THE GENERAL FORM OF ISOTROPIC TENSORS

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1. INTRODUCTION.

In this paper we find the most general form of isotropic tensors; we show that they are linear combinations, with differentiable functions as coefficients, of tensor products of the Kronecker delta. We also find the most general form of isotropic tensorial densities of any (integer) weight.

2. ISOTROPIC TENSORS.

An isotropic tensor is defined as a tensor whose components are the same in all coordinate systems. Let \( L_{k_1...k_s}^{h_1...h_r} \) be the components of an isotropic tensor of type \((r,s)\); then for a change in the coordinates:

\[
\bar{x}^i = \bar{x}^j(x^i)
\]

(2.1)

it must be:

\[
L_{k_1...k_s}^{h_1...h_r} = B_{k_1}^{h_1}...B_{k_s}^{h_s} A_{i_1}^{h_1}...A_{i_r}^{h_r} L_{j_1...j_s}^{i_1...i_r}
\]

(2.2)

where \( B_j^i = \frac{3x^i}{3x^j} \), \( A_j^i = \frac{3x^i}{2x_j} \) and use is made of the summation convention. Making the particular change \( x^i = \lambda x^i \) \((\lambda \neq 0)\), we obtain from (2.2):

\[
L_{k_1...k_s}^{h_1...h_r} = \lambda^{r-s} L_{k_1...k_s}^{h_1...h_r}
\]

and so we see that, if \( r \neq s \), the tensor must be the null tensor.

Let us consider then the case \( r = s \); then (2.2) is written as:

\[
L_{k_1...k_s}^{h_1...h_s} = B_{k_1}^{j_1}...B_{k_s}^{j_s} A_{i_1}^{h_1}...A_{i_s}^{h_s} L_{j_1...j_s}^{i_1...i_s}
\]

(2.3)
Following Rund [3], we consider all possible changes of the type (2.1); then, for a fixed point in the manifold, the $B^i_j$ may be considered as points in $GL(n, R)$. We differentiate with respect to $B^a_b$ and evaluate at $B^a_b = \delta^a_b$ to obtain:

$$0 = \delta_k^1 L_{ak_2}^{h_1} k^s + \delta_k^2 L_{k_1 a}^{h_1} k^s + \ldots + \delta_0^s L_{k_1 \ldots k_{s-1} a}^{h_1} k^s - \delta_a^h L_{k_1 \ldots k_s}^{h_1} k^s + \ldots - \delta_a^h L_{k_1 \ldots k_s}^{b_1} k^s$$

(2.4)

As a preliminary step for our final result, we prove:

**Lemma 1.** If $L_{k_1 \ldots k_s}^{h_1 \ldots h_s}$ are the components of an isotropic tensor and if $(h_1, \ldots, h_s)$ is not a permutation of $(k_1, \ldots, k_s)$, then $L_{k_1 \ldots k_s}^{h_1 \ldots h_s} = 0$.

**Proof.** First we observe that the sets $\{h_1, \ldots, h_s\}$ and $\{k_1, \ldots, k_s\}$ must be the same if the corresponding component is different from zero. For if $h_1 \notin \{k_1, \ldots, k_s\}$, we take $b = h_1$ in (2.4); all the first terms are null being $b$ different from $k_1, \ldots, k_s$ and we obtain $-m L_{k_1 \ldots k_s}^{h_1 \ldots h_s} = 0$, where $m$ is the number of times $h_1$ is repeated in $(h_1, \ldots, h_s)$. Now, if $h_1$ is repeated $m$ times in $(h_1, \ldots, h_s)$ and $n$ times in $(k_1, \ldots, k_s)$, taking $b = a = h_1$ we obtain $(n-m) L_{k_1 \ldots k_s}^{h_1 \ldots h_s} = 0$, and so the lemma is proved.

