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# FROM MY CHEST OF EXAMPLES OF FOURIER TRANSFORMS

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## INTRODUCTION.

The Riemann-Lebesgue theorem guarantees that the Fourier transform $\hat{f}$ of a function $f \in L^1(\mathbf{R})$ is continuous and tends to 0 at infinity. Indeed, the Fourier transform is a bounded linear mapping with dense range from $L^1(\mathbf{R})$ into the space $C_\infty(\mathbf{R})$ of all complex-valued continuous functions defined on $\mathbf{R}$, such that $g(t) \to 0$ ($|t| \to \infty$). However, it is not so immediate to check that the range of this transformation is not all of $C_\infty(\mathbf{R})$. More is actually true: not every compactly supported continuous function is the Fourier transform of some $L^1$-function.

This can be shown, either by some general Baire category argument, or by exhibiting a concrete example. Perhaps, the following "butterfly" (look at the graph of B!) is the simplest example that one can imagine. Define

$$B(s) = \begin{cases} (1/k) \sin 4^k s, & \text{if } \pi/2^k \leqslant |s| \leqslant \pi/2^{k-1} \\ 0, & \text{if } s = 0, \text{ or } |s| > \pi. \end{cases}$$

Clearly, $B \in C_o(\mathbf{R})$ (= the space of all compactly supported complex-valued functions defined on $\mathbf{R}$), and a simple computation shows that, for $4^k + 2^{k-1} - 2^{k-2} \leqslant s \leqslant 4^k + 2^{k-1} + 2^{k-2}$, we have

$$b(t) = \check{B}(t) = \frac{1}{2\pi} \int_{\mathbf{R}} B(s)\, e^{ist}\, ds =$$

$$= \frac{1}{2\pi} \{\int_{[-\pi/2^{k-1},-\pi/2^k] \cup [\pi/2^k,\pi/2^{k-1}]} B(s) e^{ist}\, ds + \text{"leftover"}\} \geqslant$$

$$\geqslant \frac{1}{2k\pi} \left\{ \frac{\sin 4^k(t-4^k)}{t - 4^k} - \frac{1}{t + 4^k} - k \sum_{n \neq k} \left(\frac{1}{t + 4^n} + \frac{1}{|t - 4^n|}\right)\right\} >$$

$$> \sqrt{2}/(2^k.3k\pi) - \frac{1}{2\pi} \left\{\frac{1}{4^k} + \frac{3k(k-1)}{4^k} + 3k \sum_{n=k+1}^{\infty} \frac{1}{4^k}\right\} > (1/3k\pi)2^{-k},$$

for all $k \geq k_o$ $(>4)$.

It readily follows that

$$\int_{\mathbf{R}} |b(t)|\ dt \geq \sum_{k=k_o}^{\infty} \int_{|t-(4^k+2^{k+1})| \leq 2^{k-2}} b(t)\ dt \geq$$

$$\geq \sum_{k=k_o}^{\infty} 2.2^{k-2}\ (1/3k\pi)2^{-k} = \frac{1}{6\pi} \sum_{k=k_o}^{\infty} \frac{1}{k} = \infty.$$

Hence, B is not the Fourier transform of any $L^1$-function.

However, since $C_o(\mathbf{R}) \subset L^2(\mathbf{R})$, it is immediate (from Plancherel's theorem) that b and $B = \hat{b}$ belong to $L^2(\mathbf{R})$; furthermore, it is not difficult to check that b is the restriction to the real axis of an *entire* function, and that $b(t) \to 0$ $(|t| \to \infty)$.

On the other hand, B is a real-valued function whose image is equal to $B(\mathbf{R}) = [-1,1]$. If

$$k(t) = \frac{1}{2\pi}\left(\frac{\sin\ t/2}{t/2}\right)^2 ,$$

then $\hat{k}(s) = \max\{1 - |s|,0\}$ , and therefore the function $2\hat{k}(s/2) - 3\hat{k}(s) \in C_o(\mathbf{R})$ is the Fourier transform of the $L^1$-function $k(2t) - 3k(t)$, and has exactly the same image as B.

These examples suggest two questions:

(I) According to a famous theorem of Hahn and Mazurkiewicz, a non-empty subset X of the complex plane $\mathbf{C}$ is the continuous image of a closed interval if and only if X is compact, connected and locally connected [3,IV.9]. Thus, if $f \in L^1(\mathbf{R})$, then $(\text{Image } (\hat{f}))^- = \hat{f}(\mathbf{R}) \cup \{0\}$ necessarily satisfies the above conditions, and contains the origin. Does there exist a compact, connected, locally connected subset X of $\mathbf{C}$, containing the origin, such that X is not closure of the image of a Fourier transform of an $L^1$-function?. (That is, $X \neq (\text{Image } (\hat{f}))^-$ for all f in $L^1(\mathbf{R})$).

