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TWO-DIMENSIONAL REAL DIVISION ALGEBRAS

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ABSTRACT. We classify completely non-associative real division algebras of dimension two with left identity. Each algebra of this type is isomorphic to exactly one member of four infinite families.

In this paper we take "algebra" to mean a non-associative algebra over the field of real numbers R, that is, a real vector space A, with a product which is distributive vis-à-vis addition, and satisfies $(\alpha x)y = x(\alpha y) = \alpha(xy)$ for all $\alpha \in R$, $x,y \in A$. An algebra is called a division algebra if the equation in A $ax = b$ (resp., $xa = b$) has a unique solution whenever $a \neq 0$.

The fundamental work of Milnor and Bott [6], as well as Kervaire [4], showed that all such finite-dimensional division algebras have dimension 1, 2, 4 or 8. Here we classify completely those of dimension two which have a left identity.

The classification of associative algebras of low dimension has been studied for over a century. See [3] for a historical overview. Recently associative unitary algebras of dimension 5 [2,5] have been classified successfully. The corresponding problem in higher dimensions gives rise to combinatorial difficulties.

In the general non-associative case, less has been accomplished. An algebra A is called flexible if $(xy)x = x(yx)$ for all $x,y \in A$. Finite-dimensional flexible division algebras are classified in [1].

First we would like to consider the general situation of an algebra A of dimension two. Let $[x_1,x_2]$ be a basis of A. The product in A is determined by the multiplication table

(1)

	x_1	x_2
x_1	$\alpha' x_1 + \beta' x_2$	$\gamma' x_1 + \delta' x_2$
x_2	$\alpha x_1 + \beta x_2$	$\gamma x_1 + \delta x_2$

THEOREM 1. *The algebra* A *determined by* (1) *is a division algebra if and only if*

$$(2) \quad 4(\alpha'\delta' - \beta'\gamma')(\alpha\delta - \beta\gamma) > (\alpha\delta' + \alpha'\delta - \beta\gamma' - \beta'\gamma)^2$$

and

$$(3) \quad 4(\alpha'\beta - \alpha\beta')(\gamma'\delta - \gamma\delta') > (\alpha\delta' - \alpha'\delta - \beta\gamma' + \beta'\gamma)^2 .$$

Proof. Indeed A is a division algebra iff the maps $T(x) = ax$ and $U(x) = xa$ are non-singular for all $0 \neq a \in A$. If $a = \pi x_1 + \sigma x_2$ then $ax_1 = (\alpha'\pi + \alpha\sigma)x_1 + (\beta'\pi + \beta\sigma)x_2$ and $ax_2 = (\gamma'\pi + \gamma\sigma)x_1 + (\delta'\pi + \delta\sigma)x_2$. Now T is non-singular iff $\{ax_1, ax_2\}$ is a basis, i.e.

$$(\alpha'\delta' - \beta'\gamma')\pi^2 + (\alpha\delta' + \alpha'\delta - \beta\gamma' - \beta'\gamma)\pi\sigma + (\alpha\delta - \beta\gamma)\sigma^2 =$$
$$= (\alpha'\pi + \alpha\sigma)(\delta'\pi + \delta\sigma) - (\beta'\pi + \beta\sigma)(\gamma'\pi + \gamma\sigma) \neq 0 .$$

By the theory of quadratic forms, this is true for all $0 \neq a \in A$ whenever (2) is satisfied. Similarly, the invertibility of U guarantees (3).

We note that a division algebra A can have at most one left identity. Moreover, if the division algebras A and A' have left identities x_1 and y_1 (resp.) and $T : A \to A'$ is an isomorphism then $T(x_1) = y_1$.

THEOREM 2. *Suppose that the algebra* A *determined by* (1) *is a division algebra. Then it has left identity if and only if*

$$(4) \quad \frac{\beta}{\alpha'\beta - \alpha\beta'} = \frac{-\gamma}{\gamma'\delta - \gamma\delta'} \quad \textit{and} \quad \frac{\beta'}{\alpha'\beta - \alpha\beta'} = \frac{-\gamma'}{\gamma'\delta - \gamma\delta'} .$$

Proof. Let $e = \pi x_1 + \sigma x_2$ be the left identity. Then $ex_1 = x_1$ and $ex_2 = x_2$ result in the systems

$$\alpha'\pi + \alpha\sigma = 1 \qquad \gamma'\pi + \gamma\sigma = 0$$
$$\beta'\pi + \beta\sigma = 0 \qquad \delta'\pi + \delta\sigma = 1 .$$

Now (3) assures that $\alpha'\beta - \alpha\beta' \neq 0$ and $\gamma'\delta - \gamma\delta' \neq 0$. Hence both systems have a unique solution, namely

$$(5) \quad \frac{\beta}{\alpha'\beta - \alpha\beta'} = \pi = \frac{-\gamma}{\gamma'\delta - \gamma\delta'} \quad \text{and} \quad \frac{-\beta'}{\alpha'\beta - \alpha\beta'} = \sigma = \frac{\gamma'}{\gamma'\delta - \gamma\delta'} .$$

Conversely, if π and σ are defined by (5), then $e = \pi x_1 + \sigma x_2$ is the left identity.

