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OPERATOR DIFFERENTIAL EQUATIONS OF n-th ORDER WITH n BOUNDARY CONDITIONS(*)

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ABSTRACT. In this paper we study the boundary operator differential problem of a n-th order linear operator differential equation with n boundary operator conditions. Conditions for the existence and uniqueness and an explicit expression of the solution are given.

0. INTRODUCTION.

If we consider the boundary value problem for a n-th order differential equation

$$(0.1)\quad \begin{cases} y^{(n)}(t) + A_{n-1}(t)y^{(n-1)}(t) + \dots + A_o(t)y(t) = f(t) \\ \sum_{j=1}^{n} (N_{ij}^{(1)} y^{(j-1)}(0) + N_{ij}^{(2)} y^{(j-1)}(T)) = 0 \\ \qquad 0 \leqslant t \leqslant T \;,\; 1 \leqslant i \leqslant n \end{cases}$$

where the given function f and the unknown y are vector-functions with values in $\mathbf{C}^m$ and the coefficients $A_o(t),\dots,A_{n-1}(t)$ are $m \times m$ matrices with entries that are integrable on $[0,T]$, and for i,j the matrices $N_{ij}^{(1)}$ and $N_{ij}^{(2)}$ appearing in the boundary conditions are constant $m \times m$ matrices. The boundary conditions in (0.1) are said to be well set if the corresponding homogeneous equation has the trivial solution only, and in this case the solution of (0.1) can be written in the form

$$y(t) = \int_0^T G(t,s)f(s)ds \quad , \quad 0 \leqslant t \leqslant T$$

where $G(t,s)$ is the Green's function of the equation (0.1), see [3], chapter 7.

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This paper is concerned with the infinite-dimensional analogous problem, that is, $A_i(t)$ is a bounded linear operator on a complex Hilbert space X, for all t in $[0,T]$ and $0 \leqslant i \leqslant n-1$.
We assume that $N_{ij}^{(i)}$, $i = 1,2$, $1 \leqslant i,j \leqslant n$ are bounded linear operators on X and f is a Bochner integrable function with values in X, that we can suppose continuous in order to obtain solutions everywhere defined in $[0,T]$.
For convenience we define the following operators that will be used in the following

$$A(t) = \begin{bmatrix} 0 & I & \cdots & 0 \\ 0 & 0 & \cdots & \vdots \\ \vdots & & & I \\ -A_o(t) & -A_1(t) & \cdots & -A_{n-1}(t) \end{bmatrix} \quad ; \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix}$$

$$C = [I \ 0 \ \cdots \ 0] \quad ; \quad N_\nu = \begin{bmatrix} N_{11}^{(\nu)} & \cdots & N_{1n}^{(\nu)} \\ \vdots & & \\ N_{n1}^{(\nu)} & \cdots & N_{nn}^{(\nu)} \end{bmatrix} \ , \quad \nu = 1,2$$

1. PROOFS AND RESULTS.

We recall that for the evolution equation $(du/dt)(t) = A(t)u(t)$ on the interval $[0,T]$, if $A(.)$ is a continuous function, generator of a fundamental operator $U_A(t,s)$, defined on the triangle $\Delta = \{(t,s) \ ; \ 0 \leqslant s \leqslant t \leqslant T\}$, such that if $A(t) \in L(H,H)$, then $U_A(t,s) \in L(H,H)$, strongly continuous jointly in s,t and that satisfies:

(i) The partial derivative $\partial U_A(t,s)/\partial t$ exists in the strong topology, belongs to $L(H,H)$ for $(s,t) \in \Delta$, and is strongly continuous in t.

(ii) $\partial U_A(t,s)/\partial t = A(t)U_A(t,s)$, $U_A(t,t) = I$ for (s,t) in Δ.

(iii) $U_A(t,s)U_A(s,u) = U_A(t,u)$, for $0 \leqslant u \leqslant s \leqslant t \leqslant T$.

