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ON AN INEQUALITY IN THE THEORY OF PARABOLIC H^p SPACES

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ABSTRACT. In this paper we prove (see Theorem 3) that if f belongs to a parabolic H^p space, $0 < p \leqslant 2$ and if $p \leqslant q$ and $1/p + 1/q \geqslant 1$, then

$$(*) \qquad \left\{\int |\hat{f}(x)|^q \rho^*(x)^{-\gamma(q/p-q+1)} dx\right\}^{1/q} \leqslant c \, \|f\|_{H^p} ,$$

where c is a constant which depends on p and q. This theorem generalizes a result of Calderón and Torchinsky ([2], Theorem 4.4), where formula (*) is proved under the considerable more restrictive hypothesis: $0 < p \leqslant q/q-1 \leqslant 2$.

Our theorem contains, as a particular case, an n dimensional analogue of a classical theorem of Hardy and Littlewood [5].

1. INTRODUCTION.

In this section we review the basic facts of the theory of parabolic H^p spaces. For details we refer the reader to [2] and [3].

Let $A_t = t^P$ $(0 < t < \infty)$ be a group of linear transformations on R^n with infinitesimal generator P. The group A_t satisfies the differential equation

$$(1.1) \qquad t \frac{dA_t}{dt} = PA_t .$$

With (x,y) we denote the ordinary inner product of two vectors x and y in R^n and with $|x| = (x,x)^{1/2}$ the norm of the vector x. The transpose of A_t, with respect the ordinary inner product, will be denoted by A_t^*.

It is well known that

$$(1.2) \qquad \det A_t = \det A_t^* = t^\gamma \quad , \ \gamma = \text{trace } P.$$

We assume that the infinitesimal generator P of the group satisfies

$$(1.3) \qquad (Px,x) \geqslant (x,x).$$

LEMMA 1. *For every* $x \in R^n$:

$$\text{(i)}\quad |A_t x| \geqslant t|x| \quad \text{if} \quad t \geqslant 1$$
$$\text{(ii)}\quad |A_t x| \leqslant t|x| \quad \text{if} \quad 0 < t \leqslant 1.$$

Proof. We may suppose that $x \neq 0$. If $t > 0$, by (1.1) and (1.3)

$$t \frac{d}{dt} (A_t x, A_t x) = 2(PA_t x, A_t x) \geqslant 2|A_t x|^2 > 0.$$

Hence, if we define the function $\phi(t) = |A_t x|^2$, we have $t\phi'(t) \geqslant 2\phi(t)$. Thus

$$\frac{d}{dt} (\phi(t)/t^2) = \phi'(t)/t^2 - (2/t^3)\phi(t) \geqslant 0.$$

This implies that the function $\phi(t)/t^2$ is monotonically increasing in $t > 0$. Hence, if $t \geqslant 1$, then

$$|A_t x|^2 = \phi(t) \geqslant \phi(1)t^2 = |A_1 x|^2 |t|^2 = |x|^2 |t|^2.$$

This proves (i). The proof of (ii) is completely similar.

For each $x \in R^n$, $x \neq 0$, let $\rho(x)$ be the unique $t > 0$ such that $|A_t^{-1} x| = 1$. We put $\rho(0) = 0$. Then the function $\rho(x)$, $x \in R^n$, is a norm in R^n which satisfies:

$$(1.4)\quad \begin{aligned} &\text{(i)} && \rho(A_t x) = t\rho(x) \quad (t > 0)\ ; \\ &\text{(ii)} && \rho(x) \leqslant 1 \text{ if and only if } |x| \leqslant 1\ ; \\ &\text{(iii)} && |x| \leqslant \rho(x) \quad \text{if} \quad \rho(x) \leqslant 1 \quad \text{or} \quad |x| \leqslant 1\ ; \\ &\text{(iv)} && |x| \geqslant \rho(x) \quad \text{if} \quad \rho(x) \geqslant 1 \quad \text{or} \quad |x| \geqslant 1\ . \end{aligned}$$

For the proof of (i) and (ii) see [2]. Formula (iii) follows from Lemma 1. Indeed, if $0 < t = \rho(x) \leqslant 1$, then

$$1 = |A_t^{-1} x| \geqslant t^{-1}|x|.$$

Thus $\rho(x) = t \geqslant |x|$. The proof of (iv) is completely analogous.

