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THE RELATIVE GENERALIZED JACOBIAN MATRIX IN THE SUBDIFFERENTIAL CALCULUS

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1. INTRODUCTION.

The purpose of this article is the characterization of the relative generalized jacobian matrix of a locally Lipschitz function which is used in optimization problems with non differentiable data.

An important tool in that way was the concept of generalized jacobian matrix of F at x_o in Clarke's sense [1], we denote by $\tilde{J}F(x_o)$. From its basic definition (recalled at the beginning of Section II), one can see that $\tilde{J}F(x_o)$ takes into account the behaviour of F all around x_o. Actually, it has appeared that, for many purposes, the knowledge of all this information was not quite necessary; what needs to be known is the contribution of F restricted to a subset Q in the construction of $\tilde{J}F(x_o)$.
These considerations gave rise to what we call the generalized jacobian matrix of F relative to Q, denoted (at x_o) by $\tilde{J}_QF(x_o)$. This concept which is defined through a "lim sup" - operation on the collection $\{\tilde{J}F(x): x \in Q\}$ was firstly used in [2] for particular purposes. In Section II, we go into details of the study of $\tilde{J}_QF(.)$: conditions ensuring the connectedness of $\tilde{J}_QF(x_o)$, properties of $\tilde{J}_QF(.)$ as a set-valued mapping and chain rules. Of course, all these properties and calculus rules are mainly derived from those already exhibited for the usual generalized jacobian matrix (which turns out to be $\tilde{J}_QF(.)$ for Q the whole space) [1].

2. THE RELATIVE GENERALIZED JACOBIAN MATRIX.

Let $F: R^n \longrightarrow R^m$ be a locally Lipschitz function on O an open subset of R^n.

The generalized jacobian matrix of F at $x_o \in O$ denoted by $\tilde{J}F(x_o)$ is the set of matrices defined as the convex hull of all matrices M of the form $M = \lim_{n\to\infty} JF(x_n)$ where x_n converges to x_o in dom F'.

In this definition, dom F' denotes the subset of full measure of O where F is differentiable.

As readily seen, this definition takes into account the behaviour of F all around x_o. In many problems it has appeared that the information is not quite necessary; what it is needed is essentially the behaviour of F relative to a given subset Q.

That let us to introduce a relative generalized jacobian matrix which can be defined on all of cl(Q) (clousure of Q).

DEFINITION 2.1. The generalized jacobian matrix of F relative to Q is defined at x_o by $\bigcap_{V \in E(x_o)} \mathrm{cl}\{\bigcup_{x \in Q\cap V} JF(x)\}$ (2.2)

where $E(x_o)$ denotes the collection of neighborhoods of x_o. The set of matrices defined in (2.2) will be denoted by $\tilde{J}_Q F(x_o)$.

Let $F = (f_1,\ldots,f_m)^t$ be a locally Lipschitz function defined on an open subset $O \subset R^n$, let $x_o \in O$ and Q be a subset of R^n such that $x_o \in \mathrm{cl}(Q)$.

We denote by $E_Q(x_o)$ the filter of neighborhoods of x_o for the topology induced on Q. The collection $\{\tilde{J}F(x) : x \in O;\ E_Q(x_o)\}$ is a filtered family [3, pág.126]. For this family, we may consider the "lim sup" which we will denote by $\tilde{J}_Q F(x_o)$ and to obtain the definition (2.2).

In other words, $\tilde{J}_Q F(x_o)$ consists of all cluster points of sequences of matrices $M_n \in \tilde{J}F(x_n)$; x_n converging to x_o in Q. Clearly, $\tilde{J}_Q F(x_o)$ is empty if $x_o \notin \mathrm{cl}(Q)$ and coincides with $\tilde{J}F(x_o)$ whenever $x_o \in Q$. If x_o lies on the boundary of Q (bdQ), $\tilde{J}_Q F(x_o)$ is, as often as not, strictly smaller than $\tilde{J}F(x_o)$.

