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FINITE TETRAVALENT MODAL ALGEBRAS

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ABSTRACT. We prove that a finite tetravalent modal algebra is determined, up to an isomorphism, by its determinant system, applying the results of [4].

INTRODUCTION.

It is well known that a finite distributive lattice A is determined, up to an isomorphism, by the ordered set π of all its prime elements [1]. Similarly, a finite De Morgan algebra A is determined by its determinant system [5,6,8]. The aim of this paper is characterize the determinant system of a finite tetravalent modal algebra A and obtain from it the structure of A.

Recalling from [3,4] we have:

1. DEFINITION. A *tetravalent modal algebra* $\langle A;\wedge,\vee,\sim,\nabla,1\rangle$ or, simply A, is an algebra of type (2,2,1,1,0) satisfying the following axioms:

$A_1)$ $x\wedge(x\vee y) = x$, $A_2)$ $x\wedge(y\vee z) = (z\wedge x)\vee(y\wedge x)$

$A_3)$ $\sim\sim x = x$, $A_4)$ $\sim(x\wedge y) = \sim x\vee\sim y$

$A_5)$ $\sim x\vee\nabla x = 1$, $A_6)$ $x\wedge\sim x = \sim x\wedge\nabla x$.

Let A be a finite tetravalent modal algebra and $\langle\pi,\Phi\rangle$ its prime spectrum [4]. In this case, it is well known that a prime filter P of A is a principal filter $P = F(p)$ where p is a prime element of A [2]. Therefore we shall identify the set π with the family of all prime elements of A. We can also identify the Birula-Rasiowa transformation associated with A [4], Φ, with a map ϕ from the set π of all prime elements of A, into itself. If $p \in \pi$, $\phi(p)$ is the generator of the principal prime filter $\Phi(F(p)) = F(q)$, i.e., $\phi(p) = q \in \pi$. Thus ϕ has the following properties:

1) $\phi(\phi(p)) = p$ for each $p \in \pi$.

2) If $p_1,p_2 \in \pi$ and $p_1 \leqslant p_2$ then $\phi(p_2) \leqslant \phi(p_1)$.

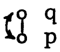
2. DEFINITION. The couple $<\pi,\phi>$ is the *determinant system* of the finite tetravalent modal algebra A.

An immediate consequence of theorem 3.8 of [4] is the following result, which gives us the characterization of the determinant system of a finite tetravalent modal algebra:

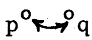
3. THEOREM. *The determinant system* $<\pi,\phi>$ *of a finite tetravalent modal algebra* A, *has* ϕ*-connected components of the three following types:*

Type I: 

with $\phi(p) = p$.

Type II: 

with $\phi(p) = q$ *and* $\phi(q) = p$.

Type III: 

with $\phi(p) = q$ *and* $\phi(q) = p$.

Following the work of A.Monteiro in [5,6,8], let us show that it is possible to recover the operator ∇ from the knowledge of the determinant system of a finite tetravalent modal algebra A.

From [4] we recall the following lemma, that will simplify the proofs of next results:

4. LEMMA [4]. *Let* A *be a tetravalent modal algebra,* $a \in A$. *If* P *is a prime filter in* A, *then* $\nabla a \in P$ *iff* $a \in P$ *or* $a \in \phi(P)$.

We have then:

5. THEOREM. *In a finite tetravalent modal algebra* A *with determinant system* $<\pi,\phi>$, *if* $p \in \pi$, *then* $\nabla p = p \vee \phi(p)$.

Proof. Let us prove that we have (a) $p\vee\phi(p) \leqslant \nabla p$.
From [4] we know that (b) $p \leqslant \nabla p$. Since $p \in \pi$, $P = F(p)$ is a prime filter in A. Let us suppose that (c) $\phi(p) \not\leqslant \nabla p$; it follows then (d) $\nabla p \notin F(\phi(p)) = \Phi(P)$, From (d), by lemma 4, it follows $p \notin \Phi(\Phi(p)) = P$, which is a contradiction. So we get $\phi(p) \leqslant \nabla p$ and we have (a) as wished.

Let us suppose that (e) $p\vee\phi(p) < \nabla p$ holds. It is well known, in lattice theory, that in this condition, there is a prime filter $Q=F(q)$ in A such that:

$$\text{(f)} \quad \nabla p \in Q \quad \text{and} \quad \text{(g)} \quad p\vee\phi(p) \notin Q.$$

From (f) and lemma 4, it follows either (h) $p \in Q$ or (i) $p \in \Phi(Q)$. Since (h) contradicts (g), we have (i), which is equivalent to (j) $P \subseteq \Phi(Q)$. Applying lemma 2.4 of [4] to condition (j), we get either

(ℓ) $P = \Phi(Q)$ or (m) $\Phi(P) = \Phi(Q)$. From (ℓ), we have $p = \phi(q)$, thus $\phi(p) = q$ and so $\phi(p) \in Q$, which contradicts (g). From (m) we get $P = Q$, so $p \in Q$, that also contradicts (g). Therefore we cannot have condition (e); hence, from (a) it follows that $p \vee \phi(p) = \nabla p$.

