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THE GENERAL FORM OF ISOTROPIC TENSORS

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1. INTRODUCTION.

In this paper we find the most general form of isotropic tensors; we show that they are linear combinations, with differentiable functions as coefficients, of tensor products of the Kronecker delta. We also find the most general form of isotropic tensorial densities of any (integer) weight.

2. ISOTROPIC TENSORS.

An isotropic tensor is defined as a tensor whose components are the same in all coordinate systems. Let $L^{h_1\ldots h_r}_{k_1\ldots k_s}$ be the components of an isotropic tensor of type (r,s); then for a change in the coordinates:

$$\bar{x}^i = \bar{x}^i(x^j) \tag{2.1}$$

it must be:

$$L^{h_1\ldots h_r}_{k_1\ldots k_s} = B^{j_1}_{k_1}\ldots B^{j_s}_{k_s}\, A^{h_1}_{i_1}\ldots A^{h_r}_{i_r}\, L^{i_1\ldots i_r}_{j_1\ldots j_s} \tag{2.2}$$

where $B^i_j = \dfrac{\partial x^i}{\partial \bar{x}^j}$, $A^i_j = \dfrac{\partial \bar{x}^i}{\partial x^j}$ and use is made of the summation convention. Making the particular change $\bar{x}^i = \lambda x^i$ ($\lambda \neq 0$), we obtain from (2.2):

$$L^{h_1\ldots h_r}_{k_1\ldots k_s} = \lambda^{r-s}\, L^{h_1\ldots h_r}_{k_1\ldots k_s}$$

and so we see that, if $r \neq s$, the tensor must be the null tensor. Let us consider then the case $r = s$; then (2.2) is written as:

$$L^{h_1\ldots h_s}_{k_1\ldots k_s} = B^{j_1}_{k_1}\ldots B^{j_s}_{k_s}\, A^{h_1}_{i_1}\ldots A^{h_s}_{i_s}\, L^{i_1\ldots i_s}_{j_1\ldots j_s} \tag{2.3}$$

Following Rund [3], we consider all posible changes of the type (2.1); then, for a fixed point in the manifold, the B^i_j may be considered as points in GL(n,R). We differentiate with respect to B^a_b and evaluate at $B^a_b = \delta^a_b$ to obtain:

$$\begin{aligned} 0 = \delta^{b}_{k_1} L^{h_1\dots h_s}_{ak_2\dots k_s} + \delta^{b}_{k_2} L^{h_1\dots h_s}_{k_1 a\dots k_s} + \dots + \delta^{b}_{k_s} L^{h_1\dots h_s}_{k_1\dots k_{s-1}a} - \\ - \delta^{h_1}_{a} L^{bh_2\dots h_s}_{k_1\dots k_s} - \delta^{h_2}_{a} L^{h_1 b\dots h_s}_{k_1\dots k_s} - \dots - \delta^{h_s}_{a} L^{h_1 h_2\dots b}_{k_1\dots k_s} \end{aligned} \qquad (2.4)$$

As a preliminary step for our final result, we prove:

LEMMA 1. *If* $L^{h_1\dots h_s}_{k_1\dots k_s}$ *are the components of an isotropic tensor and if* $(h_1,\dots,h_s)$ *is not a permutation of* $(k_1,\dots,k_s)$, *then* $L^{h_1\dots h_s}_{k_1\dots k_s} = 0$.

Proof. First we observe that the sets $\{h_1,\dots,h_s\}$ and $\{k_1,\dots,k_s\}$ must be the same if the corresponding component is different from zero. For if $h_1 \notin \{k_1,\dots,k_s\}$, we take $b = h_1 = a$ in (2.4); all the first terms are null being b different from $k_1,\dots,k_s$ and we obtain $-m\, L^{h_1\dots h_s}_{k_1\dots k_s} = 0$, where m is the number of times h_1 is repeated in $(h_1,\dots,h_s)$. Now, if h_1 is repeated m times in $(h_1,\dots,h_s)$ and n times in $(k_1,\dots,k_s)$, taking $b = a = h_1$ we obtain $(n-m)L^{h_1\dots h_s}_{k_1\dots k_s} = 0$, and so the lemma is proved.

