Revista de la
Unión Matemática Argentina
Volumen 31, 1984.

REDUCTION OF CODIMENSION OF ISOMETRIC IMMERSIONS BETWEEN INDEFINITE RIEMANNIAN MANIFOLDS

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1. INTRODUCTION.

Let $f: M^n_s \longmapsto Q^{n+p}_t(c)$ be an isometric immersion of a connected indefinite Riemannian manifold of dimension n and signature (s,n-s) into an indefinite manifold of constant curvature c. If s=1 or s = n-1, we say that M^n_s is a Lorentz manifold. By changing the sign in the inner products we may assume that s=1. We say that the immersion f is m-*regular* if the k^{th} normal space of the immersion N_k satisfies: dim N_k = constant for k = 1,...,m (see Section 2 for further definitions). The aim of this paper is to extend the main result of [1] to the indefinite Riemannian case. We prove the following result

1.1. THEOREM. *Let* $f: M^n_1 \longmapsto Q^{n+p}_t(c)$ *be an isometric immersion. Assume that the curvature tensor of the normal connection satisfies* $(\nabla^\perp)^m R^\perp|_{N_m^\perp} = 0$ *and that the mean curvature vector satisfies* $(\nabla^\perp)^m H \subset N_m$. *Then there exists a totally geodesic submanifold* Q^* *of* $Q^{n+p}_t(c)$ *of dimension* n+k, *where* k = dim N_m, *such that* $f(M^n) \subset Q^*$.

2. PRELIMINARIES.

We denote by M^n_s a differentiable manifold whose tangent spaces have a nondegenerate metric of signature (s,n-s). Let us consider an isometric immersion, $f: M^n_s \longmapsto \widetilde{M}^{n+\ell}_t$, of one indefinite Riemannian manifold into another. Given $p \in M$, we identify the tangent space T_pM to M at p with $df(T_pM)$. The normal space $T_pM^\perp$ is the subspace of $T_p\widetilde{M}$ consisting of all vectors $\xi(p) \in T_p\widetilde{M}$ which are normal to T_pM with respect to the metric < , > of $\widetilde{M}$. Let $\overline{\nabla}$ (resp. ∇) be the co-

variant differentiation of the Levi-Civita connection in $\widetilde{M}$ (resp. M) and $\nabla^{\perp}$ the covariant differentiation in the normal bundle of f.
Given $\xi(p) \in T_pM^{\perp}$, we define the second fundamental form of f relative to $\xi(p)$

$$A_{\xi(p)}: T_pM \longmapsto T_pM$$

by the Weingarten equation:

$$\overline{\nabla}_X\xi = -A_\xi X + \nabla^{\perp}_X\xi \quad ,$$

where $X \in T_pM$ and ξ is any normal extension of $\xi(p)$.
We shall denote the curvature tensor of $\overline{\nabla}$ by $\overline{R}$ and that of $\nabla^{\perp}$ by $R^{\perp}$, i.e.

$$\overline{R}(X,Y) = \overline{\nabla}_X \overline{\nabla}_Y - \overline{\nabla}_Y \overline{\nabla}_X - \overline{\nabla}_{[X,Y]}$$

and

$$R^{\perp}(X,Y) = \nabla^{\perp}_X \nabla^{\perp}_Y - \nabla^{\perp}_Y \nabla^{\perp}_X - \nabla^{\perp}_{[X,Y]} \quad .$$

We define the bilinear symmetric form

$$\alpha: T_pM \times T_pM \longrightarrow T_pM^{\perp}$$

by the Gauss equation:

$$\overline{\nabla}_X Y = \nabla_X Y + \alpha(X,Y) \quad .$$

Then, the condition

$$\langle\alpha(X,Y),\xi\rangle = \langle A_\xi X,Y\rangle$$

is satisfied.

