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THE DISTANCE IN COMPACT RIEMANNIAN MANIFOLDS

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ABSTRACT. In a connected Riemannian manifold one has a natural distance. A subset of the manifold is called basic if every point in M is uniquely determined by its distances to the points in the subset. We prove that every compact Riemannian manifold has a finite basic set and obtain some consequences.

1. INTRODUCTION.

Let (M,g) be a connected Riemannian manifold. Let d be the distance associated to g.

A set $B \subset M$ is called basic if every point in M is uniquely determined by its distances to the points in B.

We see (Theorem (2.1)) that every compact Riemannian manifold has a finite basic set. As a consequence a "metric number" can be associated to M. We compute some of these numbers.

2. THE DISTANCE IN A RIEMANNIAN MANIFOLD.

In what follows (M,g) will be a connected complete Riemannian manifold with metric g and d(x,y), $x,y \in M$ will be the distance function naturally associated to g.

(2.1) THEOREM. *Let* (M,g) *be a compact Riemannian manifold. Then there exists in* M *a finite set of points* $\{x_1,\dots,x_N\}$ *such that if* x,y *are in* M *and* $d(x,x_i) = d(y,x_i)$, $i = 1,2,\dots,N$, *then* $x = y$.

Proof. Let p_0 be a point in M and consider $Br(p_0)$ a convex, normal, minimizing ball arround p_0. Let $X_1,\dots,X_n$ (n = dim M) be a basis of M_{p_0} such that $\exp_{p_0} X_i = p_i \in Br(p_0)$ and consider the functions

$$f_i(x) = (d(x,p_i))^2 \quad i = 0,1,\dots,n$$

We can start with r small enough in order to assure that

$d^2: Br(p_0) \times Br(p_0) \longrightarrow R$ be C^∞.

With the f_i's we form the function $F: Br(p_0) \longrightarrow R^{n+1}, F = (f_0,\dots,f_n)$ then we clearly have $dF|_{p_0}$ has rank n since $\operatorname{gr} f_{i|p_0} = 2X_i$ $i = 1,2,\dots,n$. By the rank theorem we have that there exists a neighborhood $B_\varepsilon(p_0)$, $\varepsilon \leqslant r$, such that $F|B_\varepsilon(p_0)$ is one to one.

Now we can do this for each point $p_0 \in M$ obtaining an open covering by balls $B_{\varepsilon_{p_0}}(p_0)$ and obtain by compactness a finite number of points $\{p_0^1, p_0^2, \dots, p_0^k\}$ such that $\bigcup_{u=1}^{k} B_{\varepsilon_{p_0^j}}(p_0^j) = M$.

For each of the points we have the corresponding balls $B_{r_{p_0^j}}(p_0^j)$ and in each one of them the n points $p_1^j,\dots,p_n^j$ determined by the basis.

We have then $k(n+1)$ points in M.

Now take two points x,y in M such that

$$d(x,p_i^j) = d(y,p_i^j) \quad j = 1,\dots,k \; , \; i = 0,1,\dots,n.$$

Since the balls form a covering we have, for some j, $x \in B_{\varepsilon_{p_0^j}}(p_0^j)$ i.e. $d(x,p_0^j) < \varepsilon_{p_0^j}$ and therefore also $y \in B_{\varepsilon_{p_0^j}}$. Now the function

$$F^j: B_{r_{p_0^j}}(p_0^j) \longrightarrow R^{n+1}$$

is one to one in $B_{\varepsilon_{p_0^j}}(p_0^j)$ and since $F^j(x) = F^j(y)$ we must have $x=y$ and the theorem is proven.

An arbitrary subset $B \subset M$ with the property of the previous theorem will be called *basic in* M i.e. B is *basic in* M if for each pair $x,y \in M$

$$d(x,b) = d(y,b) \quad \forall\, b \in B \;\Rightarrow\; x = y.$$

If there exists in (M,g) a finite set which is basic in (M,g) we define $N(M,g)$ as the minimum of the cardinalities of such sets. Otherwise we write $N(M,g) = \infty$.

If a given manifold M admits a Riemannian metric g such that $N(M,g) < \infty$ we may consider all such metrics and define $LN(M) = \min_g N(M,g)$.

We can make now the simple but important

(2.2) OBSERVATION. If (M,g) has a basic set $A \subset M$ then there is a one to one continuos function $\phi: M \longrightarrow R^A$ defined by

$$[\phi(x)]\ (a) = [d(a,x)]^2 \quad \text{for each} \quad a \in A.$$

In particular if M is compact and we write IN(M) for the minimum of the dimensions of the euclidean spaces in which M can be topologically imbedded ($\exists$. ϕ: M $\longrightarrow$ E^k one to one continuous) we have

(2.3) COROLLARY. If M is compact IN(M) $\leqslant$ N(M,g) for each Riemannian metric g on M. In particular IN(M) $\leqslant$ LN(M).

We also have the following

(2.4) COROLLARY. If (M,g) is a Riemannian manifold and dim M = n then N(M,g) $>$ n.