In a similar fashion we define $\rho^*(x)$ with A_t^*, and the following relations hold .

$$(1.4')\quad \begin{aligned} &\text{(i)} && \rho^*(A_t x) = t\rho^*(x)\ . \\ &\text{(ii)} && \rho^*(x) \leqslant 1 \text{ if and only if } |x| \leqslant 1\ . \\ &\text{(iii)} && |x| \leqslant \rho^*(x) \quad \text{if} \quad \rho^*(x) \leqslant 1 \quad \text{or} \quad |x| \leqslant 1\ . \\ &\text{(iv)} && |x| \geqslant \rho^*(x) \quad \text{if} \quad \rho^*(x) \geqslant 1 \quad \text{or} \quad |x| \geqslant 1\ . \end{aligned}$$

Let $d(x,y) = \rho(x-y)$ be the metric associated with ρ. For $x \in R^n$ and $r > 0$, $B_r(x) = \{y : d(x,y) < r\}$ denotes the ball of center x and radius r. The Lebesgue measure of $B_r(x)$ is $|B_r(x)| = \omega_n\, r^\gamma$, where ω_n is the volume of the unit ball in R^n.

The following two lemmas will be useful in the proof of the results below.

LEMMA 2. *If $\lambda < \gamma$ the following integral converges and*

$$(1.5)\qquad \int_{\rho^*(x)<t} \rho^*(x)^{-\lambda}dx = \left(\frac{\omega_n\gamma}{\gamma-\lambda}\right)t^{\gamma-\lambda} .$$

Proof. By a well-known formula (see [7], p.162), we have

$$I = \int_{\rho^*(x)<t} \rho^*(x)^{-\lambda}dx = \int_0^\infty |\{x \in R^n : \rho^*(x) < t,\ \rho^*(x) < s^{-1/\lambda}\}|\ ds .$$

Taking into account that

$$|\{x \in R^n:\ \rho^*(x) < t,\ \rho^*(x) < s^{-1/\lambda}\}| = \begin{cases} \omega_n t^\gamma & \text{if } s \leqslant t^{-\lambda} \\ \omega_n s^{-\gamma/\lambda} & \text{if } s > t^{-\lambda} , \end{cases}$$

we obtain

$$I = \omega_n \int_0^{t^{-\lambda}} t^\gamma ds + \omega_n \int_{t^{-\lambda}}^\infty s^{-\gamma/\lambda}\ ds = \left(\frac{\gamma\omega_n}{\gamma-\lambda}\right) t^{\gamma-\lambda} .$$

LEMMA 3. *If $\lambda > \gamma$ and $t > 0$, the following integral converges and*

$$(1.6)\qquad \int_{\rho^*(x)>t} \rho^*(x)^{-\lambda}\ dx = \left(\frac{\omega_n\gamma}{\lambda-\gamma}\right) t^{\gamma-\lambda} .$$

Proof. In a similar manner as in the proof of Lemma 2, we have

$$I = \int_{\rho^*(x)>t} \rho^*(x)^{-\lambda}dx = \int_0^\infty |\{x \in R^n:\ \rho^*(x) > t,\ \rho^*(x) < s^{-1/\lambda}\}|\ ds .$$

Taking into account that

$$|\{x \in R^n:\ \rho^*(x) > t,\ \rho^*(x) < s^{-1/\lambda}\}| = \begin{cases} \omega_n(s^{-\gamma/\lambda}-t^\gamma) & \text{if } s \leqslant t^{-\lambda} \\ 0 & \text{if } s > t^{-\lambda}. \end{cases}$$

we obtain

$$I = \int_0^{t^{-\lambda}} \omega_n(s^{-\gamma/\lambda}-t^\gamma)ds = \left(\frac{\omega_n\gamma}{\lambda-\gamma}\right) t^{\gamma-\lambda} .$$