If the ambient space has constant curvature, the following relations hold:

$$(\nabla_X A)_\xi(Y) = (\nabla_Y A)_\xi(X) \quad , \quad \text{Codazzi's equation}$$

and

$$\langle R^{\perp}(X,Y)\xi,\eta\rangle = \langle [A_\xi,A_\eta]X,Y\rangle \quad , \quad \text{Ricci's equation} \; .$$

A basis $X_1,\dots,X_n$ of an indefinite inner product space with signature $(s,n-s)$ is called orthonormal if $\langle X_i,X_j\rangle = -\delta_{ij}$ $1 \leqslant i,j \leqslant s$, $\langle X_r,X_t\rangle = \delta_{rt}$ $s+1 \leqslant r,t \leqslant n$ and $\langle X_i,X_r\rangle = 0$ $1 \leqslant i \leqslant s$, $s+1 \leqslant r \leqslant n$.
If the vector space is a Lorentz space, then a pseudo-orthonormal basis is one of the form $Z,\overline{Z},X_1,\dots,X_{n-2}$, such that $\langle Z,Z\rangle = 0 = \langle\overline{Z},\overline{Z}\rangle$, $\langle Z,\overline{Z}\rangle = 1$, $\langle X_i,Y_j\rangle = \delta_{ij}$ $1 \leqslant i,j \leqslant n-2$ and $\langle Z,X_i\rangle = 0 = \langle\overline{Z},X_i\rangle$ $1 \leqslant i \leqslant n-2$.
We define the *mean curvature vector* as

$$H = \frac{1}{n} \sum_{i=1}^{n} \langle X_i, X_i \rangle \, \alpha(X_i, X_i) ,$$

where $X_1, \ldots, X_n$ is an orthonormal basis of T_pM.
We say that the immersion is *totally geodesic* if $A_\xi = 0$ for all $\xi \in TM^\perp$.

Given $p \in M$, we define the *first normal space* as

$$N_1(p) = \text{Span} \{\alpha(X,Y)(p): X,Y \in T_pM\} .$$

We define the k^{th} *normal space* as

$$N_k(p) = \text{Span} \{\alpha(X,Y)(p), \nabla^\perp_{w_1} \alpha(X,Y)(p), \ldots, \nabla^\perp_{w_{k-1}} \cdots \nabla^\perp_{w_1} \alpha(X,Y)(p)\}$$

for $k = 2,3,\ldots$, where $X,Y,w_1,\ldots,w_{k-1}$ are vector fields tangent to M.

A *normal subbundle* of dimension k is a family $L(p)$, for all $p \in M$, of vector subspaces of $T_pM^\perp$ of dimension k with the property that, for all $q \in M^n$, there are an open neighborhood U of q and k differentiable fields $\xi_1,\ldots,\xi_k$ defined in U such that, for all $p \in U$, $\xi_1(p),\ldots,\xi_k(p)$ generate $L(p)$.

An immersion is said to be m-*regular* if each $N_k(p)$, for $k = 1,\ldots,m$ and for all $p \in M$, has constant dimension. It is easily seen that if an immersion is m-regular, then each N_k for $k = 1,\ldots,m$ is a normal subbundle.

If L is a normal subbundle, by $(\nabla^\perp)^m R^\perp|_L = 0$ it is to be understood that

$$((\nabla^\perp)^m R^\perp)(X_1;X_2;\ldots;X_{m+2})(\xi) = 0$$

for all $X_1,\ldots,X_{m+2} \in TM$ and all $\xi \in L$.

Finally, if η is a section of the normal subbundle L, then $(\nabla^\perp)^m \eta \in L$ means that

$$\nabla^\perp_{X_1} \cdots \nabla^\perp_{X_m} \eta \in L \quad \text{for all} \quad X_1,\ldots,X_m \in TM.$$

3. PROOF OF THEOREM 1-1.

First, we recall the following indefinite version of a theorem of Allendoerfer-Erbacher (see [2], [3]).

