitopological space (X,P,Q) into a bitopological space (X*,P*,Q*) is said to be (P $\longrightarrow$ Q*) [resp. (Q $\longrightarrow$ P*)] - closed if and only if the image of every P (resp. Q)-closed subset of (X,P,Q) is a Q* (resp. P*)-closed subset of (X*,P*,Q*).

DEFINITION 2.4. A function f mapping a bitopological space (X,P,Q) into a bitopological space (X*,P*,Q*) has at worst a *removable discontinuity* at a point $p \in X$ if there exists a point $y \in X^*$ such that for each P*-open neighbourhood V_{P^*} and Q*-open neighbourhood V_{Q^*} of y, there are a P-open neighbourhood U_P and Q-open neighbourhood U_Q of p such that

$$f(U_P - \{p\}) \subset V_{P^*} \quad \text{and} \quad f(U_Q - \{p\}) \subset V_{Q^*} .$$

REMARK 2.1. If a function f is continuous at a point of (X,P,Q), then f has at worst a removable discontinuity at that point but the converse is not true as shown by the following example.

EXAMPLE 2.1. Let (X,P,Q) and (X*,P*,Q*) be two bitopological spaces where $X = \{\alpha,\beta,\gamma\}$, $X^* = \{a,b,c\}$, $P = \{X,\emptyset,\{\alpha,\beta\}\}$, $Q = \{X,\emptyset,\{\alpha,\gamma\}\}$ and $P^* = \{X^*,\emptyset,\{a,b\},\{c\}\} = Q^*$.

Let f: (X,P,Q) $\to$ (X*,P*,Q*) be given by f: $\alpha \to c$, $\beta \to b$ and $\gamma \to a$.

Here f has a removable discontinuity at α, because there is a point in X*, viz., the point a such that for every P*-open neighbourhood V_{P^*} and for every Q*-open neighbourhood V_{Q^*} of a, there are P-open neighbourhood $\{\alpha,\beta\}$ and Q-open neighbourhood $\{\alpha,\gamma\}$ of α such that

$$f(\{\alpha,\beta\}-\{\alpha\}) \subset V_{P^*} \quad \text{and} \quad f(\{\alpha,\gamma\}-\{\alpha\}) \subset V_{Q^*} .$$

But f is not continuous at α, because {c} is a P*-open neighbourhood of $f(\alpha)$ and there is no P-open neighbourhood U_P of α such that $f(U_P) \subset \{c\}$ and so the induced mapping of f from (X,P) to (X*,P*) is not continuous at α and consequently f: (X,P,Q) $\to$ (X*,P*,Q*) is not continuous at α.

3. LEMMAS AND THEOREMS.

LEMMA 3.1. *Let* A *and* B *be respectively two non-empty disjoint* P-*open and* Q-*open subsets of* (X,P,Q). *If* D *is a connected non-empty subset of* (X,P,Q) *such that* $D \subset A \cup B$, *then either* $D \cap A = \emptyset$ *or* $D \cap B = \emptyset$.

Proof. Since $A \cap B = \emptyset$, $A \subset CB$. Also as B is Q-open, CB is Q-closed and so $cl_Q(A) \subset CB$. Hence $B \cap cl_Q(A) = \emptyset$. Similarly, $A \cap cl_P(B) = \emptyset$. As D is a connected subset of (X,P,Q), the space (D,P/D,Q/D) is connected. Let P/D = P* and Q/D = Q*. We write

$$D = (D \cap A) \cup (D \cap B) \text{ , then}$$

$$(D \cap A) \cap cl_{P*}(D \cap B) = (D \cap A) \cap [D \cap cl_P (D \cap B)] \subset$$
$$\subset (D \cap A) \cap cl_P(B) = \emptyset .$$

Thus $(D \cap A) \cap cl_{P*}(D \cap B) = \emptyset$. Similarly $(D \cap B) \cap cl_{Q*}(D \cap A) = \emptyset$. Hence if each of the sets $D \cap A$ and $D \cap B$ is non-empty, then the space (D,P^*,Q^*) i.e., the space $(D,P/D,Q/D)$ has a separation and consequently D cannot be a connected subset of (X,P,Q). Hence either $D \cap A = \emptyset$ or $D \cap B = \emptyset$. This proves the lemma.