(II) Does there exist a function f, defined on $\mathbf{R}$, such that both f and its Fourier transform $\hat{f}$ are restrictions to the real axis of entire functions, $f, \hat{f} \in L^2(\mathbf{R}) \setminus L^1(\mathbf{R})$, and both f and $\hat{f}$ tend to 0 at the infinity?.

The author does not know the answer to the first question, but several examples point in the affirmative direction. On the other hand, the answer to the second question is YES, and $f(t) = (1/t)\sin t^2$ is a concrete example of such a function.

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## IMAGES OF FOURIER TRANSFORMS OF $L^1$-FUNCTIONS.

PROPOSITION 1. *There exists* $g \in L^1(\mathbf{R})$ *such that* $\hat{g} \in C_o(\mathbf{R})$ *and* Image $(\hat{g})$ *is the closed square* S *of vertices* 0, 1, 1+i *and* i; *that is,* $\hat{g}$ *is a "Peano curve" mapping a closed interval continuously onto a square.*

Let $A(\mathbf{T})$ be the class of all continuous periodic functions (period $2\pi$) with absolutely convergent Fourier series; that is, $G \in A(\mathbf{T})$ if and only if $G(s) = \sum_{-\infty}^{\infty} g_n e^{ins}$ $(s \in \mathbf{R})$, where $\|G\|_{A(\mathbf{T})} = \sum_{-\infty}^{\infty} |g_n| < \infty$.

LEMMA 2. ([5,Lemma $6_7$]). *Let* $F \in C_o(\mathbf{R})$ *be a function such that* $F(s) = 0$ *for* $|s| > \pi-\varepsilon$ *(for some* $\varepsilon$, $0 < \varepsilon < \pi$*), and let*

$$f(t) = \frac{1}{2\pi}\int_{\mathbf{R}} F(s)\, e^{ist}\, ds .$$

*Then the three following statements are equivalent:*

(1) $\sum_{-\infty}^{\infty} |f(n)| < \infty$ ;

(2) $\sum_{-\infty}^{\infty} \max\{|f(t)|: n \leqslant t \leqslant n+1\} < \infty$ ;

(3) $f \in L^1(\mathbf{R})$.

A well-known result of S.Bernstein [2,Theorem 6.3], [6,Vol.I,p.240], says that $A(\mathbf{T})$ contains every Lipschitz function of order $\alpha$, for each $\alpha > 1/2$. Thus, if $F \in C_o(\mathbf{R})$ vanishes outside $(-\pi+\varepsilon,\pi-\varepsilon)$ (for some $\varepsilon$, $0 < \varepsilon < \pi$) and $F \in \mathrm{Lip}_\alpha$ for some $\alpha > 1/2$, then Lemma 2 guarantees that $F = \hat{f}$ for some $f \in L^1(\mathbf{R})$. Unfortunately, this observations does not help too much in order to prove Proposition 1, because a Peano curve mapping a closed interval continuously onto a square can be chosen to be $\mathrm{Lip}_{1/2}$, but not $\mathrm{Lip}_\alpha$ for any $\alpha > 1/2$ [1]. However, we can still use an important part of the proof of Lemma 2: if $g \in L^1(\mathbf{R})$ and $F \in A(\mathbf{T})$, then the product $F(s)\hat{g}(s)$ is the Fourier transform of an $L^1$-function. Indeed, if $F(s) = \sum_{-\infty}^{\infty} f_n e^{ins}$,

$\sum_{-\infty}^{\infty} |f_n| < \infty$, then $F(s)\hat{g}(s) = \sum_{-\infty}^{\infty} f_n e^{ins} \hat{g}(s)$ is the Fourier transform of the function

$$\sum_{-\infty}^{\infty} f_n\, g(t-n) \in L^1(\mathbf{R}) \ .$$

Let $J \in A(\mathbf{T})$ be any function such that $0 \leqslant J(s) \leqslant 1$ for all $s \in \mathbf{R}$, $J(s+\pi) = J(s)$ , and

$$J(s) = \begin{cases} 0, & \text{if } 0 \leqslant s \leqslant 3\pi/8 \ , \\ 1, & \text{if } \pi/2 \leqslant s \leqslant 7\pi/8 \ . \end{cases}$$

(For instance, we can define J so that its graph is continuous, equal to a linear segment between $3\pi/8$ and $\pi/2$, and equal to another linear segment between $7\pi/8$ and $\pi$).