Given a function in the class S of rapidly decreasing infinitely differentiable functions of L. Schwartz in R^n, we define $\phi_t(x) = t^{-\gamma}\phi(A_t^{-1}x)$. If $\int\phi(x)dx \neq 0$ and if f is a tempered distribution we define the function

$$F(x,t) = (f*\phi_t)(x) \ , \ x \in R^n \ , \quad t > 0$$

and the maximal function

$$M_\alpha f(x) = \sup_{\rho(y)<\alpha t} |F(x+y,t)| \quad , \ \alpha > 0 \ .$$

We say that $f \in H^p$ $(0 < p < \infty)$ if $M_\alpha f \in L^p$ and we put $\|f\|_{H^p} = \|M_\alpha f\|_p$. For any other choice of ϕ and α we obtain the same space H^p and an equivalent norm $\|.\|_{H^p}$.

An atom is defined as follows: A p-atom $(0 < p \leqslant 1)$ is a measurable function $a(x)$, $x \in R^n$, which is supported in a ball $B_r(x_0)$ and which satisfies:

(i) $|a(x)| \leqslant |B_r(x_0)|^{-1/p}$;

(ii) $\int x^\alpha a(x)dx = 0$, for every multi-index α such that $|\alpha| \leqslant [\gamma(1/p-1)]$.

The following theorem of atomic decomposition, which is an extension of a previous result of A.P.Calderón [1], was obtained by R.H.Latter and Akihito Uchiyama [6] and A.B.E.Gatto [4] .

THEOREM A. *Let* $f \in H^p$ $(0 < p \leqslant 1)$. *Then there exist a sequence* a_i *of* p-*atoms and a sequence* $\lambda_i \geqslant 0$ *such that*

$$f = \sum_{i=1}^{\infty} \lambda_i a_i$$

and

$$\sum_{i=1}^{\infty} \lambda_i^p \leqslant B\|f\|_{H^p}^p \ .$$

Conversely, if $f = \sum_{i=1}^{\infty} \lambda_i a_i$, *where* a_i *is a* p-*atom and* $\{\lambda_i\} \in \ell^p$, *then* $f \in H^p$ *and*

$$A\|f\|_{H^p}^p \leqslant \sum_{i=1}^{\infty} |\lambda_i|^p .$$

The constants A *and* B *depend only on the choice of norm for* H^p.

Throughout this paper we use the letter c to denote a constant which need not be the same in different occurrences.

2. CASE $0 < p \leqslant 1$.

In this section we apply the atomic decomposition theorem, Theorem A, to prove Theorem 1, which will be used in the proof of Theorem 3.

Previously we prove the following two lemmas.

LEMMA 4. *If* a *is a* p-*atom*, $0 < p \leqslant 1$, *supported on the ball* $B_r(0)$, *then*

$$(2.1)\qquad a(x) = r^{-\gamma/p}\, b(A_r^{-1}x) \;,$$

where b *is a* p-*atom supported on the ball* $B_1(0)$, *and the following equality holds.*

$$(2.2)\quad \int |\hat{a}(x)|^q \rho^*(x)^{-\gamma(q/p-q+1)} dx = \int |\hat{b}(x)|^q \rho^*(x)^{-\gamma(q/p-q+1)} dx \;, \quad (q \geqslant p).$$

Proof. Let b be the function

$$b(x) = r^{\gamma/p}\, a(A_r x) \;.$$

Then, as it is easy to see, (2.1) holds, $|b(x)| \leqslant |B_1(0)|^{-1/p}$, Supp $b \subset B_1(0)$ and

$$\int b(x)x^\alpha = r^{\gamma/p}\int a(A_r x)x^\alpha dx = r^{-\gamma(1-1/p)}\int a(z)(A_r^{-1}z)^\alpha\, dx = 0 \quad ,$$

for every multi-index α such that $|\alpha| \leqslant N = [\gamma(1/p-1)]$. Therefore b is a p-atom. From formula (2.1), we have

$$\hat{a}(x) = r^{-\gamma/p} \int b(A_r^{-1}y)e^{-2\pi i(x,y)}\, dy \;.$$

By means the change of variable $y = A_r z$, we obtain

$$\hat{a}(x) = r^{-\gamma(1/p-1)} \int b(z)e^{-2\pi i(x,A_r z)}dz = r^{-\gamma(1/p-1)}\hat{b}(A_r^* x) \;.$$

Hence

$$\int |\hat{a}(x)|^q \rho^*(x)^{-\gamma(q/p-q+1)}dx = r^{-\gamma(q/p-q)} \int \hat{b}(A_r^* x)\rho^*(x)^{-\gamma(q/p-q+1)}\, dx.$$

Making the change of variable $x = A_r^{*-1}z$ in last integral, we obtain formula (2.2). This proves the lemma.