3-1 PROPOSITION. *Let* $f: M^n_s \longmapsto Q^{n+p}_t(c)$ *be an isometric immersion of a connected indefinite Riemannian manifold into a space form. If the* *re exists a* k*-dimensional parallel normal subbundle* L(p) *which contains the first normal space* $N_1(p)$ *for all* $p \in M^n_s$*, then there exists a* (n+k)*-dimensional totally geodesic submanifold (possible degenerate)* Q* *of* $Q^{n+p}_t(c)$ *such that* $f(M^n_s) \subset Q^*$.

The following result is the main part of the proof of Theorem 1.1.

3-2. PROPOSITION. *Let* $f\colon M^n_1 \longmapsto Q^{n+\ell}_t(c)$ *be an isometric immersion that is* m*-regular in an open neighborhood* U *of a point* p *of* M^n_1. *Then*

$$N^{\perp}_{m+1}(p) = \{\xi \in N^{\perp}_1(p): \quad ((\nabla^{\perp})^k\, R^{\perp}(\xi)) = 0\ , \ (\nabla^{\perp})^k\, H(p) \perp \xi \quad \text{for} \quad 0 \leqslant k \leqslant m\}.$$

The proof of the following four lemmas is the same as in the positive definite case (see [1]).

3-3. LEMMA. *Let* M *be an indefinite Riemannian manifold and* ξ,η *vector fields defined in an open neighborhood* U *of a point* p *of* M. *Then, we have that*

i) $<(\nabla)^k\, \xi,\eta> = 0$ *for* $0 \leqslant k \leqslant m$ *if and only if*
$<\xi,(\nabla)^k\, \eta> = 0$ *for* $0 \leqslant k \leqslant m$.

ii) $\begin{cases} <(\nabla)^k\, \xi,\eta> = 0 & \textit{for} \quad 0 \leqslant k \leqslant m \\ <(\nabla)^{m+1}\, \xi,\eta>(p) = 0. \end{cases}$ *if and only if*

$\begin{cases} <\xi,(\nabla)^k\, \eta> = 0 & \textit{for} \quad 0 \leqslant k \leqslant m \\ <\xi,(\nabla)^{m+1}\, \eta>(p) = 0. \end{cases}$

3-4. LEMMA. *Let* $f\colon M \longmapsto \widetilde{M}$ *be an isometric immersion that is* m*-regular in an open set* U *of* M. *Then, in* U*, we have that for* r = 1,...,m
$\xi \in N^{\perp}_r$ *if and only if* $A_{(\nabla^{\perp})^k \xi} = 0$ *for* $0 \leqslant k \leqslant r-1$.

3-5. LEMMA. *Let* $f\colon M \longrightarrow \widetilde{M}$ *be an isometric immersion that is* m*-regular in an open neighborhood* U *of a point* p *of* M. *Then,*
$\eta_p \in N^{\perp}_{m+1}(p)$ *if and only if there exists a local extension* η *of* η_p *such that* i) $A_{(\nabla^{\perp})^k \eta} = 0$ *for* $0 \leqslant k \leqslant m-1$ and ii) $A_{(\nabla^{\perp})^m\, \eta(p)} = 0$.

3-6. LEMMA. *Let* $f\colon M \longmapsto \widetilde{M}$ *be an isometric immersion and* η *a normal vector field defined in an open neighborhood* U *of a point* p *of* M. *Then, we have that*

i) $((\nabla^{\perp})^k\, R^{\perp})(\eta) = 0$ *for* $0 \leqslant k \leqslant m-1$ *if and only if*

$R^{\perp}((\nabla^{\perp})^k\,\eta) = 0$ *for* $0 \leqslant k \leqslant m-1$.