When $U_A(t,s)$ is invertible for (s,t) in Δ, we can extend $U_A(\cdot,\cdot)$ to the rectangle $[0,T] \times [0,T]$, in the following way $U_A(s,t) = (U_A(t,s))^{-1}$ for $0 \leqslant s \leqslant t \leqslant T$. In this case it follows that $\partial U_A(s,t)/dt = -U_A(s,t)A(t)$ and the property (iii) is verified for

$0 \leqslant t \leqslant s \leqslant u \leqslant T$. Several different conditions on $A(t)$ in order to ensure that $\{A(t)\}$ generates a fundamental operator can be found in [7], [8], [9], [11], and [12].

THEOREM 1. *Let us consider the boundary problem* (0.1) *where* $A_i(\ )$ *are continuous functions with values in* $L(X,X)$, f *is a continuous* X *valued function and* $N_{ij}^{(1)}$, $N_{ij}^{(2)}$ *are operators in* $L(X,X)$. *If* $\{A(t)\}$ *is generator of an invertible fundamental operator* $U_A(t,s)$ *such that*

$$N_1 + N_2U_A(T,0) \quad \textit{is invertible} \tag{1.1}$$

and we denote

$$P = (N_1+N_2U_A(T,0))^{-1}N_2U_A(T,0)$$

$$G(t,s) = \begin{cases} C\ U_A(t,0)(I-P)U_A(0,s)B & , \quad 0 \leqslant s \leqslant t \leqslant T \\ -C\ U_A(t,0)PU_A(0,s)B & , \quad 0 \leqslant t < s < T \end{cases}$$

then the only solution of (0.1) *is given by*

$$y(t) = \int_0^T G(t,s)f(s)ds \quad , \quad 0 \leqslant t \leqslant T \tag{1.2}$$

Proof. Given the problem (0.1) and taking

$$x(t) = \begin{bmatrix} y(t) \\ \vdots \\ y^{(n-1)}(t) \end{bmatrix}$$

it is clear that this problem is equivalent to the following boundary problem

$$\begin{cases} \dot{x}(t) = A(t)x(t) + Bf(t) \\ y(t) = C\ x(t) \\ N_1x(0) + N_2x(T) = 0 \end{cases} \tag{1.3}$$

From the hypothesis imposed on $\{A(t)\}$, the Cauchy problem associated with the differential equation in (1.3) has only one solution for a given value $x(0)$ in $t=0$, and this solution is given by

$$x(t) = U_A(t,0)x(0) + \int_0^t U_A(t,s)Bf(s)ds$$

see [11], p.19 for details. We define now the following other function

$$x(t) = U_A(t,0)(I-P)\int_0^t U_A(0,s)Bf(s)ds - U_A(t,0)P\int_t^T U_A(0,s)Bf(s)ds$$

It follows that

$$x(0) = -P\int_0^T U_A(0,s)Bf(s)ds$$

$$x(T) = U_A(T,0)(I-P)\int_0^T U_A(0,s)Bf(s)ds$$

(1.4) $$(I-P)x(0)+P\,U_A(0,T)x(T) = 0$$

From (1.1) and the definition of P one gets

$$\begin{aligned} I-P &= I - (N_1+N_2U_A(T,0))^{-1}N_2U_A(T,0) = \\ &= (N_1+N_2U_A(T,0))^{-1}\{N_1+N_2U_A(T,0)-N_2U_A(T,0)\} = \\ &= (N_1+N_2U_A(T,0))^{-1}N_1 \end{aligned}$$

Thus the boundary condition in (1.3) is equivalent to (1.4). From the definition of x(t) and by differentiation it follows

$$\begin{aligned} \dot{x}(t) &= A(t)U_A(t,0)(I-P)\int_0^t U_A(0,s)Bf(s)ds + U_A(t,0)(I-P)U_A(0,t)Bf(t) - \\ &- A(t)U_A(t,0)P\int_t^T U_A(0,s)Bf(s)ds + U_A(t,0)PU_A(0,t)Bf(t) = A(t)x(t)+f(t) \end{aligned}$$

From (1.3) it follows that $y(t) = Cx(t)$. From here and the expression which defines x(t), the result is concluded.

