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ON THE SPACE OF RIEMANNIAN HOROSPHERES

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ABSTRACT. Let M be a Hadamard manifold of dimension $n \geq 2$, and $\mathcal{H}$ be the set of oriented horospheres of M. We show that $\mathcal{H}$ can be identified topologically with the cylinder $R \times S^{n-1}$. In the constant curvature case, this identification gives rise to a diffeomorphism, and we employ this fact to compute explicitely the density of horospheres. As one application we get a unified formula which appears in Integral Geometry separately in the Euclidean and the hyperbolic space. In contrast with the geodesic case, we show that the concept of measure of horospheres, in the constant curvature case, cannot be extended to the class of Hadamard manifolds.

INTRODUCTION.

Let M be a Hadamard manifold, i.e., a simply connected complete C^∞-Riemannian manifold of dimension $n \geq 2$ and sectional curvature $K \leq 0$. Let $\mathcal{H} = \mathcal{H}(M)$ be the set of horospheres of M, where a horosphere is regarded as a C^2-oriented hypersurface.
In [5] we show that if, instead of $\mathcal{H}$, we consider the set of oriented geodesics $\vec{G}$ of M, then $\vec{G}$ is a differentiable manifold, diffeomorphic to the tangent bundle of S^{n-1}, and admits on it a volume which is invariant under the group of isometries.
Hence, a natural question arises: What can be said about $\mathcal{H}$?.
This question turns out to be of interest from the point of view of Integral Geometry since, if M is symmetric, the set $\mathcal{H}$ is different from the space of horocycles defined by Helgason (see for example [4]), except in the constant curvature case.
This paper is divided into three sections. The first one, under the heading Preliminaries sketches some basic material about Busemann functions. We show that these functions give rise to a natural homeomorphism between $\mathcal{H}$ and the cylinder $R \times S^{n-1}$, where S^{n-1} is the standard n-1-dimensional unit sphere. This natural identification leads us to say "when an almost everywhere differentiable structure on $\mathcal{H}$ is defined"; and consequently what we understand by a measure on $\mathcal{H}$ as analogous to the geodesic case.

In section two, we apply the above situation when M has constant sectional curvature to compute explicitely the density of horospheres, which does not appear in the literature except in special cases. As a consequence we get a unified formula which appears in Integral Geometry separately in the Euclidean and the hyperbolic space. In section three, we give an example of a 3-dimensional (symmetric) Hadamard manifold, with the property that on $\mathcal{H}$ an almost everywhere differentiable structure is defined, but does not admit a non-trivial measure invariant under the group of isometries. Consequently, the concept of measure of horospheres in the constant curvature case, cannot be extended to the class of symmetric Hadamard manifolds.

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1. PRELIMINARIES.

Throughout let M be a Hadamard manifold, SM the sphere bundle of M and S_p the fibre of SM over $p \in M$. For each $v \in SM$, we denote the geodesic with initial velocity v by c_v, and the tangent vector to c_v at "t" by $c_v'(t) = v(t)$. Let d be the distance function on M. For any $q \in M$ and each $v \in SM$, we define $b_{v\_}(q)$: $[0,\infty) \to R$ by $b_{vt}(q) = t - d_{vt}(q)$, where $d_{vt}(q) = d(q,c_v(t))$. These functions are smooth except at $c_v(t)$ and due to the triangular inequality are increasing with "t" and absolutely bounded by $d(q,c_v(0))$. Thus the function $b_v = \lim_{t\to\infty} b_{vt}$ is defined everywhere and also continuous on M; b_v is called the Busemann function determined by v. It follows clearly that for any $t \in R$

$$b_{v(t)} = b_v - t \tag{1}$$

Since for arbitrary $q \in M$, the triangular inequality yields

$$\lim_{t\to\infty} \frac{d(q,c_v(t))}{t} = 1 \text{ , one also gets}$$

$$b_v = \lim_{t\to\infty} \frac{t^2-d_{vt}^2}{2.t} \tag{2}$$

Call $H_v = b_v^{-1}(0)$ the horosphere and $B_v = b_v^{-1}([0,\infty))$ the horoball associated with v. Horospheres generalize in a natural way the affine hyperplanes of the Euclidean space, since H_v is the boundary of B_v, and