As a second step, we prove:

LEMMA 2. *Let* σ *be a permutation of* $\{1,2,\dots,s\}$. *Then:*

$$L^{h_{\sigma(1)}\dots h_{\sigma(s)}}_{k_{\sigma(1)}\dots k_{\sigma(s)}} = L^{h_1\dots h_s}_{k_1\dots k_s}$$

Proof. Make the change $\bar{x}^i = x^{\sigma(i)}$ and use (2.3).

As a result of the previous lemmas, we see that each of the components of an isotropic tensor is equal to one of the quantities:

$$L^{h_{\sigma(1)}\dots h_{\sigma(s)}}_{h_1\dots h_s}, \quad h_1 \leq h_2 \leq \dots \leq h_s, \quad \sigma \in S_s \qquad (2.5)$$

Next we prove that we can replace a given index in (2.5) by any other index k not belonging to $\{h_1,\dots,h_s\}$. For the sake of simplicity, we prove $L^{h_1 \ldots h_s}_{h_1 \ldots h_s} = L^{kh_2 \ldots h_s}_{kh_2 \ldots h_s}$ for $k \notin \{h_1, \ldots, h_s\}$. Changing indices in (2.4), we obtain:

$$0 = \delta^b_{k_1} L^{h_1 \ldots h_s}_{ah_2 \ldots h_s} + \delta^b_{h_2} L^{h_1 \ldots h_s}_{k_1 a h_2 \ldots h_s} + \ldots + \delta^b_{h_s} L^{h_1 \ldots h_s}_{k_1 h_2 \ldots h_{s-1} a} - - \delta^{h_1}_a L^{bh_2 \ldots h_s}_{kh_2 \ldots h_s} - \delta^{h_2}_a L^{h_1 b h_3 \ldots h_s}_{kh_2 \ldots h_s} - \ldots - \delta^{h_s}_a L^{h_1 \ldots h_{s-1} b}_{kh_2 \ldots h_s} \tag{2.6}$$

Taking $b = k_1$ and $a = h_1$ in (2.6) (no summation convention here) we obtain $L^{h_1 \ldots h_s}_{h_1 \ldots h_s} = L^{kh_2 \ldots h_s}_{kh_2 \ldots h_s}$. Similarly it follows:

$$L^{h_{\sigma(1)} \ldots h_{\sigma(i)} \ldots h_{\sigma(s)}}_{h_1 \ldots h_{\sigma^{-1}(i)} \ldots h_s} = L^{h_{\sigma(1)} \ldots k \ldots h_{\sigma(s)}}_{h_1 \ldots k \ldots h_s} \tag{2.7}$$

for $k \notin \{h_1, \ldots, h_s\}$.

According to (2.7), any quantity in (2.5) is equal to a quantity of the form

$$L^{\sigma(1)\ \sigma(2) \ldots \sigma(s)}_{1\ 2 \ldots s}$$

for $s \leqslant$ dimension of the manifold. Then the dimension of isotropic tensors at a fixed point of the manifold is $s!$ if $s \leqslant d$ = dimension of the manifold. Similarly, it can be proved that this dimension is $d!$ if $s > d$, but the proof is too involved to be written out in full detail here. The problem is that we have to replace indices already appearing in $\{h_1, \ldots, h_s\}$. We give an example from which the idea of the proof will be clear. Suppose we want to replace h_1 by h_2 in the quantity

$$L^{h_1 \ldots h_1 h_2 \ldots h_2 h_3}_{h_1 \ldots h_1 h_2 \ldots h_2 h_3} \tag{2.8}$$

(no summation convention here). Changing indices in (2.4), we obtain:

$$0 = \delta^b_{h_2} L^{h_1 \ldots h_1 h_2 \ldots h_2 h_3}_{ah_1 \ldots h_1 h_2 \ldots h_2 h_3} + \delta^b_{h_1} L^{h_1 \ldots h_1 h_2 \ldots h_2 h_3}_{h_2 a \ldots h_1 h_2 \ldots h_2 h_3} + \ldots +$$

$$+ \delta^b_{h_1} L^{h_1 \ldots h_1 h_2 \ldots h_2 h_3}_{h_2 \ldots h_1 a h_2 \ldots h_2 h_3} + \delta^b_{h_2} L^{h_1 \ldots h_1 h_2 \ldots h_2 h_3}_{h_2 \ldots h_1 a h_2 \ldots h_2 h_3} + \ldots +$$