From the above result, we then have:

6. THEOREM. *Let* A *be a finite tetravalent modal algebra whose determinant system is* $<\pi, \phi>$. *If* $x \in A$, *we have:*

1) *If* $x = 0$, *then* $\nabla x = 0$.

2) *If* $x \neq 0$, *then* $\nabla x = \bigvee_{p \in \pi(x)} (p \vee \phi(p))$, *where* $\pi(x) = \{p \in \pi : p \leqslant x\}$.

Proof. Let $x \in A$. 1) If $x=0$, by definition $0 = {\sim}1$ [3]. Using axiom A_6) we have $0 \wedge 1 = 1 \wedge \nabla 0$, thus $0 = \nabla 0$, so $\nabla x = 0$.

2) Let $x \neq 0$. It is well known that: (a) $x = \bigvee_{p \in \pi(x)} p$ [2].

Since $\nabla(a \vee b) = \nabla a \vee \nabla b$ [3], from (a) it follows: (b) $\nabla x = \bigvee_{p \in \pi(x)} \nabla p$.

From (b) and theorem 5, we finally have:

$$\nabla x = \bigvee_{p \in \pi(x)} (p \vee \phi(p)).$$

Now we can prove the main result of this paper, which justifies the given name of determinant system of a finite tetravalent modal algebra:

7. THEOREM. *Let* $<\pi, \phi>$ *be a couple formed by a finite ordered set* $\pi(\leqslant)$ *and an anti-isomorphism* ϕ *from* π *into* π *which is an involution of* π, *such that its* ϕ*-connected components are of the three types of theorem 3. Then, there is up to an isomorphism, a finite tetravalent modal algebra* A *whose determinant system is* $<\pi, \phi>$.

Proof. In these conditions, from [1,5,6,8] we have at once that there is, up to an isomorphism, a finite De Morgan algebra A whose determinant system is $<\pi, \phi>$. Define an operator ∇ over A:

Let $x \in A$:

∇_1) If $x=0$, let $\nabla 0 = 0$,

∇_2) If $x \neq 0$, let $\nabla x = \bigvee_{p \in \pi(x)} (p \vee \phi(p))$, where $\pi(x) = \{p \in \pi : p \leqslant x\}$.

These formulas make sense, because $\pi(x)$ is a finite set. From the definition of the operator ∇, we get at once (1) $x \leqslant \nabla x$.

We must prove that this operator ∇ satisfies the two axioms A_5) and A_6) from the definition of a tetravalent modal algebra.

a) Axiom A_5) $\sim x \vee \nabla x = 1$ is verified:

Let us suppose that we had (2) $\sim x \vee \nabla x \neq 1$. By [7], from (2) it follows that there is a prime filter P of A, such that (3) $\sim x \vee \nabla x \notin P$. From (3) we get: (4) $\sim x \notin P$; (5) $\nabla x \notin P$. Condition (4) is equivalent to $x \notin \sim P$, which is equivalent to (6) $x \in \Phi(P)$. But, applying lemma 4 to condition (5), we obtain $x \notin P$ and $x \notin \Phi(P)$ which contradicts (6). Thus, condition (2) cannot hold and so axiom A_5) is fulfilled.

b) Axiom A_6) $x \wedge \sim x = \sim x \wedge \nabla x$ is verified:

From (1) it follows at once (I) $x \wedge \sim x \leqslant \sim x \wedge \nabla x$. Let us suppose that we had (7) $\sim x \wedge \nabla x \nleqslant x \wedge \sim x$. Then it should be a prime filter P of A such that (8) $\sim x \wedge \nabla x \in P$ and (9) $x \wedge \sim x \notin P$. From (8) it follows (10) $\sim x \in P$ and (11) $\nabla x \in P$. Applying lemma 4 to (11) we get either (12) $x \in P$ or (13) $x \in \Phi(P)$. Conditions (10) and (12) imply $x \wedge \sim x \in P$, which is against (9), so (12) cannot hold and we have (13). But this one is equivalent to $x \notin \sim P$ which is equivalent to $\sim x \notin P$, which contradicts (10). Therefore we cannot have (7) and we get (II) $\sim x \wedge \nabla x \leqslant x \wedge \sim x$.

From (I) and (II) it follows that axiom A_6) $x \wedge \sim x = \sim x \wedge \nabla x$ is verified.

Therefore the operator ∇ gives to A the required structure of tetravalent modal algebra.

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