From now on we consider only real algebras A of dimension two with left identity. Let us take a basis $\{x_1, x_2\}$ of A where x_1 is the left identity. Hence the multiplication table has the form

(6)

	x_1	x_2
x_1	x_1	x_2
x_2	$\alpha x_1 + \beta x_2$	$\gamma x_1 + \delta x_2$

Let $A(\alpha,\beta,\gamma,\delta)$ be the algebra with this multiplication table. By Theorem 1, it is a division algebra iff $-4\beta\gamma > (\alpha - \delta)^2$. Our principal theorem follows.

THEOREM 3. *Let* A *be a real division algebra of dimension two with left identity. Then* A *is isomorphic to precisely one of the following algebras:*

	(class I)	$A(0,\beta,-1,0)$	*for some*	$\beta > 0$
or	(clas II)	$A(0,\beta,1,0)$	*for some*	$\beta < 0$
or	(class III)	$A(1,\beta,\gamma,0)$	*for some*	$\beta \neq -1$ *and* γ *with* $\beta\gamma < -\frac{1}{4}$
or	(class IV)	$A(1,-1,\gamma,1)$	*for some*	$\gamma > 0$.

Clearly an analogous theorem can be formulated for real division algebras of dimension two with right identity. Note that C, the algebra of complex numbers, appears in class I with $\beta = 1$. It is the only algebra in our list with a two-sided identity. In addition, C is the only associative algebra - indeed, the only flexible algebra - to appear.

We prove Theorem 3 in three steps. First we show, in Lemmata 5 and 6, that any algebra $A(\alpha,\beta,\gamma,\delta)$ is isomorphic to an algebra in our list. Then we show that no two algebras in the same class are (non-trivially) isomorphic (Lemma 7). Finally we show that an algebra of one class cannot be isomorphic to an algebra of another class (Lemma 8).

LEMMA 4. β *and* sgn γ *are invariants among division algebras, that is, if* $A = A(\alpha,\beta,\gamma,\delta)$ *and* $A' = A(\alpha',\beta',\gamma',\delta')$ *are isomorphic division algebras, then* $\beta = \beta'$ *and* $\operatorname{sgn}\gamma = \operatorname{sgn}\gamma'$.

Proof. Suppose $T: A \to A'$ is an isomorphism. Then the image of the left identity, x_1, of A is the left identity, y_1, of A'. Let $T(x_2) = \pi y_1 + \sigma y_2$ $(\sigma \neq 0)$. Then

$$(\alpha + \beta\pi)y_1 + \beta\sigma y_2 = T(x_2x_1) = T(x_2)T(x_1) = (\pi + \alpha'\sigma)y_1 + \beta'\sigma y_2,$$

whence $\beta\sigma = \beta'\sigma$ and $\beta = \beta'$. Moreover, since $\beta\gamma, \beta'\gamma' < 0$, then $\operatorname{sgn}\gamma = \operatorname{sgn}\gamma'$.

LEMMA 5. *If* $\beta \neq -1$ *the division algebra* $A(\alpha,\beta,\gamma,\delta)$ *is isomorphic to either*

(class I) $A(0,\beta,-1,0)$ *with* $\beta > 0$

or (class II-A) $A(0,\beta,1,0)$ *with* $\beta < 0$ *and* $\beta \neq -1$
or (class III) $A(1,\beta,\gamma',0)$ *with* $\beta \neq -1$ *and* γ' *with* $\beta\gamma' < -\frac{1}{4}$.

Proof. Without loss of generality we can take $\delta = 0$. For if $\delta \neq 0$, the change of basis to $y_1 = x_1$, $y_2 = x_1 + \varepsilon x_2$ ($\varepsilon = \frac{-1-\beta}{\delta}$) gives a new multiplication table with $\delta = 0$. This works because

$$y_2^2 = (1 + \alpha\varepsilon + \gamma\varepsilon^2)x_1 + \varepsilon(1 + \beta + \delta\varepsilon)x_2 \text{ and } 1 + \beta + \delta\varepsilon = 0.$$

Now, if $\alpha \neq 0$, then the transformation T defined by $T(x_1) = y_1$, $T(x_2) = \alpha y_2$ yields an isomorphism $A(\alpha,\beta,\gamma,0) \cong A(1,\beta,\gamma',0)$ (with $\gamma' = \frac{\gamma}{\alpha^2}$). This T preserves products because
$T(x_2x_1) = \alpha y_1 + \alpha\beta y_2 = T(x_2)T(x_1)$ and $T(x_2^2) = \gamma y_1 - \alpha^2\gamma' y_1 = (T(x_2))^2$.
On the other hand, if $\alpha = 0$, then the transformation T given by $T(x_1) = y_1$, $T(x_2) = \sqrt{|\gamma|}\,y_2$ produces an isomorphism $A(0,\beta,\gamma,0) \cong A(0,\beta,\pm 1,0)$.