$$B_v = \bigcup_{t>0} B(c_v(t)) \tag{3}$$

where $B(c_v(t))$ is the geodesic ball of radius t centered at $c_v(t)$. For any $q \in M$ and all real $t > 0$ with $c_v(t) \neq q$, let $\nabla d_{vt}(q)$ be the gradient of d_{vt} at q. A vector $w \in S_q$ is called asymptotic to v if $-\nabla d_{vt_i}(q) \to w$ for some sequence $t_i \to \infty$. The "asymptotic" is an equivalence relation on SM, since as it is proved in [1], a vector $v \in SM$ is asymptotic to $w \in SM$ if and only if the difference $b_v - b_w$ is constant on M. Moreover (see [3]), Busemann functions are C^2 and for any $v \in SM$, the gradient of b_v at a point $q \in M$ satisfies

$$\nabla b_v(q) = -\lim_{t\to\infty} \nabla d_{vt}(q) \tag{4}$$

Therefore, the C^1-normal vector field on H_v defined by $q \to \nabla b_v(q)$ with $q \in H_v$, points into B_v. Since generally $H_v \neq H_{-v}$ we shall distinguish two such horospheres by considering them as oriented hypersurfaces. More precisely, let $\mathcal{H} = \mathcal{H}(M)$ be the set of equivalence classes on SM where $v \sim w$ if and only if $b_v = b_w$; and denote with [v] the equivalence class of v. The set $\mathcal{H}$ will be called the set of horospheres of M. In other words, a horosphere is identified with its normal vectors pointing into its horoball. If M is the Euclidean space, each horosphere (as a hyperplane) is counted twice in $\mathcal{H}$, so $\mathcal{H}$ is just the set of oriented affine hyperplanes.

REMARK. The boundary $M(\infty)$ which makes $M \cup M(\infty)$ homeomorphic to the closed unit disk in R^n (see [1]), is obtained from $\mathcal{H}$ identifying parallel horospheres, i.e., $[v] \sim [w]$ if and only if $b_v - b_w$ is constant.

The following lemma defines in a natural way a model for $\mathcal{H}$.

LEMMA 1. *For any* $p \in M$, *let* $\varphi_p\colon R\times S_p \to \mathcal{H}$ *defined by* $\varphi_p(t,v) = [v(t)]$; *then:*

a) φ_p *is a bijective map.*

b) *For any* $q \in M$, *the transition map* $\varphi_q^{-1} \circ \varphi_p\colon R\times S_p \to R\times S_q$ *is given by* $\varphi_q^{-1} \circ \varphi_p(t,v) = (t - b_v(q), \nabla b_v(q))$.

Proof. (a). If $\varphi_p(t,v) = \varphi_p(s,w)$ then $b_{v(t)} = b_{w(s)}$ or equivalently by (1), $b_v - t = b_w - s$. Since $b_v(p) = b_w(p) = 0$, then $t = s$. On the other hand, $b_v = b_w$ so by (4) we have $v = \nabla b_v(p) = \nabla b_w(p) = w$. Let now $[w] \in \mathcal{H}$ and set $t = -b_w(p)$, $v = \nabla b_v(p)$. Since $b_v - b_w$ is constant on M with $b_v(p) = 0$, then by (1) it follows that $b_{v(t)} = b_w$. Thus $v(t) \sim w$ and consequently φ_p is bijective with inverse $\varphi_p^{-1}([w]) =$ $= (-b_w(p), \nabla b_w(p))$.

For (b). Let $w = v(t)$, then by part (a) we have $\varphi_q^{-1} \circ \varphi_p(t,v) =$ $= (-b_w(q), \nabla b_w(q))$. Since $b_w = b_v - t$, then $-b_w(q) = t - b_v(q)$ and $\nabla b_w(q) = \nabla b_v(q)$.

REMARK. Let $v,w \in SM$, then $b_v - b_w$ is constant if and only if $d(c_v(t), c_w(t))$ is bounded for $t \geqslant 0$. Moreover, if v and w are asymptotics the function $t \to d(c_v(t), c_w(t))$ with $t \geqslant 0$ is non increasing. Due to the these facts, it follows that for any pair of points $p,q \in M$, the functions $v \to b_v(q)$ and $v \to \nabla b_v(q)$ with $v \in S_p$ are continuous. Consequently one gets:

PROPOSITION 2. *If* $p \in M$, *the topology on* $\mathcal{H}$ *deduced from* $R \times S_p$ *via* φ_p *does not depend on* p.