ii) $\begin{cases} ((\nabla^{\perp})^k\,R^{\perp})(\eta) = 0 & 0 \leqslant k \leqslant m-1 \\ ((\nabla^{\perp})^m\,R^{\perp})(\eta(p)) = 0. \end{cases}$ *if and only if*

$$\begin{cases} R^{\perp}((\nabla^{\perp})^k\,\eta) = 0 & \text{for } 0 \leqslant k \leqslant m-1 \\ R^{\perp}((\nabla^{\perp})^m\,\eta(p)) = 0 . \end{cases}$$

PROOF OF PROPOSITION 3-2. First of all, we note that if $\xi \in N_1^{\perp}(p)$, then $R^{\perp}(\xi) = 0$ and $H(p) \perp \xi$. The proof will be divided in two parts, each of them showing one of the inclusions.

i) Let $\eta_p \in N_{m+1}^{\perp}(p)$. By Lemma 3-5, there exists a local extension η of η_p such that

$A_{(\nabla^{\perp})^k\eta} = 0$ for $0 \leqslant k \leqslant m-1$ and $A_{(\nabla^{\perp})^m\,\eta(p)} = 0$.

By the Ricci equation, we may write

$R^{\perp}((\nabla^{\perp})^k\,\eta) = 0$ for $0 \leqslant k \leqslant m-1$ and $R^{\perp}((\nabla^{\perp})^m\,\eta(p)) = 0$.

By Lemma 3-6 it follows that

3-7 $\qquad ((\nabla^{\perp})^k\,R^{\perp})(\eta_p) = 0$ for $0 \leqslant k \leqslant m$.

By definition, one has that $H \in N_1$. Therefore, it is immediate that

3-8 $\qquad (\nabla^{\perp})^k\,H(p) \perp \eta_p$ for $0 \leqslant k \leqslant m$.

Then, the first inclusion follows from (3-7) and (3-8).

ii) For this part of the proof we shall use induction. Using i), we may suppose that the proposition holds for $N_j^{\perp}$ for $1 \leqslant j \leqslant h$. Let $\eta_p \in N_1^{\perp}(p)$, which satisfies

3-9 $\qquad ((\nabla^{\perp})^k\,R^{\perp})(\eta_p) = 0$ for $0 \leqslant k \leqslant h$

and

3-10 $\qquad (\nabla^{\perp})^k\,H(p) \perp \eta_p$ for $0 \leqslant k \leqslant h$.

By the induction hypothesis together with (3-9) and (3-10), we thus have $\eta_p \in N_h^{\perp}(p)$. Let η be a local extension of η_p in $N_h^{\perp}$. Then,

3-11 $\qquad ((\nabla^{\perp})^k\,R^{\perp})(\eta) = 0$ for $0 \leqslant k \leqslant h-1$

and

3-12 $\qquad (\nabla^{\perp})^k\,H \perp \eta$ for $0 \leqslant k \leqslant h-1$.

From Lemma 3-6, (3-9) and (3-11), it follows that

$$R^{\perp}((\nabla^{\perp})^{h}\ \eta(p)) = 0.$$

But this means that

$<R^{\perp}(X,Y)(\nabla^{\perp}_{z_h} \dots \nabla^{\perp}_{z_1}\ \eta(p)),\xi> = 0$ for all $X,Y,Z_1,\dots,Z_h \in TM$, $\xi \in T_pM^{\perp}$.

By Ricci's equation

$$<[A_{\nabla^{\perp}_{z_h}\dots\nabla^{\perp}_{z_1}\ \eta(p)}\ ,\ A_{\xi}]\ X,Y> = 0\ .$$

In particular,

3-13 $\quad [A_{\nabla^{\perp}_{z_h}\dots\nabla^{\perp}_{z_1}\ \eta(p)}\ ,\ A_{\nabla^{\perp}_{Y_h}\dots\nabla^{\perp}_{Y_1}\ \eta(p)}] = 0$ for all $Z_1,\dots,Z_h,Y_1,\dots,Y_h \in TM$.

