Revista de la
Unión Matemática Argentina
Volumen 31, 1984.

MAPPING THEOREMS IN PARANORMED SPACES

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0. INTRODUCTION.

Let X be a metrizable topological space, Y a linear space, X_o a subset of X and T an application from X_o to Y. By a *mapping theorem* involving these elements we mean the problem of determining sufficient metric-linear conditions in order that the equation

(E) $$Tx = 0$$

should have a solution in X_o. The prototype of all mapping results of this kind must be considered the 1971 Browder's theorem [6] proved by a specific *asymptotic direction* technique. As refinements of Browder's original result we quote the 1976 Altman's contribution [1] obtained by a *transfinite induction* argument combined with a *contractor direction* technique, as well as the 1977 Downing-Kirk result [10] proved by a Caristi fixed point procedure (see also Kirk and Caristi [14]). Finally, as further developments in this direction, we must quote those of Altman [2] and, respectively, Cramer and Ray [9] based, essentially, on Ekeland's variational principle [11] and, respectively, Brézis-Browder ordering principle [3]. A basic assumption of all these contributions is that the range of the application is a Banach space with respect to a suitable norm; it's therefore natural to ask whether this condition cannot be removed. The main aim of the present note is to give a positive answer to this question - more exactly, to state and prove a mapping theorem for applications taking values in a *paranormed space* - the basic instrument of our investigations being a *maximality principle* on (partially) ordered metric spaces appearing (under its quasi-ordered semi-metrical version) as a generalization of the Brézis-Browder ordering principle we quoted before. As useful particular cases, a couple of mapping theorems for applications whose range is a normed (respectively, a Fréchet) space is given, extending in this way the similar Cramer-Ray result (see the above reference) and, respectively, completing, under this perspective, the result of Turinici [21] obtained by a specific *variable drop* technique. Some further extensions of this last result to *non-metrizable* uniform spaces will be given elsewhere.

1. PRELIMINARIES.

Let Y be a *linear space* (over the real or complex numbers). A function $x \longmapsto \|x\|$ from Y to $[0,\infty)$ will be called a *paranorm* on Y when (a) $\|x\| = 0$ if and only if $x = 0$, (b) $\|x+y\| \leqslant \|x\|+\|y\|$, $x,y \in Y$, (c) $\|-x\| = \|x\|$, $x \in Y$; correspondingly, the couple $(Y,\|\cdot\|)$ will be termed a paranormed space. Evidently, $\|\cdot\|$ induces a *metric* structure on Y by the standard construction $d(x,y) = \|x-y\|$, $x,y \in Y$. An important class of paranorms - largely used in the sequel - is that introduced by the following convention. Letting $r > 0$ and $Z \subset Y$, we shall say the paranorm $\|\cdot\|$ is r-*superadditive* on Z when

$$(1)\qquad r\|\varepsilon x\| + \|(1-\varepsilon)x\| \leqslant \|x\| \quad , \quad 0 \leqslant \varepsilon \leqslant 1 \ , \ x \in Z.$$

Note that (1) necessarily implies $r \leqslant 1$ because, when $r > 1$, we have (for $x\neq 0$ in Z and $\varepsilon\neq 0$ in $[0,1]$) by the triangle inequality (b)

$$r\|\varepsilon x\|+\|(1-\varepsilon)x\| = (r-1)\|\varepsilon x\|+\|\varepsilon x\|+\|(1-\varepsilon)x\| > \|\varepsilon x\|+\|(1-\varepsilon)x\| \geqslant \|x\| ,$$

a contradiction with respect to (1). Afirst example of such paranorms is contained in the evident

LEMMA 1. *Every norm is* 1 - *superadditive on every subset of* Y.

