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A NOTE ABOUT THE CONSISTENCY OF AN INFINITE LINEAR INEQUALITY SYSTEM

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ABSTRACT. The consistency of an infinite linear inequality system is formulated through an optimization problem which, in some particular cases, is a simple nonlinear programming problem.

1. INTRODUCTION.

Let $\{a_t'x \leqslant \beta_t,\ t \in T\}$ be a system, generally infinite, of linear inequalities over $\mathbf{R}^n$ ($a_t \in \mathbf{R}^n$, $\beta_t \in R$). Let us denote by S the set of solutions of this system. If $S \neq \emptyset$, the system is said to be consistent.

A relation $a'x \leqslant \beta$ is a "consequence" of the system $\{a_t'x \leqslant \beta_t,\ t \in T\}$ if it is satisfied for all $x \in S$.

We have proved the following characterization of the consequence relations: "$a'x \leqslant \beta$ is a consequence of the consistent system $\{a_t'x \leqslant \beta_t,\ t \in T\}$ if and only if $\begin{bmatrix} a \\ \beta \end{bmatrix} \in \mathrm{cl}\ K_c$", where $K_c = K\{\begin{bmatrix} a_t \\ \gamma_t \end{bmatrix}, \gamma_t \geqslant \beta_t,\ t \in T\}$ denotes the convex cone generated by such vectors, cl K_c being its closure. Different proofs of the last statement are given in [2] and [3].

We have also obtained, for the homogeneous case, the following characterization: "$a'x \leqslant 0$ is a consequence of the system $\{a_t'x \leqslant 0,\ t \in T\}$ if and only if $a \in \mathrm{cl}K\{a_t,\ t \in T\}$".

We shall consider sets included in some space $\mathbf{R}^p$, $\|x\|$ being the corresponding euclidean norm of x, i.e., $\|x\| = [\sum_{i=1}^{p} (x_i)^2]^{1/2}$.

Given a non empty set $T \subset \mathbf{R}^p$, we shall denote by int T, ri T and bdry T the topological interior of T, the relative interior of T and the boundary set of T, respectively.

2. THE CONSISTENCY AS AN OPTIMIZATION PROBLEM.

LEMMA 1. *The system* $\{a_t'x \leqslant \beta_t,\ t \in T\}$ *is consistent if and only if* $\begin{bmatrix}0_n\\-1\end{bmatrix} \notin \text{cl } K_c$.

Proof. Let us suppose that $\begin{bmatrix}0_n\\-1\end{bmatrix}$ belongs to cl K_c. This means that the relation $0_n'x \leqslant -1$ is a consequence of the given system, if we assume $S \neq \emptyset$. But this constitutes a contradiction.
Let us suppose now $S = \emptyset$. Then, the system $\left\{\begin{matrix}a_t'x+\beta_t x_{n+1} \leqslant 0 \ , \ t \in T\\ x_{n+1} < 0\end{matrix}\right\}$

is not consistent. Therefore $-x_{n+1} \leqslant 0$ is a consequence relation of $\{a_t'x+\beta_t x_{n+1} \leqslant 0,\ t \in T\}$, or equivalently, $\begin{bmatrix}0_n\\-1\end{bmatrix} \in \text{cl}\, K\{\begin{bmatrix}a_t\\ \beta_t\end{bmatrix},\ t \in T\} \subset$ $\subset \text{cl } K_c$.

REMARK. By means of this result, it is possible to give simpler proofs of some properties of inconsistent systems already known, such as a theorem due to Blair [1] and the lemma 1 of Jeroslow and Kortanek [4].

THEOREM 1. *Let* σ *be defined as* $\inf\{x_{n+1};\ \bar{x} = \begin{bmatrix}x\\x_{n+1}\end{bmatrix} \in K_c,\ \|x\| = 1\}$. *Then* $S \neq \emptyset$ *if and only if* $\sigma > -\infty$.

Proof. If $\sigma = -\infty$, then there is a sequence $\bar{x}^r$, $r = 1,2,\ldots$, inclu ded in K_c, such that $\|x^r\| = 1$ and $\lim_r x_{n+1}^r = -\infty$.
We can admit, with no loss of generality, that $x_{n+1}^r < 0$, $r = 1,2,\ldots$
Since $|x_{n+1}^r|^{-1}\, \bar{x}^r$, $r = 1,2,\ldots$ is also contained in K_c and converges to $\begin{bmatrix}0_n\\-1\end{bmatrix}$, we can assert that $\begin{bmatrix}0_n\\-1\end{bmatrix} \in \text{cl } K_c$, i.e., the system is not consistent.
Let us suppose now that $\begin{bmatrix}0_n\\-1\end{bmatrix} \in \text{cl } K_c$. The set ri (cl K_c) is non-vacuous. If the given system is not trivial there is a point $\bar{y} \in$ ri (cl K_c) such that $y \neq 0_n$.
Since $\lambda\bar{y} + (1-\lambda)\begin{bmatrix}0_n\\-1\end{bmatrix} \in K_c$ for all real number λ, $0 < \lambda \leqslant 1$ (lemma of accesibility), we have $\bar{x}^r := \|y\|^{-1}[\bar{y} + (r-1)\begin{bmatrix}0_n\\-1\end{bmatrix}] \in K_c$, $r = 1,2,\ldots$
But $\|x^r\| = 1$ and $x_{n+1} = \|y\|^{-1}(y_{n+1}+1-r)$, $r = 1,2,\ldots$. Hence $\sigma = -\infty$.