On the other hand, by Lemma 3-4

3-14 $$A_{(\nabla^{\perp})^{r}\ \eta} = 0 \quad \text{for} \quad 0 \leqslant r \leqslant h-1\ .$$

By Codazzi's equation applied to the normal vector field $\nabla^{\perp}_{X_{h-1}} \dots \nabla^{\perp}_{X_1}\ \eta$, we obtain

3-14 $\quad A_{\nabla^{\perp}_{Z}\ \nabla^{\perp}_{X_{h-1}}\ \dots\ \nabla^{\perp}_{X_1}\eta}\ Y = A_{\nabla^{\perp}_{Y}\ \nabla^{\perp}_{X_{h-1}}\ \dots\ \nabla^{\perp}_{X_1}\eta}\ Z$ for all $Z,Y,X_1,\dots,X_h \in TM$.

From Lemma 3-3 together with (3-10) and (3-12), it is clear that

3-15 $$<H(p)\ ,\ (\nabla^{\perp})^{h}\ \eta(p)> = 0.$$

We shall show that (3-13), (3-15) and (3-16) imply that $A_{(\nabla^{\perp})^{h}\ \eta(p)} = 0$. Then, from (3-14) and Lemma 3-5, we obtain that $\eta_p \in N^{\perp}_{h+1}(p)$ and the proposition follows.

Let $Z_1,\dots,Z_n$ be an orthonormal basis and i_o a fixed index. From Codazzi's equation applied to the normal vector field $\delta = \nabla^{\perp}_{X_{h-1}}\ \dots\ \nabla^{\perp}_{X_1}\ \eta$, we obtain

$$A_{\nabla^{\perp}_{Z_{i_o}}\ \delta}\ Z_j = A_{\nabla^{\perp}_{Z_j}\ \delta}\ Z_{i_o}$$

so that

$$\sum_{j=1}^{n} \langle Z_j,Z_j\rangle\langle A_{\nabla^{\perp}_{Z_{i_o}}\delta} Z_j,Z_j\rangle = \sum_{j=1}^{n} \langle Z_j,Z_j\rangle\langle A_{\nabla^{\perp}_{Z_j}\delta} Z_{i_o},Z_j\rangle .$$

Then

$$\sum_{j=1}^{n} \langle Z_j,Z_j\rangle\langle \alpha(Z_j,Z_j),\nabla^{\perp}_{Z_{i_o}}\delta\rangle = \sum_{j=1}^{n} \langle Z_j,Z_j\rangle\langle A_{\nabla^{\perp}_{Z_j}\delta} Z_j,Z_{i_o}\rangle .$$

Thus

$$n\langle H,\nabla^{\perp}_{Z_{i_o}}\delta\rangle = \langle \sum_{j=1}^{n} \langle Z_j,Z_j\rangle A_{\nabla^{\perp}_{Z_{i_o}}\delta} Z_j,Z_{i_o}\rangle .$$

From (3-16), we obtain that, at p

(3-17) $$\sum_{j=1}^{n} \langle Z_j,Z_j\rangle A_{\nabla^{\perp}_{Z_j}\delta} Z_j = 0$$

If $Y_1,\ldots,Y_n$ is a pseudo-orthonormal basis, put in (3-17): $Z_1 = \frac{Y_1-Y_2}{\sqrt{2}}$, $Z_2 = \frac{Y_1+Y_2}{\sqrt{2}}$ and $Z_j = Y_j$ for $3 \leqslant j \leqslant n$. Then, it follows that

(3-18) $$2 A_{\nabla^{\perp}_{Y_1}\delta} Y_2 + \sum_{j=3}^{n} A_{\nabla^{\perp}_{Z_j}\delta} Z_j = 0 .$$

By Theorem 0.4 and Proposition 0.5 of [3], we need to consider four cases.