As another specific example of r-superadditive paranorms, let $S = \{|\cdot|_i;\ i \in N\}$ be a (denumerable) *sufficient* family of *seminorms* on Y ($|x|_i = 0$, all $i \in N$ imply $x=0$), $L = (\lambda_i;\ i \in N)$ a sequence of strict positive numbers and $A = (\alpha_i;\ i \in N)$ a *summable* family of strict positive numbers ($\alpha_1+\alpha_2+\ldots<\infty$). Define a function $\|\cdot\| = \|\cdot\|(S,L,A)$ from Y to $[0,\infty)$ by

$$(2)\qquad \|x\| = \sum_{i\in N} \alpha_i |x|_i/(\lambda_i + |x|_i) \quad , \quad x \in Y.$$

LEMMA 2. *The function* $x \longmapsto \|x\|$ *defined by* (2) *is a paranorm on* Y. *Moreover,* $(Y,\|\cdot\|)$ *and* (Y,S) *are equivalent as (metrizable) topological spaces (i.e., a sequence* $(y_n;\ n \in N)$ *in* Y *converges (modulo* $\|\cdot\|$) *to* y *in* Y *if and only if it converges (modulo* S) *to* y).

Proof. The first part of the statement is clear if we observe that, for any $i \in N$,

$$|x+y|_i/(\lambda_i+|x+y|_i) \leqslant |x|_i/(\lambda_i+|x|_i)+|y|_i/(\lambda_i+|y|_i) \ , \ x,y \in Y.$$

To prove the second part, let $\varepsilon > 0$ be arbitrary fixed. As $A = (\alpha_i;\ i \in N)$ is a summable family, there exists $m = m(\varepsilon) \in N$ such that

$$\alpha_{m+1} + \alpha_{m+2} + \ldots < \varepsilon/2$$

in which case, denoting

$$\delta = \varepsilon/2 \sum_{i\leq m} \alpha_i/\lambda_i$$

we have at once

$$|x|_i < \delta,\ i \leqslant m \quad \text{implies } \|x\| < \varepsilon\ ;$$

conversely, given any $i \in N$, and putting

$$\eta_i = \alpha_i\varepsilon/(\lambda_i + \varepsilon)$$

one clearly obtains

$$\|x\| < \eta_i \quad \text{implies} \quad |x|_i < \varepsilon$$

proving our assertion. Q.E.D.

LEMMA 3. *Let* r *in* (0,1) *be arbitrary fixed. Then, the paranorm* $\|\cdot\|$ *given by* (2) *is* r - *superadditive on*

$$Y(L,r) = \{y \in Y;\ |y|_i \leqslant \lambda_i(r^{-1/2} - 1),\ i \in N\}.$$

Proof. Evidently, in order that (1) be valid it suffices that, for any $i \in N$,

$$(1)' \qquad r\varepsilon|x|_i(\lambda_i+\varepsilon|x|_i) + (1-\varepsilon)|x|_i/(\lambda_i + (1-\varepsilon)|x|_i) \leqslant$$
$$\leqslant |x|_i/(\lambda_i + |x|_i)\ ,\ 0 \leqslant \varepsilon \leqslant 1\ ,\ |x|_i \leqslant \lambda_i(r^{-1/2} - 1)$$

or equivalently (denoting $|x|_i = \xi$ and $\lambda_i = \lambda$) that

$$(3) \qquad r\varepsilon/(\lambda+\varepsilon\xi) + (1-\varepsilon)/(\lambda+(1-\varepsilon)\xi) \leqslant 1/(\lambda+\xi)\ ,$$
$$0 \leqslant \varepsilon \leqslant 1\ ,\ 0 \leqslant \xi \leqslant \lambda(r^{-1/2} - 1).$$

Let $f(\varepsilon)$ denote the left member of this inequality. A simple computation yields

$$f'(\varepsilon) = (-\xi(s+1)\varepsilon + s\xi - \lambda(1-s))g(\varepsilon) \quad ,\quad 0 \leqslant \varepsilon \leqslant 1,$$

where $g(\varepsilon)$ is strictly positive and $s = r^{1/2}$ so that, a sufficient condition for (3) (equivalent - under the above notation - with $f(\varepsilon) \leqslant f(0)\ ,\ 0 \leqslant \varepsilon \leqslant 1$) to be valid is that

$$s\xi - \lambda(1-s) \leqslant 0 \text{ (or, equivalently, } \xi \leqslant \lambda(s^{-1} - 1))$$

which is just condition appearing in the final part of (3), proving our assertion. Q.E.D.