$$+ \delta^b_{h_2} L^{h_1 \ldots h_1 h_2 \ldots h_2 h_3}_{h_2 \ldots h_1 h_2 \ldots a h_3} + \delta^b_{h_3} L^{h_1 \ldots h_1 h_2 \ldots h_2 h_3}_{h_2 \ldots h_1 h_2 \ldots h_2 a} -$$

$$- \delta^{h_1}_a L^{bh_1 \ldots h_1 h_2 \ldots h_2 h_3}_{h_2 h_1 \ldots h_1 h_2 \ldots h_2 h_3} - \delta^{h_1}_a L^{h_1 b \ldots h_1 h_2 \ldots h_2 h_3}_{h_2 h_1 \ldots h_1 h_2 \ldots h_2 h_3} - \ldots -$$

$$- \delta_a^{h_1} L^{h_1 \ldots h_1 b h_2 \ldots h_2 h_3}_{h_2 h_1 \ldots h_1 h_2 \ldots h_2 h_3} - \delta_a^{h_2} L^{h_1 \ldots h_1 b h_2 \ldots h_2 h_3}_{h_2 \ldots h_1 h_2 h_2 \ldots h_2 h_3} - \ldots -$$

$$- \delta_a^{h_2} L^{h_1 \ldots h_1 h_2 \ldots h_2 b h_3}_{h_2 h_1 \ldots h_1 h_2 \ldots h_2 h_3} - \delta_a^{h_3} L^{h_1 \ldots h_1 h_2 \ldots h_2 b}_{h_2 \ldots h_1 h_2 \ldots h_2 h_3}$$

Making $b = h_2$ and $a = h_1$, we find that $L^{h_1 \ldots h_1 h_2 \ldots h_2 h_3}_{h_1 \ldots h_1 h_2 \ldots h_2 h_3}$ can be written as a sum of terms of the form (2.8) with the first h_1 replaced by h_2 in the upper indices and the lower ones and with permutations on the upper indices. Generalizing this procedure, we see that the components of an isotropic tensor for $d < s$ are linear combinations of quantities of the form:

$$L^{\sigma(1)\ \sigma(2) \ldots \sigma(d)\ d \ldots d}_{1\ 2 \ldots d\ d \ldots d}$$

and so the dimension of isotropic tensors is $d!$ for $d < s$.

Now we observe that we have at our disposal $s!$ linearly independent isotropic tensors for $s \leqslant d$, namely

$$\delta_{k_1}^{h_{\sigma(1)}} \delta_{k_2}^{h_{\sigma(2)}} \ldots \delta_{k_s}^{h_{\sigma(s)}} \quad , \quad \sigma \in S_s \tag{2.9}$$

and $d!$ linearly independent isotropic tensors for $d < s$, namely

$$\delta_{k_1}^{h_{\sigma(1)}} \delta_{k_2}^{h_{\sigma(2)}} \ldots \delta_{k_d}^{h_{\sigma(d)}} \delta_{k_{d+1}}^{h_{d+1}} \ldots \delta_{k_s}^{h_s} \quad , \quad \sigma \in S_d \tag{2.10}$$

Since (2.10) is a subset of (1.9) for $d < s$, we obtain (2.9) as a set of generators in any case. Thus we conclude:

THEOREM 1. *Let* $L^{h_1 \ldots h_r}_{k_1 \ldots k_s}$ *be the components of an isotropic tensor. Then:*

a) $L^{h_1 \ldots h_r}_{k_1 \ldots k_s} = 0$ *for* $r \neq s$

b) *If* $r = s$, *then*

$$L^{h_1 \ldots h_s}_{k_1 \ldots k_s} = \sum_{\sigma \in S_s} f_\sigma\, \delta_{k_1}^{h_{\sigma(1)}} \delta_{k_2}^{h_{\sigma(2)}} \ldots \delta_{k_s}^{h_{\sigma(s)}}$$

where f_σ *is a differentiable function for each permutation* σ.

As a particular case, if $f_\sigma = \varepsilon(\sigma)$ (sign of the permutation σ), we obtain the generalized Kronecker delta $\delta^{h_1 \ldots h_s}_{k_1 \ldots k_s}$ (see [1]).