As a second step, we prove:

**Lemma 2.** Let $\sigma$ be a permutation of $\{1, 2, \ldots, s\}$. Then:

$$L_{k_1, \ldots, k_s}^{h_{\sigma(1)} \ldots h_{\sigma(s)}} = L_{k_1, \ldots, k_s}^{h_1 \ldots h_s}$$

**Proof.** Make the change $x^i = x^{\sigma(i)}$ and use (2.3).

As a result of the previous lemmas, we see that each of the components of an isotropic tensor is equal to one of the quantities:

$$L_{h_1, \ldots, h_s}^{h_{\sigma(1)} \ldots h_{\sigma(s)}}$$

(2.5)

Next we prove that we can replace a given index in (2.5) by any other index $k$ not belonging to $\{h_1, \ldots, h_s\}$. For the sake of simpli-
city, we prove \( L_{h_1 \ldots h_s} = L_{kh_2 \ldots h_s} \) for \( k \notin \{h_1, \ldots, h_s\} \). Changing indices in (2.4), we obtain:

\[
0 = \delta_k b_{h_1 \ldots h_s} + \delta_{h_2} b_{hl \ldots h_s} + \cdots + \delta_s b_{kh_2 \ldots h_s-1} - \\
\delta a_{h_1 \ldots h_s} - \delta a_{kh_2 \ldots h_s} - \cdots - \delta a_{kh_2 \ldots h_s}.
\]

(2.6)

Taking \( b = k_1 \) and \( a = h_1 \) in (2.6) (no summation convention here) we obtain \( L_{h_1 \ldots h_s} = L_{kh_2 \ldots h_s} \). Similarly it follows:

\[
L_{h_1 \ldots h_s} = L_{kh_2 \ldots h_s} = L_{h_1 \ldots h_s}.
\]

(2.7)

for \( k \notin \{h_1, \ldots, h_s\} \).

According to (2.7), any quantity in (2.5) is equal to a quantity of the form

\[
L_{h_1 \ldots h_s} \sigma(1) \sigma(2) \cdots \sigma(s)
\]

for \( s \leq \text{dimension of the manifold} \). Then the dimension of isotropic tensors at a fixed point of the manifold is \( s! \) if \( s \leq d \) = dimension of the manifold. Similarly, it can be proved that this dimension is \( d! \) if \( s > d \), but the proof is too involved to be written out in full detail here. The problem is that we have to replace indices already appearing in \( \{h_1, \ldots, h_s\} \). We give an example from which the idea of the proof will be clear. Suppose we want to replace \( h_1 \) by \( h_2 \) in the quantity

\[
L_{h_1 \ldots h_2 \ldots h_3} = L_{h_1 \ldots h_1 h_2 \ldots h_3}
\]

(no summation convention here). Changing indices in (2.4), we obtain:

\[
0 = \delta_k b_{h_1 \ldots h_2 h_3} + \delta_{h_2} b_{hl \ldots h_2 h_3} + \cdots + \\
\delta a_{h_1 \ldots h_2 h_3} - \delta a_{h_2 \ldots h_1 h_2 \ldots h_3} - \cdots - \delta a_{h_2 \ldots h_1 h_2 \ldots h_3}.
\]
Making \( b = h_2 \) and \( a = h_1 \), we find that \( L_{h_1 h_2 h_3} \) can be written as a sum of terms of the form (2.8) with the first \( h_1 \) replaced by \( h_2 \) in the upper indices and the lower ones and with permutations on the upper indices. Generalizing this procedure, we see that the components of an isotropic tensor for \( d < s \) are linear combinations of quantities of the form:

\[
\delta_{k_1} \delta_{k_2} \cdots \delta_{k_s} \quad \sigma(1) \sigma(2) \ldots \sigma(d) \quad d \ldots d
\]

and so the dimension of isotropic tensors is \( d! \) for \( d < s \).

Now we observe that we have at our disposal \( s! \) linearly independent isotropic tensors for \( s \leq d \), namely

\[
\delta_{k_1} \delta_{k_2} \cdots \delta_{k_s} \quad \sigma(1) \sigma(2) \ldots \sigma(s) \quad \sigma \in S_s
\]

(2.9)

and \( d! \) linearly independent isotropic tensors for \( d < s \), namely

\[
\delta_{k_1} \delta_{k_2} \cdots \delta_{k_d} \delta_{k_{d+1}} \cdots \delta_{k_s} \quad \sigma \in S_d
\]

(2.10)