Clearly, if $J(s) = \sum_{-\infty}^{\infty} j_n e^{ins}$ , then

$$G(s) = \sum_{k=1}^{\infty} 2^{-k} J(8^k s) = \sum_{-\infty}^{\infty} (\sum \{2^{-k} j_m : 8^k m = n\})\, e^{ins}$$

and

$$G(2s) = \sum_{k=1}^{\infty} 2^{-k} J(2.8^k s) = \sum_{-\infty}^{\infty} (\sum \{2^{-k} j_m : 2.8^k m = n\})\, e^{ins}$$

are elements of $A(\mathbf{T})$.

A minor modification of the proof given in [4] indicates that

$$F(s) = G(s) + iG(2s) \in A(\mathbf{T})$$

maps $[0,\pi/4]$ continuously onto the square S.

Let $g(t) = 4k(2t) - k(t)$. Clearly, $g \in L^1(\mathbf{R})$, and $\hat{g}(s) = 2\hat{k}(s/2) - \hat{k}(s)$ satisfies $0 \leqslant \hat{g}(s) \leqslant 1$ for all $s \in \mathbf{R}$, and

$$\hat{g}(s) = \begin{cases} 1 \ , & \text{if } |s| \leqslant 1 \ , \\ 0 \ , & \text{if } |s| \geqslant 2 \ . \end{cases}$$

It readily follows from our previous observations that

$$\hat{p}(s) = F(s)\ \hat{g}(s)$$

is the Fourier transform of a function $p \in L^1(\mathbf{R})$, $\hat{p} \in C_o(\mathbf{R})$, and

$$\text{Image } (\hat{p}) = S$$

because $\hat{p}([-\pi/4,\pi/4]) = S$, $\hat{p}(s) = 0$ for all s such that $|s| \geqslant 2$, and $\hat{p}(s) \in S$ for $\pi/4 < |s| < 2$.

The proof of Proposition 1 is now complete.

What else can be obtained as the image of a Fourier transform of an $L^1$-function?.

(a) Clearly, if $\phi$ is any function analytic on a neighborhood of the real interval $[-1,1]$, such that $\phi(0) = 0$, then the Riesz-Dunford functional calculus (in the Banach algebra $L^1(\mathbb{R})$) indicates that

$$\phi[-1,1]) = \text{Image } (\phi \circ \hat{g}) \qquad (g(t) = 4k(2t) - k(t))$$

can be attained as the image of the Fourier transform of an $L^1$-function.

(b) Instead of $F(s) = G(s) + iG(2s)$ (as in the proof of Proposition 1), we can consider functions of the form $\alpha G(s) + \beta G(2s)$, $\alpha, \beta \in \mathbb{C}$, and deduce that every "solid" parallelogram with a vertex at the origin is also attainable.

(c) If $\psi(z) = [-iz/(z-2i)]^4$, then $\psi$ is analytic on a neighborhood of the square S, $\psi(0) = 0$, and $\psi(S)$ is the closed unit disk D. Hence,

$$D = \psi(S) = \text{Image } (\psi \circ [F.\hat{g}]) ,$$

and therefore there exists $m \in L^1(\mathbb{R})$ such that Image $(\hat{m}) = D$ $(\hat{m} = \psi \circ (F.\hat{g}))$.

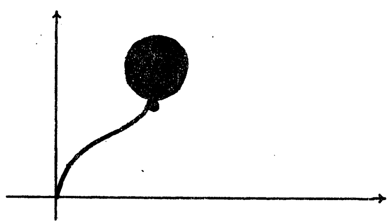
(d) Let $\gamma(z) = \sum_{n=1}^{\infty} c_n z^n$ be a Taylor series such that $\sum_{n=1}^{\infty} |c_n| < \infty$, then

$$(D) = \text{Image } \left( \sum_{n=1}^{\infty} c_n (\hat{m})^n \right) ,$$

and $\sum_{n=1}^{\infty} c_n (\hat{m})^n$ is the Fourier transform of an $L^1$-function.

By using a classical result on conformal mappings (see, e.g., [6, Vol.I,p.293]), it readily follows that if $\Omega$ is an open simply connected neighborhood of the origin such that $\partial\Omega$ is a *rectifiable* Jordan curve, then $\Omega^-$ is attainable as the image of the Fourier transform of an $L^1$-function.

Does there exist $f \in L^1(\mathbb{R})$ such that $(\text{Image } (\hat{f}))^-$ coincides with, for instance,



or with $\bigcup_{n=1}^{\infty} \{re^{i\pi/n} : 0 \leq r \leq 1/n\}$?.