REMARK. Since all properties that we have used regarding Busemann functions are also true if M is simply connected and without focal points (see [2]), it follows that the above result holds for this class of Riemannian manifolds.

REMARK. In what follows, we shall consider $\mathcal{H}$ endowed with this topology. Assume that N is isometric to M, and let $f: M \to N$ be an isometry. Then f induces a map $\hat{f}: \mathcal{H}(M) \to \mathcal{H}(N)$ defined by $\hat{f}([v]) =$ $= [f_*(v)]$, where $f_*: SM \to SN$ is the differential map of f. Consequently, for any $p \in M$ one gets the following commutative diagram

$$\begin{array}{ccc} \mathcal{H}M & \xrightarrow{\hat{f}} & \mathcal{H}N \\ \varphi_p \uparrow & & \uparrow \varphi_q \\ R \times S_p & \xrightarrow{\tilde{f}} & R \times S_q \end{array}$$

where $q = f(p)$ and $\tilde{f}(t,v) = (t, f_*(v))$. Since φ_p, φ_q and $\tilde{f}$ are homeomorphisms, then it also is $\hat{f}$. In particular, let I(M) be the group of isometries of M; then by the above diagram with $N = M$ one gets

COROLLARY 3. The group I(M) acts on $\mathcal{H}$ as a group of homeomorphisms.

DEFINITION 1. We shall say that $\mathcal{H} = \mathcal{H}(M)$ is an almost everywhere differentiable manifold, if there exists an open-dense set $\mathcal{H}^o$ in $\mathcal{H}$ such that for any pair of points $p,q \in M$, the map $\varphi_q^{-1} \circ \varphi_p : \varphi_p^{-1}(\mathcal{H}^o) \to \varphi_q^{-1}(\mathcal{H}^o)$ is differentiable.

REMARK. This definition is invariant by isometries. In fact, if $f: M \to N$ is an isometry and $\mathcal{H}(M)$ is an a.e.-differentiable manifold, let $\mathcal{H}^o(N) = \hat{f}(\mathcal{H}^o(M))$ and define on $\mathcal{H}^o(N)$ the differentiable structure which makes $\hat{f}$ a diffeomorphism. From the above diagram and since $\tilde{f}$ is a diffeomorphism, it follows that $\varphi_q : \varphi_q^{-1}(\mathcal{H}^o(N)) \to \mathcal{H}^o(N)$ is a diffeomorphism for any $q \in N$. Moreover, let $f \in I(M)$ and set $\mathcal{H}_f^o = \mathcal{H}^o \cap \hat{f}^{-1}(\mathcal{H}^o)$, then $\mathcal{H}_f^o$ is an open submanifold of $\mathcal{H}^o$ which is dense in $\mathcal{H}$. From the diagram it follows that $\hat{f}: \mathcal{H}_f^o \to \mathcal{H}_f^o$ is a diffeomorphism.

DEFINITION 2. We shall say that M is $\mathcal{H}$-measurable if the following conditions are satisfied:

a) $\mathcal{H} = \mathcal{H}(M)$ is an a.e.-differentiable manifold.

b) There exists a non-trivial n-form $d\mathcal{H}^o$ on $\mathcal{H}^o$, which is invariant in absolute value by the isometries of M acting on $\mathcal{H}^o$, i.e., $(\hat{f})^*(d\mathcal{H}^o) = \pm\, d\mathcal{H}^o$ on $\mathcal{H}_f^o$ for any $f \in I(M)$.

REMARK. Again it follows from the above diagram, that if M is $\mathcal{H}$-measurable and N is isometric to M, then N is $\mathcal{H}$-measurable.

2. THE CONSTANT CURVATURE CASE.

In this section, let M be a Hadamard manifold of dimension $n \geq 2$ and and constant sectional curvature $K = -r^2$ $(r \geq 0)$. It is well known from Helgason, Santaló and others that $\mathcal{H}$ is a differentiable manifold and admits on it a density which is invariant under the group of isometries. However, the explicit expression of this density does not appear in the literature except in special cases. For $n = 2,3$ and $r = 1$, (see respectively [5] or [8] P.314, [7]). Hence, we shall give it here using the results of section 1.