EXAMPLE 1. If we consider the problem (0.1) where $A(t) = A \in L(X,X)$ for all t in [0,T], being X a complex Hilbert space, then A is generator of the invertible fundamental operator

$$U_A(t,s) = \exp((t-s)A) \quad , \quad 0 \leqslant s \leqslant t \leqslant T$$

Thus the condition (1.1) takes the form

$$N_1+N_2\exp(A\,T) \quad \text{is invertible}$$

EXAMPLE 2. For the infinite-dimensional case, if A(t) satisfies the property (See [6], p.600)

$$A(t)\left(\int_0^t A(s)ds\right) = \left(\int_0^t A(s)ds\right)A(t)$$

or

$$A(t_1)A(t_2) = A(t_2)A(t_1)$$

for all t, t_1, t_2 in [0,T], then it is easy to show that

$$U_A(t,s) = \exp\left(\int_0^{t-s} A(u)du\right) , \quad 0 \leq s \leq t \leq T$$

is an invertible fundamental operator generated by $\{A(t)\}$.
The condition (1.1) takes the form

$$N_1 + N_2 \exp\left(\int_0^T A(s)ds\right) \quad \text{is invertible}$$

EXAMPLE 3. With the hypothesis of theorem 1, if $X = \mathbf{C}^m$, the matrix $A(t)$ generates an invertible fundamental operator $U_A(t,s)$ defined by the transition state matrix of the linear system

$$(du/dt)(t) = A(t)u(t)$$

See [2] , p.22.

The results of theorem 1 and the examples are related with [1], [5] and [10].

The following result is concerned with the existence problem of T-periodic solutions of the differential equation of (0.1), when $A_i(t+T) = A_i(t)$ for all t in $[0,\infty[$, and $0 \leq i \leq n-1$, and $f(t) = f(t+T)$ in this interval.

COROLLARY 1. Let us consider the operator differential equation

$$(1.5) \qquad y^{(n)}(t) + A_{n-1}(t)y^{(n-1)}(t) + \ldots + A_0(t)y(t) = f(t)$$

where $f: [0,\infty[\dashrightarrow X$, and $A_i: [0,\infty[\dashrightarrow L(X,X)$, are T-periodic continuous functions. If $\{A(t)\}$ is generator of an invertible fundamental operator $U_A(t,s)$, such that

$$(1.6) \qquad 1 \notin \sigma(U_A(T,0))$$

where $\sigma(U_A(T,0))$ denotes the spectrum of the operator $U_A(T,0)$ then there exists only one T-periodic solution of the equation (1.5) that is given by (1.2).

Proof. If we consider the problem (0.1) where $N_{ij}^{(1)} = -N_{ij}^{(2)} = I$ for $i=j$ and $N_{ij}^{(1)} = N_{ij}^{(2)} = 0$, for $i \neq j$, that is $N_1 = -N_2 = I$, then from the hypothesis (1.6), the hypothesis (1.1) of theorem 1 is satisfied. From here, the only solution of our boundary problem is T-periodic, and it is given by (1.2).

COROLLARY 2. Let us consider the equation (1.5) with $A_i(t) = A_i$ for all t in $[0,\infty[$, and f continuous and T-periodic. If the spectrum of A, $\sigma(A)$ is disjoint with the set $\{(2k\pi i)/T\ ;\ k \text{ integer}\}$, then there exists only one T-periodic solution of (1.5), and this solution is given by (1.2), taking $U_A(t,s) = \exp((t-s)A)$ in the expression which defines $G(t,s)$.

Proof. Taking $U_A(t,s)$ in the corollary 1, the result is concluded if the condition (1.6) is satisfied. Moreover, (1.6) is equivalent to the condition

$$(1.7) \qquad 1 \notin \sigma(\exp(T\,A))$$

From the spectral mapping theorem, [4], p.569, this condition is equivalent to say that $\exp(T\lambda) \neq 1$, for all λ in the spectrum of A. Thus the result is proved.

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