Since (2.10) is a subset of (1.9) for \( d < s \), we obtain (2.9) as a set of generators in any case. Thus we conclude:

**THEOREM 1.** Let \( L_{h_1 \ldots h_r} \) be the components of an isotropic tensor.

Then:

a) \( L_{h_1 \ldots h_r} = 0 \) for \( r \neq s \)

b) If \( r = s \), then

\[
L_{h_1 \ldots h_s} = \sum_{\sigma \in S_s} f_\sigma \delta_{k_1} \delta_{k_2} \cdots \delta_{k_s} \quad \sigma(1) \sigma(2) \ldots \sigma(s)
\]

where \( f_\sigma \) is a differentiable function for each permutation \( \sigma \).

As a particular case, if \( f_\sigma = \varepsilon(\sigma) \) (sign of the permutation \( \sigma \)), we obtain the generalized Kronecker delta \( \delta_{k_1 \ldots k_s} \) (see [1]).
3. ISOTROPIC TENSORIAL DENSITIES.

Let now \( L_{k_1 \ldots k_s} \) be the components of an isotropic density of weight \( M \). The transformation rule is:

\[
L_{k_1 \ldots k_s} = J^M B_{k_1} \ldots B_{k_s} A_{i_1} \ldots A_{i_r} L_{i_1 \ldots i_r}
\]

We differentiate respect to \( B^a_b \) and evaluate at \( B^a_b = \delta^a_b \). Contracting \( b = a \) it is easy to obtain:

\[
Md = s - r
\]

and so, for \( M > 0 \) (integer), the components of the isotropic density are of the form:

\[
L_{k_1 \ldots k_r \ldots k_r+1 \ldots k_{r+Md}}
\]

We obtain an isotropic tensor by multiplying this with the Levi-Civita symbols \( \epsilon \); from theorem 1 it must be:

\[
\epsilon h_{r+1} \ldots h_{r+d} \ldots \epsilon h_{r+(m-1)d+1} \ldots h_{r+Md} = \sum_{\sigma \in S_d} f_\sigma \delta_{k_1} \ldots \delta_{k_s}
\]

and since \( \epsilon_1 \ldots d \epsilon_1 \ldots d = n! \), we obtain \( L_{k_1 \ldots k_s} \) from (3.2). Following a similar procedure for \( M < 0 \), we obtain:

THEOREM 2. Let \( L_{k_1 \ldots k_s} \) be the components of an isotropic tensorial density of weight \( M \).

Then \( Md = s - r \) and:

\[
a) L_{k_1 \ldots k_s} = \sum_{\sigma \in S_d} f_\sigma \delta_{k_1} \ldots \delta_{k_s} \epsilon h_{r+1} \ldots h_{r+Md} \epsilon h_{s-d+1} \ldots h_s
\]

if \( M > 0 \), where \( \epsilon \) are the Levi-Civita symbols

\[
b) L_{k_1 \ldots k_s} = \sum_{\sigma \in S_d} f_\sigma \delta_{k_1} \ldots \delta_{k_r} \epsilon k_{s+1} \ldots k_{s+d} \ldots \epsilon k_{r-d+1} \ldots k_r
\]
if $M < 0$.

As a particular case, if $L_{k_1 \ldots k_d}$ is a tensorial isotropic density of weight 1, then it follows from (3.3) that:

$$L_{k_1 \ldots k_d} = \sum_{\sigma \in S_d} f_{\sigma} \delta_{k_1 \ldots k_d} e_{h_1 \ldots h_d} = \sum_{\sigma \in S_d} f_{\sigma} e_{h_{\sigma(1)} \ldots h_{\sigma(d)}} =$$

$$\sum_{\sigma \in S_d} e(\sigma) f_{\sigma} e_{h_1 \ldots h_d} = g(e_{h_1 \ldots h_d}),$$

where $g$ is a differentiable function.

REMARK. The knowledge of isotropic tensors is useful in concomitant theory if null tensors are included into the domain of concomitance. See [2].

ACKNOWLEDGEMENT. To Professor L.A. Santaló who suggested me this approach to the determination of isotropic tensors.

REFERENCES


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Recibido en mayo de 1984.