REMARK. We can take $K\{\begin{bmatrix}a_t\\ \beta_t\end{bmatrix},\ t \in T\}$ instead of K_c in lemma 1 and Th.1.

EXAMPLE. Let $S := \{x \in \mathbf{R}^2;\ (1+(t_1)^2+(t_2)^2)^{1/2}x_1+t_1x_2 \leqslant t_2,\ t \in \mathbf{R}^2\}$. It can be easily seen that $K\{\begin{bmatrix} a_t \\ \beta_t \end{bmatrix},\ t \in \mathbf{R}^2\} =$
$= \{x \in \mathbf{R}^3;\ -(x_1)^2+(x_2)^2+(x_3)^2 \leqslant 0\}$. Then $\|x\|^{-1}x_3 \geqslant -1$ for all $\bar{x} \in K\{\begin{bmatrix} a_t \\ \beta_t \end{bmatrix},\ t \in \mathbf{R}^2\}$, $x \neq 0_2$. Hence $\sigma \geqslant -1$ and $S \neq \emptyset$.

In some cases the optimization problem can be reduced to a Nonlinear Program (P):

$$\left.\begin{array}{l} \text{Inf. } \beta_t \\ \text{s.t. } a_t = 0_n,\ t \in T \end{array}\right\}$$

Let v be the value of P. As usually, $v = +\infty$ if P has not a feasible point.

LEMMA 2. *If* T *is a closed convex set in* $\mathbf{R}^m$, *with* $\dim T > 0$, *there are a convex function* f *and a family of linear functions* $\{h_i,\ i = 1,\dots,p\}$, $p = m - \dim T$, *such that:*

(1) $T = \{t \in \mathbf{R}^m;\ f(t) \leqslant 0\ ,\ h_i(t) = 0\ ,\ i = 1,2,\dots,p\}$, *and*

(2) $f(t^o) < 0$ *for some* $t^o \in T$.

Proof. We shall distinguish two cases in the proof.

(i) $\dim T = m$. We can suppose, with no loss of generality, that $0_n \in \text{int } T$. Let us denote by q the Minkowsky functional of T. Then, by a well known property of q, we have $T = \{t \in \mathbf{R}^m;\ q(t) \leqslant 1\}$. If we define $f(t)$: $q(t)-1$, we obtain the desired representation of T.

(ii) $\dim T = m-p$, $p > 0$. Choosing a point $t^1 \in \text{ri } T$ and denoting by L_1 the linear subspace of $\mathbf{R}^m$ generated by $T-t^1$, we can write $\mathbf{R}^m = L_1 \oplus L_2$. By (i), there is a convex function $g: L_1 \longrightarrow \mathbf{R}$ such that $T-t^1 = \{t \in L_1;\ g(t) \leqslant 0\}$ and $g(0_{m-p}) < 0$. If we define $f: \mathbf{R}^m \longrightarrow \mathbf{R}$ such that $f(t) = g(t_1^\pi)$, where t_1^π is the projection of t on L_1, we can easily obtain the desired representation.

THEOREM 2. *Let* $\{a_t'x \leqslant \beta_t,\ t \in T\}$ *be a system such that:*

(i) T *is a compact convex set in* $\mathbf{R}^m$.

(ii) β_t *is convex and continuous on* T.

(iii) a_t *is linear*.

Then, the system is consistent if and only if $v \geqslant 0$.

Proof. First we shall prove that $v < 0$ implies $S = \emptyset$. Under the hypothesis, there is some $\bar{t} \in T$ such that $a_{\bar{t}} = 0_n$ and $v = \beta_{\bar{t}} < 0$. If there is a point $x^o \in S$, then $a_{\bar{t}}'\, x^o = 0 \leqslant \beta_{\bar{t}} < 0$.