THEOREM 3.1. *Let* f *be a connected mapping of a locally connected bitopological space* (X,P,Q) *into a pairwise Hausdorff bitopological space* (X^*,P^*,Q^*). *Then if* f *has at worst a removable discontinuity at* p*,* f *is continuous at* p.

Proof. Here the following cases come up for considerations:

(a) p is an isolated point in (X,P) as well as in (X,Q) ,
(b) p is an isolated point in (X,P) but not in (X,Q) ,
(c) p is an isolated point in (X,Q) but not in (X,P) and
(d) p is neither an isolated point in (X,P) nor in (X,Q).

CASE (a). Let V_{P*} be any P*-open neighbourhood of f(p) and let U_1 be any P-open neighbourhood of p. As p is an isolated point in (X,P), there is a P-open neighbourhood U_2 of p such that $U_1 \cap U_2 - \{p\} = \emptyset$.

Now $U_1 \cap U_2 = U_p$, say, is a P-open neighbourhood of p. Also, $f(U_p - \{p\}) = \emptyset \subset V_{P*}$ and as $f(p) \in V_{P*}$, $f(U_p) \subset V_{P*}$. Hence the induced mapping of f from (X,P) to (X^*,P^*) is continuous at p.

As p is also an isolated point in (X,Q), we get similarly that the induced mapping of f from (X,Q) to (X^*,Q^*) is also continuous at p. Hence $f\colon (X,P,Q) \longrightarrow (X^*,P^*,Q^*)$ is continuous at p.

CASE (b). Let y be the point in X* determined by the definition of removable discontinuity of f. If f is not continuous at p, $y \neq f(p)$ and so as X* is pairwise Hausdorff, there exist P*-open set V_{P*} and Q*-open set V_{Q*} such that $f(p) \in V_{P*}$, $y \in V_{Q*}$ and $V_{P*} \cap V_{Q*} = \emptyset$.

Because f has a removable discontinuity at p, there exists a Q-open neighbourhood U_Q of p such that $f(U_Q - \{p\}) \subset V_{Q*}$.

So $f(U_Q) \subset V_{P*} \cup V_{Q*}$.

Now X is locally connected at p and since p belongs to the Q-open

set U_Q, there is a connected P-open set C_P such that $p \in C_P \subset U_Q$. Similarly as p belongs to the P-open set C_P, there is a connected Q-open set D_Q such that $p \in D_Q \subset C_P$. So, $p \in D_Q \subset U_Q$.
As p is not isolated in (X,Q), $D_Q - \{p\} \neq \emptyset$. Again,
$\emptyset \neq f(D_Q - \{p\}) \subset f(U_Q - \{p\}) \subset V_{Q*}$, which implies that
$f(D_Q) \cap V_{Q*} \neq \emptyset$.
Also as $f(p) \in f(D_Q)$ and $f(p) \in V_{P*}$, $f(D_Q) \cap V_{P*} \neq \emptyset$.
Now as f is connected, $f(D_Q)$ is a connected subset of (X*,P*,Q*).
Thus as $f(D_Q) \subset V_{P*} \cup V_{Q*}$ and as V_{P*} and V_{Q*} are respectively disjoint P*-open set and Q*-open set, by Lemma 3.1, either
$f(D_Q) \cap V_{P*} = \emptyset$ or $f(D_Q) \cap V_{Q*} = \emptyset$, which is a contradiction. Hence f is continuous at p.

The cases (c) and (d) may be dealt with similarly. This proves the theorem.

LEMMA 3.2. *The following three properties are equivalent:*

(1) (X,P,Q) *is a bitopological space such that* P *is regular with respect to* Q.

(2) *For each* $x \in (X,P,Q)$ *and for each* P-*open neighbourhood* U_P *of* x, *there is a* P-*open neighbourhood* V_P *of* x *such that* $x \in V_P \subset$
$\subset cl_Q(V_P) \subset U_P$.