LEMMA 5. *If* a *is a* p-*atom*, $0 < p \leqslant 1$ *and* $p \leqslant q$, *then*

$$\int |\hat{a}(x)|^q \rho^*(x)^{-\gamma(q/p-q+1)}dx \leqslant c \;,$$

where c *is a constant which dependes on* p *and* q.

Proof. We may suppose by translation, that the atom is supported on $B_r(0)$, and by Lemma 4, that a is supported by $B_1(0)$. By formulae (1.4) and (1.4'), we have $B_1(0) = \{x\colon \rho(x) < 1\} = \{x\colon \rho^*(x) < 1\} = \{x\colon |x| < 1\}$.

Firstly, we prove that the following formula holds.

(2.5) $$|\hat{a}(x)| \leqslant c\rho^*(x)^{N+1} \quad , \quad |x| \leqslant 1 ,$$

where $N = [\gamma(1/p-1)]$.

Indeed, by definition of p-atom:

$$\hat{a}(x) = \int a(y)[e^{-2\pi i(x,y)} - \sum_{k=0}^{N} \frac{(-2\pi i(x,y))^k}{k!}] \, dy .$$

Hence

$$|\hat{a}(x)| \leqslant \omega_n^{-1/p} \int_{|y|\leq 1} |e^{-2\pi i(x,y)} - \sum_{k=0}^{N} \frac{(-2\pi i(x,y))^k}{k!}| \, dy .$$

Moreover, it is easy to show by Taylor's formula, that

$$|e^{-2\pi i(x,y)} - \sum_{k=0}^{N} \frac{(-2\pi i(x,y))^k}{k!}| \leqslant c|x|^{N+1}|y|^{N+1} .$$

Therefore, if $|x| \leqslant 1$, by formula (1.4'.iii) we have

$$|\hat{a}(x)| \leqslant c\rho^*(x)^{N+1} \int_{|y|\leq 1} \rho^*(y)^{N+1} \, dy .$$

From this inequality, taking into account that, by Lemma 3, we have

$$\int \rho^*(y)^{N+1} dy = \frac{\omega_n \gamma}{\gamma+N+1} = c ,$$

we obtain formula (2.5) which we whished to show.

Now, we prove (2.4). We have

(2.6) $$\int |\hat{a}(x)|^q \rho^*(x)^{-\gamma(q/p-q+1)} dx = \int_{\rho^*(x)\leq 1} + \int_{\rho^*(x)>1} = I_1 + I_2 .$$

Estimate of I_1: By the formula (2.5) and Lemma 2

(2.7) $$I_1 \leqslant c \int_{\rho^*(x)\leq 1} \rho^*(x)^{-\lambda} dx = \frac{c\omega_n \gamma}{\gamma - \lambda} = c ,$$

where $\lambda = \gamma(q/p-q+1) - (N+1)q$. The last integral converges since $\lambda-\gamma = q(\gamma(1/p-1) - (N+1)) < 0$.

Estimate of I_2: Consider first the case p=1. By Schwarz's inequality we have

$$\int_{\rho^*(x)>1} |\hat{a}(x)|^q \rho^*(x)^{-\gamma} dx \leqslant (\int |\hat{a}(x)|^{2q} dx)^{1/2} (\int_{\rho^*(x)>1} \rho^*(x)^{-2\gamma} dx)^{1/2} .$$

Since $p = 1 \leqslant q$, we have that $2q \geqslant 2$. Let $q' = 2q/(2q-1)$. Then $1 < q' \leqslant 2$. By Hausdorff-Young's theorem, we have

$$\left(\int |\hat{a}(x)|^{2q}dx\right)^{1/2} \leqslant \left(\int |a(x)|^{q'}dx\right)^{q/q'} \leqslant \omega_n^{q(1/q'-1)} = c$$