THEOREM 4. *With the above hypotheses,* M *is* $\mathcal{H}$*-measurable with* $\mathcal{H}^o = \mathcal{H}$, *and the* n*-form* $d\mathcal{H}$ *which satisfies condition* b) *is defined (up to a constant factor) for any* $p \in M$ *by*

$$\varphi_p^*(d\mathcal{H}) = \pm \exp(r.(n-1).t)dt \wedge dS_p . \tag{5}$$

Proof. The invariance property of $d\mathcal{H}$ follows immediately from the above diagram with N = M, and the way $d\mathcal{H}$ is represented on each cylinder $R\times S_p$. In order to prove the other statements, it follows again from the above diagram that it is sufficient to prove these statements for an adequate Hadamard manifold N of constant curvature $K = -r^2$ $(r \geqslant 0)$. Now, to show that the manifold $\mathcal{H}N$ is defined and admits an n-form $d\mathcal{H}$ which satisfies (5), it is sufficient to show that for any pair of points $p,q \in N$ the map $\varphi_q^{-1} \circ \varphi_p$ is smooth and $(\varphi_q^{-1} \circ \varphi_p)^*(dt \wedge dS_q) = J_{qp} dt \wedge dS_p$ with

(6) $$J_{qp}(t,v) = \pm \exp\,(r.(n-1).b_v(q))$$

for any $(t,v) \in R\times S_p$. In fact, for an (arbitrary) fixed $q \in N$ define

(7) $$d\mathcal{H} = (\varphi_q^{-1})^*(\exp\,(r.(n-1).t).dt \wedge dS_q)$$

then from (6) and the expression of $\varphi_q^{-1} \circ \varphi_p$ (see lemma 1), it follows that (7) holds (up to sign) replacing q by any other $p \in N$.

CASE r = 0. For $n \geqslant 2$, let $N = R^n$ be the space of n-tuples of real numbers with the usual Riemannian structure. For any $p \in N$, we identify the tangent space to R^n at p with $p+R^n$ and the unit sphere S_p with the set of $v \in R^n$ which satisfies $|v-p| = 1$. From (2) one gets trivially that for any $v \in S_p$ and $q \in R^n$, $b_v(q) = \langle q-p,v-p\rangle$; where $\langle,\rangle$ is the usual inner product. Thus $b_v(q) = v-p+q$ and $J_{qp} = \pm 1$.

CASE $r > 0$. With the usual differentiable structure on R^{n+1}, we denote with $x = (x_0,x_1,\dots,x_n)$ the points of R^{n+1}. We consider the non-degenerate symmetric bilineal form

$$F(x,y) = -r^{-2}.x_0y_0 + \sum_{i=1}^{n} x_i.y_i \;;$$

then the hypersurface N in R^{n+1} defined by $F(x,x) = -r^{-2}$ and $x_0 \geqslant 1$, is clearly diffeomorphic to R^n. Identifying for any $p \in N$ the tangent space to N at p with the set of vectors $v \in R^{n+1}$ which satisfies $F(p,v) = 0$, the restriction of F to any tangent space is positive-definite. Thus the form F restricted to the tangent space of each point of N gives rise to a Riemannian metric.
It is well known that N with this metric has constant sectional curvature $K = -r^2$. For any $p \in N$ and $v \in S_p$, the geodesic starting at p with initial velocity v is given by

(8) $$c_v(t) = p.Ch(r.t) + r^{-1}.v.Sh(r.t)$$

If d denotes the distance function on N, it follows from (8) that

$$d(p,q) = r^{-1}.\text{arc Ch}(-r^2.F(p,q)) \tag{9}$$

Since for any $p,q \in N$ and $v \in S_p$ the function $G(p,q,v) = -r^2.F(p+r^{-1}.v,q)$ satisfies $\exp(-r.d(p,q)) \leqslant G(p,q,v) \leqslant \exp(r.d(p,q))$; one obtains from (8) and (9)

$$b_v(q) = -r^{-1}.\ell n\ G(p,q,v) \tag{10}$$

An easy check also shows that

$$\nabla b_v(q) = (r.p+v).\exp(r.b_v(q))-r.q \tag{11}$$

Thus the map $\varphi_q^{-1} \circ \varphi_p$ is smooth and by virtue of (10) and (11) a brief computation shows that $J_{qp}(t,v) = \pm \exp(r.(n-1).b_v(q))$. This completes the proof.