Proof. If N(M,g) $\leqslant$ n then we can find a set A = $\{a_1,\dots,a_n\}$ which is basic in M and consider the one to one continuous function $\phi: M \to R^n$. Let (U,f) be a coordinate neighborhood arround a_1. Then $R^n \supset f(U) \xrightarrow{f^{-1}} M \xrightarrow{\phi} R^n$ is one to one continuous. By invariance of domain $\phi(U)$ is open in R^n but $\phi(a_1) = (0, r_2,\dots,r_n)$ $(r_i = d(a_1,a_i))$ is the limit in R^n of the sequence $(-\frac{1}{k}, r_2,\dots,r_n)$ which is a contradiction.

We have now

(2.5) THEOREM. $LN(R^n) = LN(S^n) = LN(H^n) = n+1$.

Proof. This follows from the last corollary and the following lemma which can be proven via the cosine theorem of the corresponding trigonometry.

(2.6) LEMMA. *For the usual constant curvature metrics in* R^n, S^n *and* H^n *we have* $N(R^n) = N(S^n) = N(H^n) = n+1$.

We also have the following

(2.7) THEOREM. *For the* n-*dimensional torus* T^n *we have* $LN(T^n) = n+1$.

Proof. It is enough to show that for some metric g on T^n $N(T^n,g) = n + 1$.

For simplicity we shall consider only the case n=2. On T^2 we can consider several flat metrics;we shall take the one of the square i.e. $T^2 = R^2/\sim$ with the lattice generated by (1,0) and (0,1).

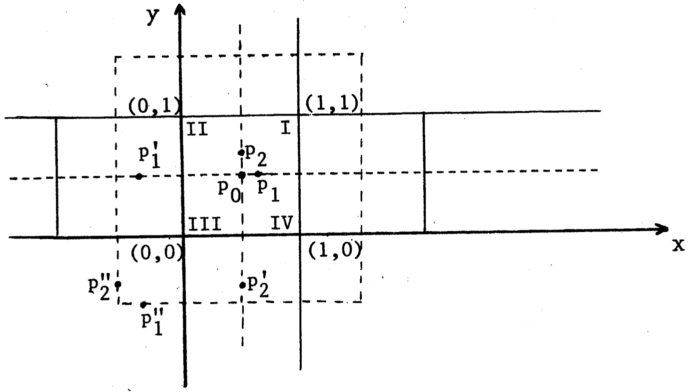

Fig.1

Let us consider the points p_0, p_1, p_2 in the torus such that the corresponding points in the plane (also denoted p_0, p_1 and p_2) be as in Fig.1 , p_0 in the center, p_1 in the parallel to the x-axis at a distance $\varepsilon < \frac{1}{2}$ from p_0 and the same p_2 on the parallel to the y-axis. We want to prove that each point in the torus is uniquely determined by its distance to these three points.

For each p_i we have its "lattice".

$$L_{p_i} = \{p_i + k_1(1,0) + k_2(0,1)\}$$

and for each p in the square we have

$$d_{T^2}(p,p_i) = d_{R^2}(p,L_{p_i})$$

where d_{T^2} and d_{R^2} mean distances in the torus and plane respectively.

Let us consider now the four regions A,B,C and D in the torus (Fig.2).

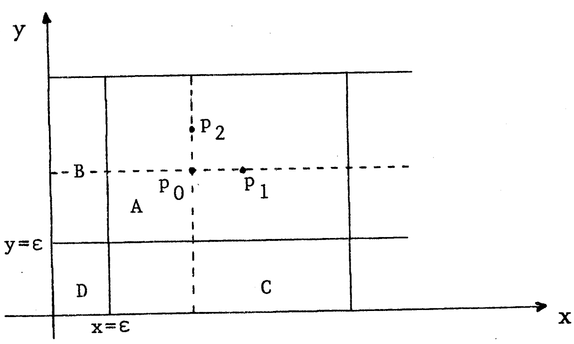

Fig.2

determined by the lines $y=\varepsilon$ and $x=\varepsilon$.

Let us take now a point p in the square and consider the following cases:

$p \in A$. In this case we have

$$d_{T^2}(p,p_i) = d_{R^2}(p,p_i) \qquad i = 0,1,2$$

then the point p is uniquely determined by these three numbers, in other words, if p and q are in A and $d_{T^2}(p,p_i) = d_{T^2}(q,p_i)$ $i = 0,1,2$ then $p = q$.

Let us take now a point $p \in B$. In this case we have the following situation

$$d_{T^2}(p,p_i) = d_{R^2}(p,p_i) \qquad i = 0,2$$

$$d_{T^2}(p,p_1) = d_{R^2}(p,p_1') \qquad \text{(see Fig.3)}$$

p_1' is the first point to the left on L_{p_1}.

There exists one and only one point q in the square such that

$$d_{R^2}(p,p_i) = d_{R^2}(q,p_i) \qquad i = 0,2$$

and clearly $q \in A$. We have to see that $d_{R^2}(p,p_1') \neq d_{R^2}(q,p_1)$.