On the other hand, by Lemma 3, we have

$$\left(\int_{\rho^*(x)>1} \rho^*(x)^{-2\gamma}dx\right)^{1/2} = \omega_n^{1/2} = c .$$

So that, we obtain

$$I_2 = \int |\hat{a}(x)|^q \rho^*(x)^{-\gamma}dx \leqslant c . \tag{2.8}$$

Consider now the case $0 < p < 1$. Obviously

$$I_2 \leqslant \| |\hat{a}|^q \|_\infty \int_{\rho^*(x)>1} \rho^*(x)^{-\gamma(q/p-q+1)} dx .$$

Since $\gamma(q/p-q+1) > \gamma$, from Lemma 3 follows that the last integral converges and that

$$\int_{\rho^*(x)>1} \rho^*(x)^{-\gamma(1/p-q+1)} dx = \frac{\omega_n}{q(1/p-1)}$$

On the other hand

$$\| |\hat{a}|^q \|_\infty \leqslant \|a\|_1^q \leqslant \left(\int_{|x|\leq 1} \omega_n^{-1/p}dx\right)^q = \omega_n^{q(1-1/p)} .$$

Therefore, we obtain

$$I_2 \leqslant \frac{\omega_n^{q(1-1/p)+1}}{q(1/p-1)} = c . \tag{2.8'}$$

Formula (2.4) follows from (2.6), (2.7) and (2.8). This proves the lemma.

THEOREM 1. *If* $f \in H^p$ $(0 < p \leqslant 1)$ *and* $q \geqslant p$, *then the Fourier transform* $\hat{f}$ *is a continuous function and*

$$\left(\int |\hat{f}(x)|^q \rho^*(x)^{-\gamma(q/p-q+1)}dx\right)^{1/q} \leqslant c\|f\|_{H^p} , \tag{2.9}$$

where c *is a constant which depends on* p *and* q *and on the choice of norm for* H^p.

Proof. Let $f \in H^p$ $(0 < p \leqslant 1)$. By the atomic decomposition theorem, Theorem A, there exist a sequence a_i of p-atoms and sequence $\lambda_i \geqslant 0$, such that

$$f = \sum_{i=1}^{\infty} \lambda_i a_i$$

and

(2.10) $$A \|f\|^p_{H^p} \leqslant \sum_{i=1}^{\infty} \lambda_i^p \leqslant B \|f\|^p_{H^p}.$$

Therefore, if $f_k = \sum_{i=1}^{k} \lambda_i a_i$, then $f-f_k \in H^p$ and

$$\|f-f_k\|_{H^p} \leqslant A^{-1/p}\left(\sum_{i=k+1}^{\infty} \lambda_i^p\right)^{1/p} \longrightarrow 0, \quad (\text{as } k \to \infty).$$

By Theorem 4.4 of [2], $\hat{f}$ is a continuous function and

$$|\hat{f}_k(x)-\hat{f}(x)| \leqslant c\rho^*(x)^{\gamma(1/p-1)}\|f-f_k\|_{H^p}.$$

This implies that $\hat{f}_k(x) \to f(x)$, as $k \to \infty$, uniformly on compact subsets of R^n. To prove the theorem it suffices to show that the inequality

(2.11) $$\left\{\int |\hat{f}_k(x)|^q \rho^*(x)^{-\gamma(q/p-q+1)}dx\right\}^{1/q} \leqslant c\ \|f\|_{H^p}$$

holds for every k. Indeed, letting $k \to \infty$ in (2.11) and applying Fatou's lemma we obtain the estimate (2.9).