Let $(X,d,\leqslant)$ be a (*partially*) *ordered* metric space. A sequence $(x_n;\ n \in N)$ in X will be said to be (a)' *monotone*, when $x_i \leqslant x_j$ for $i \leqslant j$, (b)' *asymptotic*, when $\liminf_{n\to\infty} d(x_n,x_{n+1}) = 0$, (c)' *bounded above*, in case $x_n \leqslant y$, all $n \in N$, for some y in X. Also, the element z in X will be termed *maximal*, provided that $z \leqslant y$ implies z=y. Concerning these notions, the following maximality principle established by the author in [24] will play a central role in the sequel.

LEMMA 4. *Let the ordered metric space* $(X,d,\leqslant)$ *be such that*

(i) *any monotone sequence is asymptotic*

(ii) *any monotone Cauchy sequence is bounded above.*

Then, to any x *in* X *there corresponds a maximal element* z *in* X *with* $x \leqslant z$.

As already pointed out by the author in [20], the above lemma can be formulated in the larger context of quasi-ordered semi-metric spaces, in which case, it may be viewed as a straightforward extension of the "abstract" Brêzis-Browder ordering principle [3] as well as the "uniform" Brøndsted's maximality principle [4]. Moreover, under a pattern discovered by Brøndsted [5] (see also Ekeland [11]) it's possible to formulate this lemma as a fixed point statement, in which situation it appears as an abstract counterpart of the so-called Caristi's fixed point theorem [8,13,16,19]. Finally, an extension of this maximality principle to metrizable uniform spaces may be found in Turinici [21].

2. THE MAIN RESULT.

In what follows, a precise statement of the considerations exposed in the introductory part of the note will be performed. Let (X,d) be a metric space and $(Y,\|\cdot\|)$ a paranormed space. Given a subset X_o of X, let $X_o(x_o,r)$ denote (for $x_o \in X_o$ and $r > 0$) the X_o - *closed sphere* with center x_o and radius r (the subset of all x in X_o with $d(x,x_o) \leqslant r$); also, given the application $T: X_o \longrightarrow Y$ we shall say it is *closed* [10] when $(x_n;\ n \in N)$ in X_o, $x_n \longrightarrow x$ and $Tx_n \longrightarrow y$ imply $x \in X_o$ and $Tx = y$. Suppose henceforward (X,d) and $(Y,\|\cdot\|)$ are complete in the usual sense and $T: X_o \longrightarrow Y$ is closed in the above sense. Then, as the main result of the present note, the following "local" mapping theorem can be stated and proved.

THEOREM 1. *Let the element* x_o *in* X_o *with* $Tx_o \neq 0$ *be such that, a couple of functions* $b,f: (0,\infty) \longrightarrow (0,\infty)$ *satisfying*

(4) $$(t-s)b(t) \leqslant f(t)-f(s)\ \ ,\ \ 0 < s < t$$

and a couple of numbers q,r *satisfying* $0 \leqslant q < r \leqslant 1$ *as well as*

(iii) $\|\cdot\|$ *is* r - *superadditive on* $T(X_o')$

where

(5) $$X_o' = X_o(x_o, f(\|Tx_o\|)/(r-q))$$

may be found with the property: for any x *in* X_o' *with* $Tx \neq 0$ *there exist* x' *in* X_o *and* ε *in* (0,1] *with*

(6) $$\|Tx'-(1-\varepsilon)Tx\| \leqslant q\|\varepsilon Tx\| \quad , \quad d(x,x') \leqslant \|\varepsilon Tx\| b(\|Tx\|).$$

Then, necessarily, (E) *has at least a solution in* X_o'.