The properties of the relative generalized jacobian matrix are mainly derived from those of the generalized jacobian matrix; let us some of them:

(P_1) Let $\{Q_\alpha : \alpha \in A\}$ be a finite collection of subsets whose union is Q; then

$$\tilde{J}_Q F(x_o) = \bigcup_{\alpha \in A} \tilde{J}_{Q_\alpha} F(x_o) \qquad (2.3)$$

(P_2) The set-valued mapping $x \rightrightarrows \tilde{J}_Q F(x)$ is locally bounded, that is to say: there exist a neighborhood V of x_o and a constant K such that $\max\{|||M|||,\ M \in \tilde{J}_Q F(x),\ x \in V\} \leq K$ where $|||.|||$ is the norm used for topologize the vector space of mxn matrices. (2.4)

(P_3) Clearly, $\tilde{J}_Q F(x_o)$ is a nonempty compact subset of the vector space of mxn matrices. (2.5)

(P_4) $\tilde{J}_Q F(.)$ is an upper semicontinuous set-valued mapping in the sense that if x_n converges to x_o in cl(Q) and M_n converges to M_o with

$$M_n \in \tilde{J}_Q F(x_n) \quad \text{for all } n \quad , \quad \text{then } M_o \in \tilde{J}_Q F(x_o) \tag{2.6}$$

Indeed, $\tilde{J}_Q F(.)$ is the smallest upper semicontinuous extension to cl(Q) of the set-valued mapping $\tilde{J}F(.)$ restricted to Q.

(P_5) Note that, unlike $\tilde{J}F(x_o)$, $\tilde{J}_Q F(x_o)$ is not generally convex when $x_o \in$ bd(Q).

However, its convex hull can be expressed in a way generalizing the basic definition (2.1) of the generalized jacobian matrix. F is actually differentiable except on a set of null measure; let E denote the set contained in dom F' such that its complementary set in O is of null measure. Denoting by Q' the set $Q \cap E$, the collection $\{\tilde{J}F(x) : x \in Q' \ ; \ E_Q(x_o)\}$ is also a filtered family. Therefore we may carry out for this collection the "lim sup" process as done in definition (2.1).

If Q is open, we have that cl(Q') = cl(Q) and the basic definition of the $\tilde{J}F(x)$ yields:

$$\text{co}\{\tilde{J}_Q F(x_o)\} = \text{co}\{J_{Q'} F(x_o)\} \tag{2.7}$$

for all x_o.

When Q is not open, a more general relation would be that:

$$\text{co}\{\tilde{J}_Q F(x_o)\} = \bigcap_{\substack{Q \subset U \\ U \text{ open}}} \text{co}\{J_{U'} F(x_o)\} \tag{2.8}$$

(P_6) In default of convexity, $\tilde{J}_Q F(x_o)$ enjoys a weaker topological property: namely connectedness, under precisely a local connectedness assumption on Q at x_o.

PROPOSITION 2.2. *Suppose there exists a basis* $\{V_n\}_{n \in N}$ *for the neighborhood system of* x_o *such that* $Q \cap V_n$ *is connected for all* n. *(When* $x_o \in Q$ *, this assumption merely means that* Q *is locally connected at* x_o*). Then* $\tilde{J}_Q F(x_o)$ *is connected.*

Proof. Without loss of generality, we can suppose that $V_{n+1} \subset V_n$ for all n. Now, $\tilde{J}F(x)$ is nonempty and connected (since convex) for all $x \in Q \cap V_n$.

Moreover, $\widetilde{J}F(.)$ is an upper semicontinuous set-valued mapping. Therefore, the image set $A_n = \bigcup_{x\varepsilon Q\cap V_n} JF(x)$ is connected [4, Theorem 5]. Thus, $\widetilde{J}_Q F(x_o)$ which is the intersection of a decreasing sequence $\{cl\, A_n\}_{n\varepsilon N}$ of compact sets is connected. (q.e.d.)

Concerning chain rules on relative generalized jacobian matrices, we state the following results.

THEOREM 2.3. *Let* $f = \varphi \circ F$ *where* $\varphi: R^m \longrightarrow R$ *is continuously differentiable at* $F(x_o)$. *Then*

$$\partial_Q(\varphi \circ F)(x_o) = \{M^t.\nabla\varphi(F(x_o)) \ : M \in \widetilde{J}_Q F(x_o)\} \tag{2.9}$$

Proof. This property is directly derived from the corresponding one on generalized jacobian matrices [5,6].

Interesting properties are the following:

(P_7) Let φ be continuously differentiable at $F(x_o)$ with $\nabla\varphi(F(x_o))\neq 0$, let $\widetilde{J}_Q F(x_o)$ be surjective. Then $0 \notin \widetilde{J}_Q(\varphi \circ F)(x_o)$.