Consider first the case $0 < p \leqslant q \leqslant 1$. From $|\hat{f}_k(x)| \leqslant \sum_{i=1}^{k} \lambda_i |\hat{a}_i(x)|$, we have

$$\int |\hat{f}_k(x)|^q \rho^*(x)^{-\gamma(q/p-q+1)}dx \leqslant \sum_{i=1}^{k} \lambda_i^q \int |\hat{a}_i(x)|^q \rho^*(x)^{-\gamma(q/p-q+1)}\ dx$$

Hence, by Lemma 5 and (2.10), we obtain

$$\left\{\int |\hat{f}_k(x)|^q \rho^*(x)^{-\gamma(q/p-q+1)}dx\right\}^{1/q} \leqslant c\left(\sum_{i=1}^{k} \lambda_i^q\right)^{1/q} \leqslant c\left(\sum_{i=1}^{\infty} \lambda_i^p\right)^{1/p} \leqslant c\|f\|_{H^p}.$$

Consider now the case $0 < p \leqslant 1 \leqslant q$. By Minkowski's inequality we have

$$\left\{\int |\hat{f}_k(x)|^q \rho^*(x)^{-\gamma(q/p-q+1)}dx\right\}^{1/q} \leqslant \sum_{i=1}^{k} \lambda_i \left\{\int |a_i(x)|^q \rho^*(x)^{-\gamma(q/p-q+1)}dx\right\}^{1/q}.$$

By Lemma 5 and (2.10), we obtain

$$\left\{\int |\hat{f}_k(x)|^q \rho^*(x)^{-\gamma(q/p-q+1)}dx\right\}^{1/q} \leqslant c \sum_{i=1}^{k} \lambda_i \leqslant c\left(\sum_{i=1}^{\infty} \lambda_i^p\right)^{1/p} \leqslant c\|f\|_{H^p}.$$

This proves the desired inequality (2.11), and so the proof of the theorem is complete.

3. CASE $1 < p \leqslant 2$.

In this section we prove the following theorem.

THEOREM 2. *If* $f \in L^p$ $(1 < p \leqslant 2)$ *and if* $p \leqslant q$ *and* $1/p+1/q \geqslant 1$, *then the Fourier transform* $\hat{f}$ *of* f *is a locally integrable function and*

$$(3.1) \qquad \{\int |\hat{f}(x)|^q \rho^*(x)^{-\gamma(q/p-q+1)} dx\}^{1/q} \leqslant c\|f\|_p ,$$

where c *is a constant which depends on* p *and* q.

Proof. The fact that $\hat{f}$ is a locally integrable function is an easy consequence of Hausdorff-Young's theorem.

For the proof we use a method similar to those of [9], page 121, and [7], pages 176-177.

Let r and t be given by the relations

$$(3.2) \qquad r = 1 + p/q \quad , \quad t = (p+q-pq)/p^2 .$$

Then it is easy to verify that r and t are solution of the system of equations

$$(3.3) \qquad \begin{cases} 1/p = (1-t)/r + t \\ 1/q = (1-t)/r' + t/(r'-1) , \end{cases}$$

where $r' = r/(r-1)$. From (3.2) and the hypothesis of theorem it easily follows that $1 < r \leqslant 2$ and $0 \leqslant t < 1$.

Consider the measure spaces $(X,\mu) = (R^n,dx)$ and $(Y,\nu) = (R^n,\rho^*(x)^{-\gamma r'}dx)$, for $S(X,\mu)$ the space of the simple functions on (X,μ) and $M(Y,\nu)$ the space of the measurable functions on (Y,ν), let $T\colon S(X,\mu) \to M(Y,\nu)$ be the operator defined by

$$(3.4) \qquad (Tf)(x) = \rho^*(x)^{\gamma}\hat{f}(x) .$$

We shall see that T is of strong type (r,r') and weak type $(1,r'-1)$. Indeed, by the Hausdorff-Young's theorem, we have

$$(\int_Y |Tf|^{r'} d\nu)^{1/r'} = \{\int |\hat{f}(x)|^{r'} dx\}^{1/r'} \leqslant \{\int |f(x)|^r dx\}^{1/r} = (\int_X |f|^r d\mu)^{1/r}.$$

This proves the strong type (r,r') of T. We now prove that T is a weak type $(1,r'-1)$. In fact, for every $\lambda > 0$, we have

$$\nu(\{|(Tf)(x)| > \lambda\}) = \nu(\{|\hat{f}(x)|\rho^*(x)^\gamma > \lambda\}) \leqslant \nu(\{\rho^*(x)^\gamma > \lambda/\|f\|_1)$$

$$= \nu(\{\rho^*(x) > \lambda'\}) ,$$

where $\lambda' = (\lambda/\|f\|_1)^{1/\gamma}$. So that, from Lemma 3

$$\nu(\{(Tf)(x) > \lambda\}) \leqslant \int_{\rho^*(x)>\lambda'} \rho^*(x)^{-\gamma r'} dx = \frac{\omega_n}{r'-1} (\|f_1\|/\lambda)^{r'-1} .$$