For the converse statement, let us assume $S = \emptyset$ or, equivalently, $\begin{bmatrix} 0_n \\ -1 \end{bmatrix} \in \mathrm{cl}\ K_c$. Then, we can find a sequence $(\bar{x}^k) \subset K_c$ with $\lim_k \bar{x}^k =$ $= \begin{bmatrix} 0_n \\ -1 \end{bmatrix}$, being $\bar{x}^k = \sum_{t \in T} \lambda_t^k \begin{bmatrix} a_t \\ \beta_t \end{bmatrix} + \mu^k \begin{bmatrix} 0_n \\ 1 \end{bmatrix}$, $\lambda^k = (\lambda_t^k)_{t \in T} \in \mathbf{R}_+^{(T)}$ and $\mu^k \geqslant 0$. It follows that $\lim_k \{ \sum_{t \in T} \lambda_t^k \begin{bmatrix} a_t \\ \beta_t \end{bmatrix} + (1+\mu^k) \begin{bmatrix} 0_n \\ 1 \end{bmatrix} \} = 0_{n+1}$. Since $S = \emptyset$, $\beta := \min_{t \in T} \beta_t < 0$. For each ε, $0 < \varepsilon < 1$, there is a k_ε such that, for all $k \geqslant k_\varepsilon$, $(\sum_{t \in T} \lambda_t^k)\beta + 1 \leqslant \sum_{t \in T} \lambda_t^k \beta_t + 1 + \mu^k \leqslant \varepsilon$.

Hence $\sum_{t \in T} \lambda_t^k \geqslant \frac{1-\varepsilon}{|\beta|} > 0$, for all $k \geqslant k_\varepsilon$, and if we define $\tilde{\lambda}_t^k := \frac{\lambda_t^k}{\sum_{t \in T} \lambda_t^k}$ and $\alpha^k := \frac{1+\mu^k}{\sum_{t \in T} \lambda_t^k}$, we have $\lim_k \{ \sum_{t \in T} \tilde{\lambda}_t^k \begin{bmatrix} a_t \\ \beta_t \end{bmatrix} + \alpha^k \begin{bmatrix} 0_n \\ 1 \end{bmatrix} \} =$ $= 0_{n+1}$. Let us define $t_k := \sum_{t \in T} \tilde{\lambda}_t^k\, t \in T$. As a consequence of the hypothesis on the functions, $\sum_{t \in T} \tilde{\lambda}_t^k a_t = a_{t_k}$ and $\sum_{t \in T} \tilde{\lambda}_t^k \beta_t = \beta_{t_k} + \gamma^k$, for a certain $\gamma^k \geqslant 0$.

Taking $\delta^k := \alpha^k + \gamma^k > 0$, we obtain $\lim_k \{ \begin{bmatrix} a_{t_k} \\ \beta_{t_k} \end{bmatrix} + \delta^k \begin{bmatrix} 0_n \\ 1 \end{bmatrix} \} = 0_{n+1}$.

Since $(t_k) \subset T$, let (t_j) be a subsequence converging to $t_o \in T$, and, by continuity, $\lim_j \begin{bmatrix} a_{t_j} \\ \beta_{t_j} \end{bmatrix} = \begin{bmatrix} a_{t_o} \\ \beta_{t_o} \end{bmatrix}$.

Therefore (δ^j) is convergent. Let $\delta^o := \lim_j \delta^j$, $\delta^o \geqslant 0$. It results $\begin{bmatrix} a_{t_o} \\ \beta_{t_o} \end{bmatrix} + \delta^o \begin{bmatrix} 0_n \\ 1 \end{bmatrix} = 0_{n+1}$, i.e., $a_{t_o} = 0_n$ (t_o is a feasible point of P) and $\beta_{t_o} = -\delta^o$.

If δ^o is greater than zero, then $v \leqslant \beta_{t_o} < 0$. If $\delta^o = 0$, then $v \leqslant 0$. We have to consider just the case $\delta^o = 0$, $v = 0$. In this case, for $t \in T$, $a_t = 0_n$ implies $\beta_t \geqslant 0$. By lemma 2, the feasible set of problem P can be represented as follows: $\{t \in \mathbf{R}^m\colon f(t) \leqslant 0,\ a_t = 0_n,\ h(t) = 0_p\}$, where f is convex, h is linear and there is a feasible point $\hat{t}$ such that $f(\hat{t}) < 0$ (Slater's qualification).

By the well known necessary optimality conditions for the non-differentiable nonlinear programming problem, there are multipliers

$y_o = (\lambda_o, x_o, u_o) \in \mathbf{R}^{1+n+p}$, $\lambda_o \geqslant 0$, such that (t_o, y_o) is a saddle point for the lagrangean function $\Psi(t,y) = \beta_t + \lambda f(t) + x'a_t + u'h(t)$.

The right hand side inequality, together with the complementarity condition, give $0 = \beta_{t_o} \leqslant \beta_t + \lambda_o f(t) + x_o' a_t + u_o' h(t)$, for all $t \in \mathbf{R}^m$. If $t \in T$, it follows $0 \leqslant \beta_t + x_o' a_t$, i.e., $-x_o \in S$. This contradiction completes the proof.

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