This situation is represented in Fig.3.

$$d_1^2 = d_0^2 + \varepsilon^2 - 2\, d_0\, \varepsilon \cos\alpha$$

$$d_1^2 = d^2 + (1-\varepsilon)^2 - 2\, d_0\, (1-\varepsilon) \cos\alpha$$

since $\varepsilon < \frac{1}{2}$ the two equations can hold simultaneously if and only if $2\, d_0 \cos\alpha = 1$.

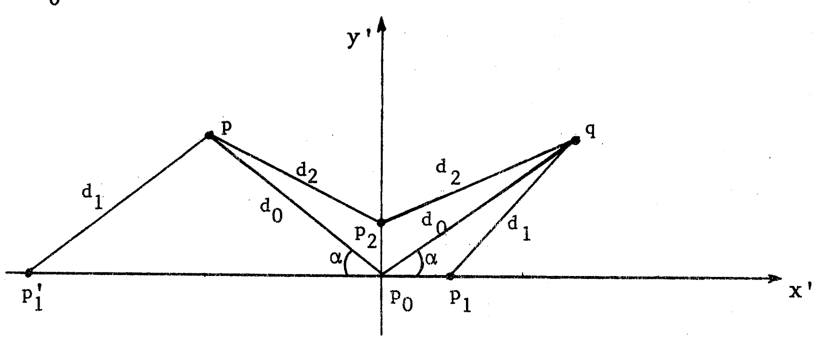

Fig.3

This is the equation of the vertical line $x' = \frac{1}{2}$ or of $x = 1$ in the system indicated in Fig.1 so the point q lies on the right boundary of the square and so $p = q$ in the torus.

If we take a point in C the situation is clearly analogous with p_2'

(Fig.1).

Let us take now a point p in D. We have

$$d_{T^2}(p,p_0) = d_{R^2}(p,p_0)$$
$$d_{T^2}(p,p_1) = d_{R^2}(p,p_1')$$
$$d_{T^2}(p,p_2) = d_{R^2}(p,p_2')$$

It is therefore clear that in D can not exist two points with the same three numbers since the points p_0, p_1' and p_2' determine uniquely every point in the plane.

Now it remains to be proven that no other point in the torus could have the same three numbers that the point p in D.

There is clearly one (and only one) point q in B such that

$$d_{R^2}(p,p_1') = d_{R^2}(q,p_1')$$
$$d_{R^2}(p,p_0) = d_{R^2}(q,p_0)$$

but clearly $d_{R^2}(q,p_2) < d_{R^2}(p,p_2')$ except when p is on the lower border of D and in this case p=q in T^2.

There is also one (and only one) point q' in C with the same distances to p_0 and p_2' but its distance to p_1 is smaller unless p is on the left border of D and in this case p = q' in T^2.

The theorem is then proven.

The results we have seen may mislead us to think that for any Riemannian manifold M admitting a constant curvature metric LN(M) = = n+1 but in fact we have the following

(2.8) THEOREM. *Let* RP^m *be the* m-*dimensional real projective space. Then for* $m = 2^r + j$, $j = 0,1,2,5,6,7$, $r > 3$ *we have*

$$2m - k < LN(RP^m) \leqslant \frac{m}{2}(m+3)$$

where k *is a function of* j *given by the following table*

j	0	1	2	5	6	7
k	1	2	4	4	6	8

Proof. We need to recall the following remarkable result due to Haefliger [2] (see also [1]).

(2.9) LEMMA (Haefliger [2]). *If* M^m *and* N^n *are two differentiable manifolds such that:*

i) M^m *imbeds topologically in* N^n *(i.e. there is* f: $M^n \to N^n$ *one to one which is a homeomorphism onto its image with the induced topology).*

ii) $n \geq \frac{3}{2}(m+1)$.

Then there exists a differentiable imbedding of M^m *into* N^n.

Now there are several well known results due to various authors concerning the impossibility of imbedding RP^m for $m = 2^r + j$ into R^n $n = 2m-k$ (k and j as above) see [1] p.91 for the corresponding table. These results and our observation (2.2) yield the left hand side inequality since $r > 3$ implies $n \geq \frac{3}{2}(m+1)$ in each case.
Now the right hand side inequality can be obtained for the usual constant curvature metric by using the well-known cell structure of RP^m.

In the topological category, on the other hand, the situation is much simpler as the following fact shows

(2.10) THEOREM. *Let* M *be a compact connected topological space. Then* M *can be imbedded in a finite dimensional euclidean space if and only if* M *is metrizable and admits a distance with a finite basic set.*

Proof. If M is metrizable and admits a distance with a finite basic set we obtain the imbedding as in (2.2).
On the other hand if M can be imbedded in R^N we may assume that it is substantial now consider on M the induced distance then M is metrizable and clearly there is a basic set with N+1 points on M.

FINAL REMARK. If M is a differentiable manifold the induced distance from R^N considered in (2.10) cannot be the distance of a Riemmannian metric on M. It seems, in fact, very hard to determine N(M,g) for the distance of the induced metric of a differentiable imbedding of M in R^N.

It is clear however that N(M,g) must be closely related to the geometry of (M,g) being small for more "regular" metrics and larger for more complicated ones.

Clearly LN(M) has a more "topological" nature but we suspect it is more related to the differentiable structure of M.

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