REMARK. The meaning of the parameter "t" which appears in the expression $(\varphi_p)^*(d\mathcal{H})$ is the following: $|t|$ is always the distance from p to the horosphere $H_{v(t)}$ $(v \in S_p)$. If $t \leqslant 0$, p lies in the horoball $B_{v(t)}$ and if $t > 0$, then p lies in the concave region determined by $H_{v(t)}$.

From equalities (5), (10) and (11) one also gets

COROLLARY 5. Let M be a Hadamard manifold of dimension $n \geqslant 2$ and constant sectional curvature $K = -r^2$ $(r \geqslant 0)$. Let c be a rectifiable curve in M of lenght L. For any $h \in \mathcal{H}$, let $\eta(h)$ be the number of intersection points of c with the horosphere h, then

$$\int_{\mathcal{H}} \eta \ d\mathcal{H} = \frac{L}{c_n} \cdot \text{vol}(S^{n-1}), \text{ where } c_n = \frac{n-1}{2} \cdot \int_0^{\pi} \text{sen}^{n-2} t.dt$$

If n = 2,3, one gets that this integral equals respectively 4L, $2\pi L$ (see [6], [7]).

3. AN EXAMPLE OF A HADAMARD MANIFOLD WHICH IS NOT $\mathcal{H}$-MEASURABLE.

It seems to be true that definition 1 is satisfied for any Hadamard manifold (at least in the symmetric case). Since we have no proof of this assumption, we shall construct a 3-dimensional (symmetric) Hadamard manifold, for which $\mathcal{H}$ is an a.e-differentiable manifold but is not $\mathcal{H}$-measurable.

Let $M = R^3$ be the space of 3-tuples of real numbers (x_1,x_2,x_3) with the usual differentiable structure. Denoting with $X_i = \frac{\partial}{\partial x_i}$ ($i = 1,2,3$) the global tangent base field on M, we define a Riemannian metric as follows:

$\langle X_1,X_1\rangle = \langle X_2,X_2\rangle = 1$, $\langle X_3,X_3\rangle = e^{2.x_2}$ and $\langle X_i,X_j\rangle = 0$ if $i \neq j$.

If $p = (x_1,x_2,x_3)$ and $v = \sum_{i=1}^{3} v_i.X_i(p)$ with $v \in S_p$, i.e., $v_1^2 + v_2^2 + v_3^2\, e^{2.x_2} = 1$, let $r(v) = (1-v_1^2)^{1/2}$.

To avoid notation we shall also write $v = (v_1,v_2,v_3)$.

For any $(t,v) \in R\times S_p$ we define $g(t,v) = 1$ if $r(v) = 0$ and $g(t,v) = Ch(r(v).t) + r^{-1}(v).v_2.Sh(r(v).t)$ if $r(v) > 0$.

If c_v is the geodesic starting at p with initial velocity v and $c_v(t) = (x_1(t,v),\ x_2(t,v),\ x_3(t,v))$, then

$x_1(t,v) = x_1 + t.v_1$, $x_2(t,v) = x_2 + \ln g(t,v)$, $x_3(t,v) = x_3$ if $r(v) = 0$ and $x_3(t,v) = x_3 + r^{-1}(v).v_3.g^{-1}(t,v).Sh(r(v).t)$ if $r(v) > 0$.

Let σ be the plane section of the tangent space to M at p, spanned by two orthonormal vectors $X = (a_1,a_2,a_3)$ and $Y = (b_1,b_2,b_3)$. The sectional curvature K_σ is given by $K_\sigma = -e^{2.x_2}(a_2.b_3-a_3.b_2)^2 \leq 0$. Hence, M is a Hadamard manifold and a simple computation shows also that M is symmetric.

Let $q = (y_1,y_2,y_3)$; then the distance function d on M is given by

(12) $$d(p,q) = ((x_1-y_1)^2 + \rho^2(p,q))^{1/2}$$

where $\rho(p,q) = \text{arc}\, Ch(Ch(x_2-y_2) + \frac{1}{2}(x_3-y_3)^2.e^{x_2+y_2})$.

