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THE UNIQUENESS OF THE COVARIANT DERIVATIVE

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1. INTRODUCTION.

It is very well known that with the components u_i of a covector, its partial derivatives $u_{i,j}$ and the components of a linear connection Γ^i_{jk} we can form a 2-covariant tensor, the covariant derivative of the covector relative to the connection, whose components are:

$$u_{i|j} = u_{i,j} - \Gamma^s_{ij}\, u_s \tag{1.1}$$

It is also known (for instance, see [4], pp.308-312) that the assumption of the product rule and (1.1) define univocally the covariant derivative of any tensor of any type. In the classical tensor analysis, the covariant derivative is motivated by the requirement that it must be linear in u_i and $u_{i,j}$, and the transformation rule for the connection is derived from the assumption that the covariant derivative is a tensor of type (1.1).

In this paper we prove a sort of a reciprocal. We show that, assuming linearity in the partial derivatives only and up to the order of the indices, the covariant derivative is the only 2-covariant tensor concomitant of a covector, its first partial derivatives and a symmetric connection. We do this essentially by working out the invariance identities [3] that tensorial concomitants must satisfy.

2. CONCOMITANTS OF A COVECTOR.

2. a) SCALARS

Let L be a scalar concomitant of a covector, i.e., $L = L(u_i)$. Then for any change of coordinates

$$x^i = x^i(\bar{x}^j) \tag{2.1}$$

it must be:

$$L(B^i_p\, u_i) = L(u_p) \tag{2.2}$$

where $B^i_p = \partial x^i/\partial \bar{x}^p$. Differentiating (2.2) with respect to B^a_b and evaluating at $B^a_b = \delta^a_b$ we have $L^{;b}\, u_a = 0$, where $L^{;b} = \partial L/\partial u_b$, and since it must be satisfied for every covector, we deduce $L^{;b} = 0$. But then:

$$L^{,h} = \frac{\partial L}{\partial x^h} = \frac{\partial L}{\partial u_b} \cdot \frac{\partial u_b}{\partial x^h} = L^{;b} . u_{b,h} = 0$$

and so L is a constant.

2. b) TENSORS OF TYPE (1.1)

Let L^h_k be a concomitant of a covector, i.e., $L^h_k = L^h_k(u_i)$. Then for the change (2.1) it must be:

$$L^h_k(B^p_s\, u_p) = B^i_k\, A^h_j\, L^j_i\, (u_s) \tag{2.3}$$

where A^h_j is the inverse matrix of B^h_j, i.e., $B^h_j\, A^j_p = \delta^h_p$. For the change of coordinates given by $\bar{x}^i = \lambda x^i$ $(\lambda \neq 0)$ we have from (2.3):

$$L^h_k\, (\lambda\, u_i) = L^h_k\, (u_i) \tag{2.4}$$

Making $\lambda \to 0$ en (2.4) we see that $L^h_k\, (u_s) = L^h_k\, (0)$, and so:

$$L^{h;i}_k = \frac{\partial L^h_k}{\partial u_i} = 0 \tag{2.5}$$

Now we differentiate (2.3) with respect to B^a_b and set $B^a_b = \delta^a_b$ to obtain, from (2.5):

$$0 = \delta^b_i\, L^h_a - \delta^h_a\, L^b_i$$

Contracting b = i, we have:

$$n\, L^h_a = \delta^h_a\, L^b_b = \alpha\delta^h_a\ ,$$

where α is a scalar concomitant of u_i and so it is a constant, i.e., α is a real number. Making $\beta = \alpha/n$ we see that it must be, for any (1,1)-tensor concomitant of a covector:

$$L^h_a = \beta\delta^b_a \tag{2.6}$$

2. c) TENSORS OF TYPE (2.2)

Let L^{hk}_{ij} be a concomitant of a covector u_i, i.e., $L^{hk}_{ij} = L^{hk}_{ij}\, (u_i)$. Then, for the change (2.1), it must be:

$$L^{hk}_{ij}\, (B^p_s\, u_p) = B^p_i\, B^m_j\, A^h_s\, A^k_t\, L^{st}_{pm}\, (u_i) \tag{2.7}$$

For the change of coordinates given by $\bar{x}^i = \lambda x^i$ ($\lambda \neq 0$), we have from (2.7):

$$L^{hk}_{ij}(\lambda u_p) = L^{hk}_{ij}(u_p) \tag{2.8}$$

Making $\lambda \to 0$ in (2.8), we see that $L^{hk}_{ij}(u_p) = L^{hk}_{ij}(0)$, and so:

$$L^{hk;p}_{ij} = \frac{\partial L^{hk}_{ij}}{\partial u_p} = 0 \tag{2.9}$$

Now we differentiate (2.7) with respect to B^a_b and evaluate at $B^a_b = \delta^a_b$ to obtain, from (2.9):

$$0 = \delta^b_i L^{hk}_{aj} + \delta^{bk}_j L^{hk}_{ia} - \delta^h_a L^{bk}_{ij} - \delta^k_a L^{hb}_{ij}$$