LEMMA 6. *The division algebra* $A(\alpha,-1,\gamma,\delta)$ *is isomorphic to either*
(class II-B) $A(0,-1,1,0)$
or (class IV) $A(1,-1,\gamma',1)$ *for some* $\gamma' > 0$.

Proof. If $\delta \neq 0$, the transformation T given by $T(x_1) = y_1$, $T(x_2) = \delta y_2$ produces an isomorphism $A(\alpha,-1,\gamma,\delta) \cong A(\alpha',-1,\gamma'',1)$ (with appropriate α' and γ''). Furthermore, if $\alpha' \neq 1$, $A(\alpha',-1,\gamma'',1) \cong A(1,-1,\gamma',1)$, using the transformation given by $T(x_1) = y_1$, $T(x_2) = \frac{1}{2}(\alpha' - 1)y_1 + y_2$. Finally, if $\delta = 0$, then the transformation defined by $T(x_1) = y_1$, $T(x_2) = \frac{1}{2}\alpha y_1 + \frac{1}{2}\sqrt{4\gamma - \alpha^2}\, y_2$ produces an isomorphism $A(\alpha,-1,\gamma,0) \cong A(0,-1,1,0)$.

LEMMA 7.
(class I) *If* $\beta,\beta' > 0$, *then* $A(0,\beta,-1,0) \cong A(0,\beta',-1,0)$ *iff* $\beta = \beta'$.
(class II) *If* $\beta,\beta' < 0$, *then* $A(0,\beta,1,0) \cong A(0,\beta',1,0)$ *iff* $\beta = \beta'$.
(class III) *If* $\beta,\beta' \neq -1$ *and* $\beta\gamma,\beta'\gamma' < -\frac{1}{4}$, *then*
$A(1,\beta,\gamma,0) \cong A(1,\beta',\gamma',0)$ *iff* $\beta = \beta'$ *and* $\gamma = \gamma'$.
(class IV) *If* $\gamma,\gamma' > 0$, *then* $A(1,-1,\gamma,1) \cong A(1,-1,\gamma',1)$ *iff* $\gamma = \gamma'$.

Proof. In classes I and II, this is a consequence of Lemma 4. Now let us consider class III. Suppose we have an isomorphism

$T: A(1,\beta,\gamma,0) \to A(1,\beta',\gamma',0)$ with $T(x_1) = y_1$, $T(x_2) = \pi y_1 + \sigma y_2$ $(\sigma \neq 0)$. Then

$$(1 + \beta\pi)y_1 + \beta\sigma y_2 = T(x_2x_1) = T(x_2)T(x_1) = (\pi + \sigma)y_1 + \beta'\sigma y_2,$$
$$\gamma y_1 = T(x_2^2) = (T(x_2))^2 = (\pi^2 + \pi\sigma + \gamma'\sigma^2)y_1 + (\pi\sigma + \beta'\pi\sigma)y_2.$$

Hence $1 + \beta\pi = \pi + \sigma$, $\beta\sigma = \beta'\sigma$, $\gamma = \pi^2 + \pi\sigma + \gamma'\sigma^2$, $0 = \pi\sigma + \beta'\pi\sigma$. Since $\sigma \neq 0$, $\beta = \beta'$ and $0 = \pi + \beta\pi$. Since $\beta \neq -1$, $\pi = 0$, and $\sigma = 1$, and $\gamma = \gamma'$. Class IV is similar.

LEMMA 8. *An algebra* A *in the class* J *cannot be isomorphic to any algebra* A' *in the class* K (J $\neq$ K ; J,K = I,II,III,IV).

Proof. Half the cases follow immediately from Lemma 4. The cases which need a separate argument are (J,K) = (I,III), (II,III), (II,IV). Here we shall consider just the last one. If $T: A(0,\beta,1,0) \to A(1,-1,\gamma,1)$ is an isomorphism with $T(x_1) = y_1$, $T(x_2) = \pi y_1 + \sigma y_2$, then

$$y_1 = T(x_2^2) = (T(x_2))^2 = (\pi^2 + \pi\sigma + \sigma^2\gamma)y_1 + \sigma^2 y_2.$$

Hence $\sigma = 0$, a contradiction.

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