CASE 1. There exists a pseudo-orthonormal basis $Y_1,\ldots,Y_n$ such that

$$A_{\nabla^{\perp}_{Y_1}\delta} = \begin{pmatrix} \lambda_o^1 & 1 & & & & \\ 0 & \lambda_o^1 & & & & \\ & & \lambda_o^1 & I_{k_o} & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & \lambda_\ell^1 I_{k_\ell} \end{pmatrix} , \quad A_{\nabla^{\perp}_{Y_j}\delta} = \begin{pmatrix} \begin{array}{cccc|} \lambda_o^j & b^j & c_3^j\ldots c_{k_o}^j & \\ 0 & \lambda_o^j & 0\ \ldots 0 & \\ 0 & c_3^j & & \\ \vdots & \vdots & A_j^o & \\ 0 & c_{k_o} & & \\ \hline \end{array} & & \\ & A_j^1 & \\ & & \ddots \ A_j^\ell \end{pmatrix}$$

An easy computation shows that $[A_i^o,A_j^o] = 0$ for $1 \leqslant i,j \leqslant n$. Let us consider the positive definite subspace $V = \mathrm{span}\,\{Y_3,\ldots,Y_{k_o}\}$ and the symmetric linear transformations $\overline{A}_i = V \rightarrow V$ for $i = 3,\ldots,n$, defined by

$$\overline{A_i}(X) = \sum_{h=3}^{k_o} \langle A_{\nabla^{\perp}_{Y_i}\delta} X, Y_h \rangle Y_h .$$

Since $A_i^o = \overline{A}_i$ for $1 \leqslant i \leqslant n$, thus there exists an orthonormal basis $\overline{Y}_3,\ldots,\overline{Y}_{k_o}$ of V which diagonalizes simultaneously the matrices $\overline{A}_i$ for $1 \leqslant i \leqslant n$. The matrix $A_{\nabla^{\perp}_{Y_1}\delta}$ does not change for the new basis $Y_1, Y_2, \overline{Y}_3, \ldots, \overline{Y}_{k_o}, Y_{k_o+1}, \ldots, Y_n$, and for the other matrices, we have (dropping the bar)

$$A_{\nabla^{\perp}_{Y_j}} = \left(\begin{array}{ccccc|ccc} \lambda_o^j & b_o^j & a_3^j & \cdots & a_{k_o}^j & & & \\ 0 & \lambda_o^j & 0 & \cdots & 0 & & & \\ 0 & a_3^j & \gamma_3^j & & & & & \\ \vdots & \vdots & & \ddots & & & & \\ 0 & a_{k_o}^j & & & \gamma_{k_o}^j & & & \\ \hline & & & & & A_j^1 & & \\ & & & & & & \ddots & \\ & & & & & & & A_j^{\ell} \end{array}\right)$$

From (3-15), we obtain

i) for $Z = Y_1$ and $Y = Y_2$

$$\lambda_o^2 = 1 .$$

ii) for $Z = Y_1$ and $Y = Y_j$, $3 \leqslant j \leqslant k_o$,

$$\lambda_o^3 = \ldots = \lambda_o^{k_o} = 0 .$$

iii) for $Z = Y_2$ and $Y = Y_j$, $3 \leqslant j \leqslant k_o$,

$$a_j^j = \gamma_j^2 .$$

From (3-18), we obtain

iv) $\gamma_j^j = 0$, $3 \leqslant j \leqslant k_o$.

v) $Z + \sum_{j=3}^{k_o} a_j^j = 0$.

From (3-16), we have that

$$A_{\nabla^{\perp}_{Y_2}\delta} A_{\nabla^{\perp}_{Y_j}\delta} = A_{\nabla^{\perp}_{Y_j}\delta} A_{\nabla^{\perp}_{Y_2}\delta} , \quad 3 \leqslant j \leqslant n .$$

In particular, comparing the $j^{\underline{th}}$ element of the first line of both matrix products, we obtain

vi) $\lambda^2 a_j^j + a_j^2 \gamma_j^j = \lambda_o^j a_j^2 + \gamma_j^2 a_j^j \quad , \quad 3 \leq j \leq k_o$.