By the relations (3.3), we may use the Marcinkiewicz's interpolation theorem to conclude that T is of strong type (p,q), i.e.

$$\{\int_Y |Tf|^q \, d\nu\}^{1/q} \leqslant c \|f\|_p ,$$

for every simple function f. Therefore, taking into account (3.4), we obtain

$$\{\int |\hat{f}(x)|^q \rho^*(x)^{\gamma q} \rho^*(x)^{-\gamma(1+q/p)} dx\}^{1/q} \leqslant c \|f\|_p .$$

Thus, we have proved formula (3.1) in the case of simple-functions. A simple argument using the density of the simple functions in L^p enables us to extend formula (3.1) to every function of L^p.

COROLLARY. *If* $f \in L^p$ $(1 < p \leqslant 2)$, *then*

$$\{\int |\hat{f}(x)|^p \rho^*(x)^{\gamma(p-2)} dx\}^{1/p} \leqslant c \|f\|_p, \tag{3.6}$$

where c *is a constant which depends on* p.

Proof. This corollary follows immediately from Theorem 2, taking p=q.

REMARK. In the particular case in which the infinitesimal generator P of the group coincides with the identity, we obtain from this Corollary a well-known theorem: If $f \in L^p$ $(1 < p \leqslant 2)$, then

$$\{\int |\hat{f}(x)|^p |x|^{n(p-2)} dx\}^{1/p} \leqslant c \|f\|_p . \tag{3.7}$$

See, for example, [7], page 175.

4. GENERAL CASE.

As consequence of Theorems 1 and 2, we obtain the following result.

THEOREM 3. *If* $f \in H^p$ $(0 < p \leqslant 2)$ *and if* $p \leqslant q$, $1/p+1/q \geqslant 1$, *then the Fourier transform* $\hat{f}$ *of* f *is a locally integrable function and*

$$(4.1) \qquad \{\int |\hat{f}(x)|^q \rho^*(x)^{-\gamma(q/p-q+1)} dx\}^{1/q} \leqslant c\|f\|_{H^p} ,$$

where c *is a constant which depends on* p *and* q *and on the choice of norm for* H^p.

Proof. If $0 < p \leqslant 1$, the theorem immediately follows from Theorem 1. If $1 < p \leqslant 2$, then $H^p = L^p$ and the norms $\|f\|_p$ and $\|f\|_{H^p}$ are equivalent ([2], Corollary 1.8 and Theorem 4.3); therefore the theorem, in this case, is an easy consequence of Theorem 2.

The following corollary is obtained taking $q=p$ in Theorem 3.

COROLLARY. *If* $f \in H^p$ $(0 < p \leqslant 2)$ *then the Fourier transform* $\hat{f}$ *of* f *is a locally integrable function and*

$$(4.2) \qquad \{\int |\hat{f}(x)|^p \rho^*(x)^{\gamma(p-2)} dx\}^{1/p} \leqslant c \|f\|_{H^p} ,$$

where c *is a constant which depends on* p *and* q *and on the choice of norm for* H^p.

REMARK. In the particular case in which the infinitesimal generator P of the group A_t coincides with the identity, formula (4.2) reads:

$$(4.3) \qquad \{\int |\hat{f}(x)|^p |x|^{n(p-2)} dx\}^{1/p} \leqslant c\|f\|_{H^p} , \quad 0 < p \leqslant 2 .$$

This last result, which was obtained by Hwai-Chiuan Wang [8] , is an R^n analogue of the following classical theorem of Hardy and Littlewood [5]: If a function analytic in the unit disk $|z| < 1$, belongs to H^p $(0 < p \leqslant 2)$ and if

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

is its Taylor's expansion, then

$$\{\sum_{k=0}^{\infty} (k+1)^{p-2} |a_k|^p\}^{1/p} \leqslant c \|f\|_p ,$$

where c is a constant which depends on p.

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