Contracting $b = i$ we have:

$$n L^{hk}_{aj} + L^{hk}_{ja} = \delta^h_a L^{ik}_{ij} + \delta^k_a L^{hi}_{ij}$$

Since L^{ik}_{ij} and L^{hi}_{ij} are tensors of type (1,1) concomitants of a covector, they must satisfy (2.6). Then:

$$n L^{hk}_{aj} + L^{hk}_{ja} = \alpha\delta^h_a \delta^k_j + \beta\delta^k_a \delta^h_j , \tag{2.10}$$

α and β being numbers. Changing h and a, we have a similar equation. Multiplying (2.10) by n and substracting y the latter, we obtain

$$(n^2-1) L^{hk}_{aj} = (n\alpha-1) \delta^k_j \delta^h_a + (n\beta-1) \delta^h_j \delta^k_a ;$$

and so, for $n \neq 1$, the concomitant L^{hk}_{aj} must be of the form:

$$L^{hk}_{aj} = \alpha\delta^k_j \delta^h_a + \beta\delta^k_j \delta^k_a , \tag{2.11}$$

with α and β real numbers. From (2.10), the same is true for $n = 1$. Others concomitants of a covector have been studied elsewhere [2], but we will only need (2.11).

3. THE COVARIANT DERIVATIVE.

Let L_{ij} be a 2-covariant tensor concomitant of a covector, its first partial derivatives and a symmetric connection, i.e.,

$$L_{ij} = L_{ij}(u_k; u_{k,h}; \Gamma^i_{kh}) \tag{3.1}$$

If we assume that L_{ij} is linear in $u_{k,h}$, then it must be:

$$L_{ij}^{;hk} = \frac{\partial L_{ij}}{\partial u_{h,k}} = L_{ij}^{;hk}(u_s, \Gamma^i_{st})$$

It is known (see [1], Theorem A.2) that then it is: $L^{;hk} = L_{ij}^{;hk}(u_s)$, and so from (2.11) we see that

$$L_{ij}^{;hk} = \alpha\delta^h_i\,\delta^k_j + \beta\,\delta^h_j\,\delta^k_i$$

Integrating we obtain:

$$L_{ij} = \alpha\, u_{i,j} + \beta\, u_{j,i} + T_{ij}(u_h, \Gamma^h_{k\ell}) \tag{3.2}$$

From the transformation rule for L_{ij} it is easy to obtain:

$$\begin{aligned}(\alpha+\beta)\, B^i_{hk}\, u_i + T_{hk}\,(B^i_s\, u_i,\ A^i_s\, B^s_{jt} + A^i_s\, B^p_j\, B^m_t\, \Gamma^s_{pm}) &= \\ = B^i_h\, B^j_k\, T_{ij}(u_s, \Gamma^i_{st})\ , &\end{aligned} \tag{3.3}$$

where $B^i_{hk} = \partial^2 x^i/\partial\bar{x}^h\,\partial\bar{x}^k$. Differentiating (3.3) with respect to $B_{bc}^{\ a}$ and setting $B^i_j = \delta^i_j$, we have:

$$(\alpha+\beta)\,\frac{1}{2}\,(\delta^b_h\,\delta^c_k + \delta^c_h\,\delta^b_k)\, u_a + T_{hk}{}^{;\ bc}_{\ \ a} = 0\ , \tag{3.4}$$

where $T_{hk}{}^{;\ bc}_{\ \ a} = \partial T_{hk}/\partial\Gamma^a_{bc}$. From (3.4) we see that, if $\{b,c\} \neq \{h,k\}$, then it is $T_{hk}{}^{;\ bc}_{\ \ a} = 0$. Also from (3.4) we have:

$$T_{hk}{}^{;\ hk}_{\ \ a} = T_{hk}{}^{;\ kh}_{\ \ a} = -\frac{1}{2}\,(\alpha+\beta)\, u_a$$

(no summation convention here for h and k). Integrating and taking into account the symmetry of the connection:

$$T_{hk} = -(\alpha+\beta)\,\Gamma^i_{hk}\, u_i + S_{hk}(u_i) \tag{3.5}$$

Replacing (3.5) in (3.3) we obtain the following:

THEOREM. *If* $L_{ij} = L_{ij}(u_i; u_{i,j}; \Gamma^i_{hk})$ *is a concomitant of a covector, its first partial derivatives and a symmetric connection, and if it is linear in the* $u_{i,j}$, *then it must be*

$$L_{ij} = \alpha\, u_{i|j} + \beta\, u_{j|i} + \gamma\, u_i\, u_j\ ,$$

where the vertical bar stands for the covariant derivative relative to the given connection.

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