Proof. Assume by contradiction $Tx \neq 0$ for all x in X_o'; so, given any x in X_o' there exist x' in X_o and ε in (0,1] such that (6) holds. By the first part of this relation

$$\|Tx-Tx'\| \leqslant (1+q)\|\varepsilon Tx\|$$

as well as (if we take (iii) into account)

$$\|Tx'\| \leqslant q\|\varepsilon Tx\|+\|(1-\varepsilon)Tx\| \leqslant (q-r)\|\varepsilon Tx\|+r\|\varepsilon Tx\|+\|(1-\varepsilon)Tx\| \leqslant$$
$$\leqslant (q-r)\|\varepsilon Tx\|+\|Tx\|$$

or, equivalently,

(7) $$\|\varepsilon Tx\| \leqslant (\|Tx\|-\|Tx'\|)/(r-q)$$

so that, by a combination of them

(8) $$\|Tx-Tx'\| \leqslant (1+q)(\|Tx\|-\|Tx'\|)(r-q)$$

(note at this moment that, again by the first part of (6), $Tx \neq Tx'$ (hence, $x \neq x'$) because, otherwise, $Tx=Tx'$ would imply $1 \leqslant q$, a contradiction). On the other hand, the second part of this relation yields, in combination with a consequence of (1) ($tb(t) \leqslant f(t)$, $t > 0$) and a consequence of (7) ($\|\varepsilon Tx\| \leqslant \|Tx\|/(r-q)$)

$$d(x,x') \leqslant \|Tx\| b(\|Tx\|)/(r-q) \leqslant f(\|Tx\|)/(r-q).$$

Now, let us denote

(5)' $$X_o'' = \{y \in X_o';\ d(x_o,y) \leqslant (f(\|Tx_o\|)-f(\|Ty\|))/(r-q)\}$$

and observe that $x \in X_o''$ plus the above inequality implies $x' \in X_o'$ (whence $Tx' \neq 0$) so that, again by the second part of (6), in combination with (4) + (7)

(9) $$d(x,x') \leqslant (f(\|Tx\|)-f(\|Tx'\|))/(r-q)$$

a relation that evidently implies $x' \in X_o''$. Let e denote the "product" metric on X_o

$$e(x,y) = \max\ (d(x,y), \|Tx-Ty\|) \ , \ x,y \in X_o$$

and let ⩽ indicate the ordering on X_o' defined as

$x \leqslant y$ *if and only if* $d(x,y) \leqslant (f(\|Tx\|)-f(\|Ty\|))/(r-q)$ and $\|Tx-Ty\| \leqslant (1+q)(\|Tx\|-\|Ty\|)/(r-q)$.

We claim conditions (i)+(ii) are fulfilled in $(X_o'',e,\leqslant)$ and this will lead us to the desired contradiction. (Indeed, it will follow then by Lemma 4 that, for the element x_o in X_o'' a maximal element z in X_o'' may be found with $x_o \leqslant z$; on the other hand, by the above developments, a $z' \in X_o''$ may be chosen with $z \leqslant z'$ and $z \neq z'$, contradicting the maximality of z in $(X_o'',\leqslant)$). To this end, let $(x_n; n \in N)$ be a monotone sequence in X_o'', that is

$$(10) \qquad d(x_n,x_m) \leqslant (f(\|Tx_n\|)-f(\|Tx_m\|))/(r-q) \quad \text{and}$$
$$\|Tx_n-Tx_m\| \leqslant (1+q)(\|Tx_n\|-\|Tx_m\|)/(r-q) \quad , \ n \leqslant m.$$

As $(f(\|Tx_n\|); n \in N)$ and $(\|Tx_n\|; n \in N)$ are descending (hence Cauchy) sequences on $(0,\infty)$ it immediately follows $(x_n; n \in N)$ and $(Tx_n; n \in N)$ are Cauchy sequences in X and Y respectively. By the completeness - closedness hypothesis, $x_n \longrightarrow x$ and $Tx_n \longrightarrow Tx$ for some x in X_o and this establishes (i) (module e). Moreover, as X_o' is relatively closed in X_o, it also follows $x \in X_o'$ (whence $Tx \neq 0$) in which situation, observing that, as a consequence of (10) (the first part)

$$d(x_n,x_m) \leqslant (f(\|Tx_n\|)-f(\|Tx\|))/(r-q) \quad , \ n \leqslant m$$

one immediately derives (letting m tend to infinity)

$$d(x_n,x) \leqslant (f(\|Tx_n\|)-f(\|Tx\|))/(r-q) \quad , \ n \in N \ ,$$

and consequently, $x \in X_o''$; in the same time, by an argument similar to the above one, (10) (the second part) gives

$$\|Tx_n-Tx\| \leqslant (1+q)(\|Tx_n\|-\|Tx\|)/(r-q) \quad , \quad n \in N$$

proving $x_n \leqslant x$, $n \in N$, and establishing (ii). Therefore, the proof is complete. Q.E.D.