(3) *For each* $x \in (X,P,Q)$ *and each* P-*closed set* A *not containing* x, *there is a* P-*open neighbourhood* V_P *of* x *with* $cl_Q(V_P) \cap A = \emptyset$.

Proof. (1) → (2). Let U_P be given. Then the P-closed set CU_P does not contain x. As in (X,P,Q), P is regular with respect to Q, there is a P-open set V_P and a Q-open set V_Q such that $x \in V_P$ and $CU_P \subset V_Q$ and $V_P \cap V_Q = \emptyset$. Thus $V_P \subset CV_Q$ and so $cl_Q(V_P) \subset CV_Q \subset U_P$.
Hence $x \in V_P \subset cl_Q(V_P) \subset U_P$.

(2) → (3). Using x and its P-open neighbourhood CA, we can find a P-open neighbourhood V_P of x such that $x \in V_P \subset cl_Q(V_P) \subset CA$. Thus $cl_Q(V_P) \cap A = \emptyset$.

(3) → (1). Let A be P-closed and $x \notin A$. We choose a P-open neighbourhood V_P of x such that $cl_Q(V_P) \cap A = \emptyset$. Thus $A \subset C[cl_Q(V_P)]$ and $C[cl_Q(V_P)] \cap V_P = \emptyset$. This proves the lemma.

LEMMA 3.3. *The following properties are equivalent:*

(1) (X,P,Q) *is a bitopological space such that* Q *is regular with respect to* P.

(2) *For each* $x \in (X,P,Q)$ *and for each* Q-*open neighbourhood* U_Q *of* x, *there is a* Q-*open neighbourhood* V_Q *of* x *such that* $x \in V_Q \subset$ $\subset cl_P(V_Q) \subset U_Q$.

(3) *For each* $x \in (X,P,Q)$ *and for each* Q-*closed set* A *not containing* x, *there is a* Q-*open neighbourhood* V_Q *of* x *with* $cl_P(V_Q) \cap A = \emptyset$.

Proof. The proof runs parallel to Lemma 3.2.

THEOREM 3.2. *Let* f *be a function that maps a pairwise regular bitopological space* (X,P,Q) *into a bitopological space* (X*,P*,Q*) *such that* f *is* $(P \to Q^*)$-*closed and* $(Q \to P^*)$-*closed and also for every* $y \in X^*$, $f^{-1}(y)$ *is* P-*closed and* Q-*closed subset of* X. *Then if* f *has at worst a removable discontinuity at* $p \in X$, f *is continuous at* p.

Proof. We should consider the following cases:

(a) p is an isolated point in (X,P) as well as in (X,Q); (b) p is not an isolated point in (X,P) but is an isolated point in (X,Q); (c) p is an isolated point in (X,P) but not an isolated point in (X,Q) and (d) p is neither an isolated point in (X,P) nor an isolated point in (X,Q).

We prove the theorem for the case (b). The other cases are similar.

Let y be the point in X* determined by the definition of removable discontinuity of f. If f is not continuous at p, $f(p) \neq y$ and so $p \notin f^{-1}(y)$. But $f^{-1}(y)$ is a P-closed set in X and as P is regular with respect to Q, there exists, by Lemma 3.2, a P-open neighbourhood U_P of p such that

$$f^{-1}(y) \cap cl_Q(U_P) = \emptyset.$$

As f is $(Q \to P^*)$-closed, $f(cl_Q(U_P))$ is P*-closed and as $y \notin f(cl_Q(U_P))$, there is a P*-open neighbourhood V_{P^*} of y such that

$$V_{P^*} \cap f(cl_Q(U_P)) = \emptyset.$$

From the definition of removable discontinuity, there exists a P-open neighbourhood W_P of p such that

$$f(W_P - \{p\}) \subset V_{P^*} .$$

Since p is not isolated in (X,P), $U_P \cap W_P - \{p\} \neq \emptyset$.

Hence $\emptyset \neq f(W_P - \{p\}) \cap f(cl_Q(U_P)) \subset V_{P^*} \cap f(cl_Q(U_P)) = \emptyset$, a contradiction. Hence f is continuous at p.

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