3. ISOTROPIC TENSORIAL DENSITIES.

Let now $L^{h_1 \ldots h_r}_{k_1 \ldots k_s}$ be the components of an isotropic density of weight M. The transformation rule is:

$$L^{h_1 \ldots h_r}_{k_1 \ldots k_s} = J^M B^{j_1}_{k_1} \ldots B^{j_s}_{k_s} A^{h_1}_{i_1} \ldots A^{h_r}_{i_r} L^{i_1 \ldots i_r}_{j_1 \ldots j_s} \qquad (3.1)$$

We differentiate respect to B^a_b and evaluate at $B^a_b = \delta^a_b$. Contracting $b = a$ it is easy to obtain:

$$Md = s - r$$

and so, for $M > 0$ (integer), the components of the isotropic density are of the form:

$$L^{h_1 \ldots h_r}_{k_1 \ldots k_r \; k_{r+1} \ldots k_{r+Md}}$$

We obtain an isotropic tensor by multiplying this with the Levi-Civita symbols $\varepsilon^{\cdots}$; from theorem 1 it must be:

$$L^{h_1 \ldots h_r}_{k_1 \ldots k_r \; k_{r+1} \ldots k_{r+Md}} \; \varepsilon^{h_{r+1} \ldots h_{r+d}} \ldots \varepsilon^{h_{r+(m-1)d+1} \ldots h_{r+Md}} =$$

$$= \sum_{\sigma \varepsilon S_s} f_\sigma \, \delta^{h_{\sigma(1)}}_{k_1} \ldots \delta^{h_{\sigma(s)}}_{k_s} \quad , \qquad (3.2)$$

and since $\varepsilon^{i_1 \ldots i_d} \; \varepsilon_{i_1 \ldots i_d} = n!$, we obtain $L^{h_1 \ldots h_r}_{k_1 \ldots k_s}$ from (3.2). Following a similar procedure for $M < 0$, we obtain:

THEOREM 2. *Let* $L^{h_1 \ldots h_r}_{k_1 \ldots k_s}$ *be the components of an isotropic tensorial density of weight* M.

Then $Md = s-r$ *and:*

a) $$L^{h_1 \ldots h_r}_{k_1 \ldots k_s} = \sum_{\sigma \varepsilon S_s} f_\sigma \, \delta^{h_{\sigma(1)}}_{k_1} \ldots \delta^{h_{\sigma(s)}}_{k_s} \, .$$

$$\cdot \; \varepsilon_{h_{r+1} \ldots h_{r+1}} \ldots \varepsilon_{h_{s-d+1} \ldots h_s} \qquad (3.3)$$

if $M > 0$, *where* $\varepsilon_{\ldots}$ *are the Levi-Civita symbols*

b) $$L^{h_1 \ldots h_r}_{k_1 \ldots k_s} = \sum_{\sigma \varepsilon S_s} f_\sigma \, \delta^{h_{\sigma(1)}}_{k_1} \ldots \delta^{h_{\sigma(r)}}_{k_r} \, .$$

$$\cdot \; \varepsilon^{k_{s+1} \ldots k_{s+d}} \ldots \varepsilon^{k_{r-d+1} \ldots k_r} \qquad (3.4)$$

if $M < 0$.

As a particular case, if $L_{k_1 \dots k_d}$ is a tensorial isotropic density of weight 1, then it follows from (3.3) that:

$$L_{k_1 \dots k_d} = \sum_{\sigma \varepsilon S_d} f_\sigma \, \delta^{h_{\sigma(1)} \dots h_{\sigma(d)}}_{k_1 \dots k_d} . \varepsilon_{h_1 \dots h_d} = \sum_{\sigma \varepsilon S_d} f_\sigma \, \varepsilon_{h_{\sigma(1)} \dots h_{\sigma(d)}} =$$

$$= \sum_{\sigma \varepsilon S_d} \varepsilon(\sigma) \, f_\sigma \, \varepsilon_{h_1 \dots h_d} = g . \varepsilon_{h_1 \dots h_d} \; ,$$

where g is a differentiable function.

REMARK. The knowledge of isotropic tensors is useful in concomitant theory if null tensors are included into the domain of concomitance. See [2].

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