From i), ii), iii), iv) and vi), it follows that

$$a_j^j = (a_j^j)^2 ,$$

which is a contradiction to v). So case 1 is not possible.

CASE 2. There exists a pseudo-orthonormal basis $Y_1,\ldots,Y_n$ such that

$$A_{\nabla^\perp_{Y_1}\delta} = \begin{pmatrix} \lambda_o^1 & 0 & 1 & & & \\ 0 & \lambda^1 & 0 & & & \\ 0 & 1 & \lambda_o^1 & & & \\ & & & \lambda_o^1 I_{k_o} & & \\ & & & & \ddots & \\ & & & & & \lambda_\ell^1 I_{k_\ell} \end{pmatrix} , \quad A_{\nabla^\perp_{Y_j}\delta} = \begin{pmatrix} \begin{array}{ccc|} \mu^j & b^j & c_1^j \ldots c_{k_o+1}^j \\ 0 & \mu^j & 0 \ldots 0 \\ 0 & c_1^j & \\ \vdots & \vdots & A_o^j \\ 0 & c_{k_o+1}^j & \\ \hline \end{array} & & & \\ & A_j^1 & & \\ & & \ddots & \\ & & & A_j^\ell \end{pmatrix}$$

From (3-15), we have

$$A_{\nabla^\perp_{Y_1}\delta} Y_2 = A_{\nabla^\perp_{Y_2}\delta} Y_1 .$$

Thus

$$\lambda_o^1 Y_2 + Y_3 = \mu^2 Y_1 ,$$

which is not possible. So Case 2 can not occur.

CASE 3. There exists an orthonormal basis $Y_1,\ldots,Y_n$ such that

$$A_{\nabla^\perp_{Y_1}\delta} = \begin{pmatrix} \alpha_1 & \beta_1 & & & & \\ -\beta_1 & \alpha_1 & & & & \\ & & \alpha_1 I_{k_o} & & & \\ & & & \lambda_1^1 I_{k_1} & & \\ & & & & \ddots & \\ & & & & & \lambda_\ell^1 I_{k_\ell} \end{pmatrix} , \quad A_{\nabla^\perp_{Y_j}\delta} = \begin{pmatrix} \alpha_j & \beta_j & & & & \\ -\beta_j & \alpha_j & & & & \\ & & A_j^o & & & \\ & & & A_s^1 & & \\ & & & & \ddots & \\ & & & & & A_s^\ell \end{pmatrix}$$

where $\beta_1 \neq 0$.

From (3-15), we obtain

i) for $Z = Y_1$ and $Y = Y_2$

$$\beta_1 = \alpha_2$$

From (3-17), we obtain

ii) $$\beta_1 = -\alpha_2$$

Then $\beta_1 = 0$, which is a contradiction. So case 3 is not possible.

CASE 4. There exists an orthonormal basis $Y_1,\ldots,Y_n$ such that

$$A_{\nabla^{\perp}_{Y_1}\delta} = \begin{pmatrix} \lambda_o I_{k_o} & & & \\ & \lambda_1 I_{k_1} & & \\ & & \ddots & \\ & & & \lambda_\ell I_{k_\ell} \end{pmatrix}$$

It is easy to see that the basis can be chosen in such a way that

$$A_{\nabla^{\perp}_{Y_j}\delta} = \begin{pmatrix} A^j & & & \\ & \gamma^j_{k_o+1} & & \\ & & \ddots & \\ & & & \gamma^j_n \end{pmatrix} \qquad \text{for} \quad 2 \leqslant j \leqslant n$$

where $A^j = (a^j_{k_\ell})$ is a $k_o \times k_o$ matrix.