By considering φ defined by $\varphi(y_1,\ldots,y_m) = y_i$ we have the following "projection" property:

$$\partial_Q f_i(x_o) = \{x_i^*:(x_1^*,\ldots,x_i^*,\ldots,x_m^*) \in \widetilde{J}_Q^t F(x_o)\} \tag{2.10}$$

When φ is simply Lipschitz around x_o, things are less pleasant. By applying earlier results [7] we obtain the following result:

THEOREM 2.4. *Let* φ *be Lipschitz in a neighborhood of* $F(x_o)$. *Then*

$$\partial_Q(\varphi \circ F)(x_o) \subset co\{M^t u: u \in \partial_{F(Q)} \varphi(F(x_o)) \ ; M \in \widetilde{J}_Q F(x_o)\} \tag{2.11}$$

Proof. We begin by recalling the following mean-value theorem:

Let $F: R^n \longrightarrow F^m$ be locally Lipschitz on an open subset O of R^n, let $[a,b]$ be a segment in O with $a\neq b$. Then there exist real numbers a_k, vectors c_k, matrices M_k; $(k = 1,\ldots,m)$ such that $a_k \geqslant 0$; $c_k \in (a,b)$; $M_k \in \widetilde{J}F(c_k)$ for all k, $\sum_{1\leq k\leq m} a_k = 1$ and

$$F(b)-F(a) = \sum_{1\leq k\leq m} a_k M_k(b-a) \tag{2.12}$$

When the segment $[a,b]$ is not in O, one can assert a formula analogous to (2.12) when the open subset O is connected. In such that a case, two points a and b in O can be joined by a locally Lipschitz path σ, $\sigma: [0,1] \longrightarrow O$ locally Lipschitz on (a,b); $\sigma(0) = a$; $\sigma(1) = b$. Then by application of a chain rule [8, Proposition 4.9] we have that $F(b)-F(a) \in co\{\tilde{J}F(\sigma(t))\partial\sigma(t): t \in (0,1)\}$.

Let us consider a sequence $\{x_n\}$ converging to x_o and a sequence $\{\lambda_n\} \in R_+^*$ converging to 0.

We set $E_n = \{\varphi(F(x_n + \lambda_n d)) - \varphi(F(x_n))\}\lambda_n^{-1}$.

According to the above theorem, there exist $F_n \in (F(x_n), F(x_n+\lambda_n d))$ and $u_n \in \partial\varphi(F_n)$ such that

$$E_n = \langle F(x_n + \lambda_n d) - F(x_n) , u_n\rangle \ \lambda_n^{-1} \tag{2.13}$$

Now, by applying the same mean-value theorem to F, we get that

$$(F(x_n + \lambda_n d) - F(x_n))\lambda_n^{-1} = \sum_{1\leq k\leq m} a_{k,n} M_{k,n} d \tag{2.14}$$

where $a_{k,n} \geqslant 0$; $\sum_{1\leq k\leq m} a_{k,n} = 1$ and $M_{k,n} \in \tilde{J}F(c_{k,n})$ for some $c_{k,n} \in (x_n, x_n + \lambda_n d)$.

Since the set-valued mappings $\partial\varphi$ and JF are upper-semicontinuous, we may suppose that $u_n \longrightarrow u \in \partial\varphi(F(x_o))$; $a_{k,n} \longrightarrow a_k$ for all k ($\sum_{1\leq k\leq m} a_k = 1$); $M_{k,n} \longrightarrow M_k \in \tilde{J}F(x_o)$ for all k.

Consequently, we derive from (2.13) and (2.14) that

$$\operatorname{L\acute{i}m\,sup}_{n\to\infty} E_n \leqslant \text{m\'ax}\{ \langle M^t u,d\rangle : u \in \partial\varphi(F(x_o)) \ ; \ M \in \tilde{J}F(x_o)\}.$$

Hence, $\partial(\varphi \circ F)(x_o) \subset co\{M^t u: u \in \partial\varphi(F(x_o)); M \in \tilde{J}F(x_o))\}$ and the re-sult (2.11) is thereby proved. (q.e.d.)

Now, let F be continuously differentiable at x_o. Then by [9, §13] the following inclusion is valid

$$\partial_Q(\varphi \circ F)(x_o) \subset J^t F(x_o)\partial_{F(Q)}\varphi(F(x_o)) \ .$$

Therefore, we note:

(P_8) Let F be continuously differentiable at x_o. If $0 \notin \partial_{F(Q)}\varphi(F(x_o))$ and if $JF(x_o)$ is surjective, then $0 \notin \partial_Q(\varphi \circ F)(x_o)$. Under the same assumption on $JF(x_o)$, if $0 \notin co\{\partial_{F(Q)}\varphi(F(x_o))\}$ then $0 \notin co\{\partial_Q(\varphi \circ F)(x_o)\}$.

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