3. SOME PARTICULAR CASES.

The main result we established in the preceding paragraph appears, at this stage of our exposition, as an "abstract" mapping theorem only, so that, for a number of practical reasons, some "concrete" realizations of it (based on the considerations of §1) were welcomed. As a first step in this direction, let (X,d) be a complete metric space, $(Y,\|\cdot\|)$ a Banach space, X_o a subset of X and $T: X_o \longrightarrow Y$ a closed application then, the following "normed" variant of the main result may be formulated.

THEOREM 2. *Let the element* x_o *in* X_o *with* $Tx_o \neq 0$ *be such that, a number* $q \in [0,1)$ *and a couple of functions* $b,f: (0,\infty) \longrightarrow (0,\infty)$ *satisfying*

(4)' $(1-s/t)b(t) \leq f(t)-f(s)$, $t,s > 0$, $qt \leq s < t$

may be found with the property: for any x *in* X_o'*, where*

(5)' $X_o' = X_o(x_o, f(\|Tx_o\|)/(1-q))$

with $Tx \neq 0$ *there exist* x' *in* X_o *and* ε *in* (0,1] *with*

(6)' $\|Tx'-(1-\varepsilon)Tx\| \leq q\varepsilon\|Tx\|$; $d(x,x') \leq \varepsilon b(\|Tx\|)$.

Then, (E) *has at least a solution in* X_o'.

Proof. It suffices to observe that, putting $t = \|Tx\|$, $s = (1-(1-q)\varepsilon)t$, one easily obtains $qt \leq s < t$ and (by (7)) $\|Tx'\| \leq s$, in which case, taking into account (4)', we established (9). The remaining part of the argument follows from the main result with r=1 and this ends the proof. Q.E.D.

Concerning condition (4)' (essentially involved in the above theorem) let us observe it is fulfilled in case $t \longmapsto b(t)$ is increasing and $c(t) = \int_0^t (b(s)/s)ds < \infty$, $t > 0$ for, letting $f(t) = c(t/q)$, $t > 0$, we have, for any couple $t,s > 0$, $qt \leq s < t$

$$(1-s/t)b(t) \leq (b(u)/u)(t/q-s/q) \ , \ s/q \leq u \leq t/q$$

in which case, the above statement reduces to Theorem 2.1 of Cramer and Ray [9] proved by a Brézis-Browder ordering procedure. Moreover, it was demonstrated by the above quoted authors their contribution represents a considerable refinement of some "abstract" (metrical) mapping theorems established by Browder [6,7], Kirk and Caristi [14], Downing and Kirk [10], Altman [1,2] (see also Turinici [22]) as well as of some "concrete" (differential) mapping theorems established by Pohozhayev [17], Kačurovskii [12], Krasnoselskii [15], Rosenholtz and Ray [18], so that our theorem also extends these results.

Passing to the second particularization, suppose further X is a complete metrizable uniform space under the (denumerable) sufficient family of semi-metrics $D = (d_i;\ i \in N)$, Y is a complete Fréchet space under the sufficient family of seminorms $S = (|\cdot|_i;\ i \in N)$ and $T: X_o \longrightarrow Y$ (X_o a subset of X) is a closed application, then, the following "global" version of the main result can be derived.