From (3-15), it follows that

$$\gamma^s_t = 0 \quad \text{if} \quad s \neq t \quad \text{and} \quad k_o+1 \leqslant s,t \leqslant n \ .$$

Then (3-17) implies that

$$\gamma^s_s = 0 \quad \text{for} \quad k_o+1 \leqslant s \leqslant n \ .$$

Using (3-15) for $Z = Y_j$, $1 \leqslant j \leqslant k_o$, and $Y = Y_t$, $k_o+1 \leqslant t \leqslant n$, we obtain

$$A_{\nabla^{\perp}_{Y_t}\delta} = 0 \quad \text{and} \quad \gamma^j_t = 0 \ .$$

Then, from (3-17), it follows that

$$\sum_{j=1}^{k_o} \langle Y_j,Y_j\rangle\langle A_{\nabla^{\perp}_{Y_j}\delta}\, Y_j,Y_1\rangle = 0 \ .$$

Thus

$$\sum_{j=1}^{k_o} \langle Y_j,Y_j\rangle\langle A_{\nabla^{\perp}_{Y_1}\delta} Y_j,Y_j\rangle = 0 .$$

So we obtain $$-\lambda_o + (k_o-1)\,\lambda_o = 0 .$$

Therefore, $\lambda_o = 0$ or $k_o = 2$.

If $\lambda_o = 0$, from (3-15) for $Z = Y_1$ and $Y = Y_k$, $1 \leq k \leq k_o$, we obtain

$$a^k_{11} = \dots = a^k_{1n} = 0 .$$

But then, all the matrices can be simultaneously diagonalized and the same argument as in the beginning of this case shows that they must vanish.

If $k_o = 2$, we have

$$A_{\nabla^{\perp}_{Y_1}\delta} = \begin{pmatrix}\lambda_o & 0\\ 0 & \lambda_o\end{pmatrix} , \qquad A_{\nabla^{\perp}_{Y_2}\delta} = \begin{pmatrix}a_{11} & a_{12}\\ -a_{12} & a_{22}\end{pmatrix}$$

From (3-15), we obtain

$$\lambda_o = -a_{12} , \quad a_{11} = 0 .$$

From (3-17), we obtain

$$\lambda_o = a_{12} , \quad a_{22} = 0 .$$

Thus

$$A_{\nabla^{\perp}_{Y_1}\delta} = 0 = A_{\nabla^{\perp}_{Y_2}\delta}$$

which concludes the proof.

PROOF OF THEOREM 1-1. From Proposition 3-2, we have

$$N^{\perp}_{m+1}(p) = \{\xi \in N^{\perp}_m(p) : ((\nabla^{\perp})^m R^{\perp})(\xi) = 0 \text{ and } (\nabla^{\perp})^m H(p) \perp \xi\} .$$

Thus

$$N^{\perp}_m(p) = N^{\perp}_{m+1}(p) \quad \text{if and only if} \quad \begin{cases} (\nabla^{\perp})^m R^{\perp}\big|_{N^{\perp}_m} = 0 \\ (\nabla^{\perp})^m H \subset N_m . \end{cases}$$

It follows that N_m is a parallel normal subbundle and we can apply Proposition 3.2 to complete the proof.

REMARK. In fact, we proved that

$$N_m \text{ is parallel if and only if } \begin{cases} (\nabla^\perp)^m R^\perp |_{N_m^\perp} = 0 \\ (\nabla^\perp)^m H \subset N_m . \end{cases}$$

REMARK. The Theorem 1-1 remains valid if the immersion is m-regular in an open connected and dense subset of M_s^n.

REFERENCES

[1] DAJCZER, M.: *Reduction of the Codimension of Regular Isometric Immersions*. Math. Z. 179 (1982), 263-286.

[2] ERBACHER, J.: *Reduction of Codimension of Isometric Immersions*. J. Diff. Geometry (1971), 333-340.

[3] MAGID, M.: *Isometric Immersions of Lorentz space with parallel second fundamental forms*. Tsukaba, J.Math.8 (1984), 31-54.

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Recibido en diciembre de 1980.
Versión final diciembre de 1984.