**A STRANGE FOURIER TRANSFORM IN $L^2(\mathbb{R}) \setminus L^1(\mathbb{R})$.** Define

$$h(t) = \int_{[0,t]} (\cos \frac{\tau^2}{2} - \sin \frac{\tau^2}{2})\, d\tau\,, \text{ for } t \geq 0\,,$$

and $h(t) = -h(-t)$ for $t < 0$.

Clearly,

(a) h is the restriction to the real axis of an entire function, and a classical exercise of calculus of residues indicates that

$$\lim_{t\to\infty} h(t) = \lim_{t\to\infty} \int_{[0,t]} (\cos \frac{\tau^2}{2} - \sin \frac{\tau^2}{2})\, d\tau = \frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} = 0.$$

On the other hand, $\cos \frac{t^2}{2} - \sin \frac{t^2}{2} = \sqrt{2}\cos(\frac{t^2}{2} + \frac{\pi}{4})$ and (by studying the behavior of this function) it is not difficul to deduce that

(b) $\limsup_{t \to \pm\infty} th(t) = \limsup_{t \to \pm\infty} -th(t) = c$, for some constant $c > 0$,

and

(c) $h \in L^2(\mathbf{R}) \setminus L^1(\mathbf{R})$.

CLAIM. $i\hat{h}$ also enjoys the properties (a), (b) and (c). Indeed,

$$\hat{h}(s) = (-2i\sqrt{\pi}/s) \sin \frac{s^2}{2}\,.$$

Clearly, it suffices to show that $\hat{h}$ has the indicated form. The easiest way to do this is to think of h as the indefinite integral of the *distribution* $\phi$ associated to the function $\sqrt{2}\cos(\frac{t^2}{2} + \frac{\pi}{4})$ (see, e.g., [2,Chapter VI]); then $\hat{h}$ is the $L^2$-function associated to the distribution

$$(\text{p.v. } \frac{1}{is}).\hat{\phi}\,.$$

Since $\phi$ is the limit, in the sense of distributions (as $R \to \infty$), of the distributions associated to the functions

$\sqrt{2}\,\chi_{(-R,R)}(t)\cos(\frac{t^2}{2} + \frac{\pi}{4})$ $(R \to 0)$, and $\hat{\phi}$ is the distribution associated to the *function* $is.\hat{h}(s)$, we have

$$\hat{\phi}(s) = \lim_{R\to\infty} \sqrt{2} \int_{(-R,R)} e^{-ist} \cos(\frac{t^2}{2} + \frac{\pi}{4})\, dt =$$

$$= \lim_{R\to\infty} \frac{\sqrt{2}}{2} e^{i\pi/4} \int_{(-R,R)} e^{-ist} e^{i(t^2/2)}\, dt +$$

$$+ \lim_{R\to\infty} \frac{\sqrt{2}}{2} e^{-i\pi/4} \int_{(-R,R)} e^{-ist} e^{-i(t^2/4)}\, dt =$$

$$= \lim_{R\to\infty} (1/2)(1+i)e^{-i(s^2/2)} \int_{(-R,R)} e^{(i/2)(t-s)^2}\, dt \; +$$

$$+ \lim_{R\to\infty} (1/2)(1-i)e^{i(s^2/2)} \int_{(-R,R)} e^{(-i/2)(t+s)^2}\, dt \; =$$

$$\overset{(*)}{=} \lim_{R\to\infty} (1/2)(1+i)e^{i(s^2/2)} \int_{(-R,R)} e^{i(t^2/2)}\, dt \; +$$

$$+ \lim_{R\to\infty} (1/2)(1-i)e^{-i(s^2/2)} \int_{(-R,R)} e^{-i(t^2/2)}\, dt \; =$$

$$= \lim_{R\to\infty} (1+i)e^{i(s^2/2)} \int_{(0,R)} e^{i(t^2/2)}\, dt \; +$$

$$+ \lim_{R\to\infty} (1-i)e^{-i(s^2/2)} \int_{(0,R)} e^{-i(t^2/2)}\, dt \; =$$

$$\overset{(**)}{=} i\sqrt{\pi}e^{-i(s^2/2)} - i\sqrt{\pi}e^{i(s^2/2)} = 2\sqrt{\pi}\,\sin\frac{s^2}{2}\,.$$

Therefore

$$\hat{h}(s) = (\text{p.v}\,\frac{1}{is})\,.\,\hat{\phi}(s) = (-2i\sqrt{\pi}/s)\,\sin\frac{s^2}{2}\,.$$

((*) follows by a straightforward estimation of the difference between the two integrals; (**) follows by using calculus of residues).

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