From (2) and (12) one gets that the Busemann function b_v with $v = (v_1,v_2,v_3)$ and $v \in S_p$ is defined by

(13) $$b_v(q) = v_1.(y_1-x_1) + r(v).(y_2-x_2) - r(v).\ln h_v(q)$$

where

$$h_v(q) = \begin{cases} e^{2(y_2-x_2)} & \text{if } v_3 = 0 \text{ and } v_2 \leq 0 \\ \frac{r(v)+v_2}{2.r(v)} \cdot (1+e^{2.y_2}.(y_3-x_3 - \frac{v_3}{r(v)+v_2})^2) & \text{in other cases.} \end{cases}$$

Let $\nabla b_v(q) = (\xi_1(v,q),\ \xi_2(v,q),\ \xi_3(v,q))$; then from (13) one gets:

(14) $$\xi_2(v,q) = \frac{r(v)+v_2-v_2.\,h_v(q)}{h_v(q)}, \quad \xi_3(v,q) = \frac{v_3-(y_3-x_3).(r(v)+v_2)}{h_v(q)}$$

and $\xi_1(v,q) = v_1$.
For any $p \in M$, let $S_p^o = S_p - \{X_1(p), -X_1(p)\}$. Since for any $q \in M$ the map $v \to h_v(q)$ with $v \in S_p^o$ is differentiable and $\xi_1(v,q) = v_1$, there follows that $\varphi_q^{-1} \circ \varphi_p : RxS_p^o \to RxS_q^o$ is a diffeomorphism for any $p,q \in M$. Hence, $\mathcal{H}^o = \varphi_p(RxS_p^o)$ is an open-dense set in $\mathcal{H}$ which clearly does not depend on p. Consequently, $\mathcal{H}$ is an a.e-differentiable manifold.
In what follows, let $p = (0,0,0)$ and $q = (y_1,y_2,y_3)$ be an arbitrary point.
If we consider the standard volume on RxS_q^o, from (13) and (14) an easy check shows that the Jacobian J_q of $\varphi_q^{-1} \circ \varphi_p$ is defined by

$$(15) \qquad J_q(v) = \exp\left(\frac{b_v(q) - y_1 . v_1}{r(v)}\right)$$

Now, assume there exists a 3-form $d\mathcal{H}^o$ on $\mathcal{H}^o$ which is invariant under I(M), and let $f.dt \wedge dS_p$ be the representation of $d\mathcal{H}^o$ on RxS_p^o via φ_p. We shall show that f is zero.
In fact, since the map $\tau: M \to M$ defined by

$$(16) \qquad \tau(x_1,x_2,x_3) = (x_1+y_1, x_2+y_2, x_3 . e^{-y_2} + y_3)$$

is an isometry which satisfies $\tau(p) = q$ and $\hat{\tau}(\mathcal{H}^o) = \mathcal{H}^o$, it follows that the function f must satisfy

$$(17) \qquad f(t,v) = f(t-b_v(q), (\tau^{-1})_{*_q}(\nabla b_v(q))).J_q(v)$$

for any $(t,v) \in RxS_p^o$ and any isometry defined by (16) with $\tau(p) = q$.
Let $v \in S_p^o$ with $v_3 \neq 0$, and let $q = (y_1,y_2,y_3)$ with $y_3 = \frac{v_3}{r(v)+v_2}$.
Recalling that p is the origin, from (14) one gets $\nabla b_v(q) = u$, where $u = (v_1, r(v), 0)$. Hence, $f(t,v) = f(t-b_v(q),u).J_q(v)$ and consequently f is zero provided that f vanishes at the points $(t,v) \in RxS_p^o$ with $v_3 = 0$.
Now, let $v = (v_1,v_2,0)$ be a fixed vector of S_p^o and let $q = (y_1,y_2,y_3)$ be an arbitrary point with $y_3 = 0$. From equalities (13), (14) and (15) there follows that $b_v(q) = v_1.y_1+v_2.y_2$, $\nabla b_v(q) = v$ and $J_q(v) = \exp\left(\frac{v_2}{|v_2|}.y_2\right)$. Equality (17) with these values, implies clearly that $f(t,v) = 0$, since y_1,y_2 are arbitrary real numbers.

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