THEOREM 3. *Suppose there exist a sequence* $L = (\lambda_i;\ i \in N)$ *in* $(0,\infty)$*, a couple of numbers* q,r *with* $0 \leq q < r \leq 1$ *and*

(iii)' $|Tx|_i \leq \lambda_i(r^{-1/2} - 1)$, $i \in N$, $x \in X_o$

a summable family $A = (\alpha_i;\ i \in N)$ *in* $(0,\infty)$ *as well as a number* $b > 0$ *with the property: for any* x *in* X_o *with* $Tx \neq 0$ *there exist* x' *in* X_o, ε *in* $(0,1]$ *and a couple of injections* φ,ψ *from* N *to itself, with*

(iv) $q\alpha_{\varphi(i)}|\varepsilon Tx|_{\varphi(i)} < \alpha_i(\lambda_{\varphi(i)}+|\varepsilon Tx|_{\varphi(i)})$ *implies*

$$|Tx'-(1-\varepsilon)Tx|_i \leqslant \lambda_i q\alpha_{\varphi(i)}|\varepsilon Tx|_{\varphi(i)}/(\alpha_i(\lambda_{\varphi(i)} + |\varepsilon Tx|_{\varphi(i)})-q\alpha_{\varphi(i)}|\varepsilon Tx|_{\varphi(i)})$$

(v) $b\alpha_{\psi(i)}|\varepsilon Tx|_{\psi(i)} < \alpha_i(\lambda_{\psi(i)}+|\varepsilon Tx|_{\psi(i)})$ *implies*

$$d_i(x,x') \leqslant \lambda_i b\alpha_{\psi(i)}|\varepsilon Tx|_{\psi(i)}/(\alpha_i(\lambda_{\psi(i)} + |\varepsilon Tx|_{\psi(i)})-b\alpha_{\psi(i)}|\varepsilon Tx|_{\psi(i)}).$$

Then, (E) *has at least a solution in* X_o.

Proof. It suffices to observe that (iv) plus (v) give

$$\alpha_i|Tx'-(1-\varepsilon)Tx|_i/(\lambda_i+|Tx'-(1-\varepsilon)Tx|_i) \leqslant$$
$$\leqslant q\alpha_{\varphi(i)}|\varepsilon Tx|_{\varphi(i)}/(\lambda_{\varphi(i)}+|\varepsilon Tx|_{\varphi(i)}) \ , \ i \in N$$

and respectively,

$$\alpha_i d_i(x,x')/(\lambda_i+d_i(x,x')) \leqslant$$
$$\leqslant b\alpha_{\psi(i)}|\varepsilon Tx|_{\psi(i)}/(\lambda_{\psi(i)}+|\varepsilon Tx|_{\psi(i)}) \ , \ i \in N \ ,$$

so that, if we introduce a metric (paranormed) structure on X (Y) by the convention

$$d(x,y) = \sum_{i\in N} \alpha_i d_i(x,y)/(\lambda_i+d_i(x,y)) \ , \ x,y \in X$$

(and, respectively,

$$\|x\| = \sum_{i\in N} \alpha_i|x|_i/(\lambda_i+|x|_i) \ , \ x \in Y) \ ,$$

conditions of the main result are fulfilled with $X_o' = X_o$, $b(t)=b$, $t > 0$, and $f(t) = bt$, $t > 0$, so that by the conclusion of that statement, the proof is complete. Q.E.D.

Concerning the elements involved in this result , a special mention must be made about the functions φ and ψ appearing in (iv) and (v) respectively. Namely, suppose, in particular that $\varphi = \psi =$ the identity, then, the above conditions become

(iv)' $|Tx'-(1-\varepsilon)Tx|_i \leqslant \lambda_i q|\varepsilon Tx|_i/(\lambda_i+(1-q)|\varepsilon Tx|_i)$, $i \in N$

(v)' $b|\varepsilon Tx|_i < \lambda_i+|\varepsilon Tx|_i$ implies

$$d_i(x,x') \leqslant \lambda_i b|\varepsilon Tx|_i/(\lambda_i+(1-b)|\varepsilon Tx|_i)$$

and the corresponding version of Theorem 3 may be compared with a similar one due to the author [21] and proved by a "variable drop" technique. Regarding this last aspect, it's not without importance to ask whether a direct treatment of the problem - based on the initial metrizable (Frêchet) structure of the ambient spaces - may not be given. A partial answer to this question may be found in Turinici [23]; some further extensions to non-metrizable uniform spaces will be given elsewhere.

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Recibido en abril de 1983.