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H. AMANN'S SADDLE POINT REDUCTION AND FIXED POINT OF SYMPLECTIC DIFFEOMORPHISMS OF THE TORUS

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In this paper we study the existence of fixed points of symplectic diffeomorphisms of the torus T^{2N} which are homologous to the identity (cf. app. 9 of [3]). It is known that such a diffeomorphism has at least N+1 geometrically different fixed points if it is not too far from the identity (cf. [6]). The main theorem here shows that the latter condition can be dropped.

At this point it should be mentioned that, as V.I.Arnold pointed out (app. 9 of [3]), this result gives another proof of H.Poincaré's geometric theorem.

In order to state the theorem, we recall first a definition.

Let $g: T^{2N} \to T^{2N}$ be a symplectic deffeomorphism of class C^1. We say that g *is homologous to the identity* if it can be connected to the identity diffeomorphism by a C^1- curve g_t consisting of symplectic diffeomorphisms such that the field of velocities g_t at each moment of time t has a single-valued hamiltonian function. More precisely, there exists a function $K: T^{2N} \times [0,1] \to \mathbf{R}$ of class C^2, such that, if

$$\phi_K: T^{2N} \times [0,1] \to T^{2N}$$

is the flow of the hamiltonian vector field X_K with initial conditions $(x,0)$, $x \in T^{2N}$, then

$$\phi_K(x,1) = g(x) \quad , \quad x \in T^{2N} .$$

Note that we do not impose a periodicity condition on K.

THE MAIN THEOREM. *Such a diffeomorphism has at least* N+1 *geometrically different fixed points.*

The strategy of the proof of this theorem is roughly as follows. In a standard way, the existence problem of periodic solution of the hamiltonian system

$$\dot{q} = K_p(q,p,t) \quad , \quad \dot{p} = -K_q(q,p,t) \quad , \quad \{q,p\} \in \mathbf{R}^N \times \mathbf{R}^N$$

(we consider K defined in $\mathbf{R}^{2N}$ in a natural way), is transformed in an existence problem of solution of an equation of the form

$$Au = F(u)$$

in a real Hilbert space H, where A is a selfadjoint linear operator, and F is a potential operator, mapping H continuously into itself.

In [1], H.Amann shows that, with certain assumptions on the operator F, it can only interact with finitely many eigenvalues of A. Roughly speaking, this means that the original problem is reduced to the study of critical points of a functional a defined in a finite-dimensional space that, in this particular case, is of the form $\mathbf{R}^m \times \mathbf{R}^{2N}$.

Next, taking advantage of the symmetries of the problem (periodicity in $\{q,p\}$), the functional a can be considered as defined in $\mathbf{R}^m \times T^{2N}$. Finally, one can show with the aid of categorical arguments that a has at least $\mathrm{cat}(T^{2N}) = N+1$ different critical points that are solution of our problem.

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§ 1. H. AMANN'S SADDLE POINT REDUCTION.

In this section we follow almost word by word the exposition in [2].

(1.1). *The basic hypotheses.* H is a real Hilbert space with inner product $<.,.>$, and we identify H with its dual.

(A) $\begin{cases} \text{A: dom(A)} \subset H \to H \text{ is a self-adjoint linear operator of discrete spectrum } \sigma(A). \\ \text{There exist numbers } \alpha < 0 < \beta \text{ such that } \alpha,\beta \notin \sigma(A) \text{ and} \\ \sigma(A) \cap (\alpha,\beta) \text{ consists of at most finitely many eigenvalues of finite multiplicity.} \end{cases}$

We denote by $\lambda_1 < \lambda_2 < \ldots < \lambda_n$ the eigenvalues of A in (α,β), and by $m(\lambda_j)$ the multiplicity of λ_j.

(F) $\begin{cases} \text{F: } H \to H \text{ is a continuous potential operator such that} \\ \alpha\|u-v\|^2 \leqslant \langle F(u)-F(v),\, u-v\rangle \leqslant \beta\|u-v\|^2 \quad \forall\, u,v \in H. \end{cases}$

We denote the potential of A by Φ, that is, $\Phi \in C^1(H,\mathbf{R})$ and $\Phi' = F$.

Let $\{E_\lambda \mid \lambda \in \mathbf{R}\}$ be the spectral resolution of A; $P_\pm$, $P \in L(H)$ the orthogonal projections defined by

$$P_- := \int_{-\infty}^{\alpha} dE_\lambda \quad , \quad P_+ := \int_{\beta}^{\infty} dE_\lambda \quad , \quad P := \int_{\alpha}^{\beta} dE_\lambda \quad ,$$

respectively, and $X := P_-(H)$, $Y := P_+(H)$, $Z := P(H)$.

It is clear that $H = X \oplus Y \oplus Z$, and Z is finite-dimensional with

$$\dim Z = \sum_{j=1}^{n} m(\lambda_j).$$

Next, one defines self-adjoint linear operators $R \in L(H,X)$, $S \in L(H,Y)$, $T \in L(H,Z)$ by

$$R := \int_{-\infty}^{\alpha} (\alpha-\lambda)^{-\frac{1}{2}} dE_\lambda \quad , \quad S := \int_{\beta}^{\infty} (\lambda-\alpha)^{-\frac{1}{2}} dE_\lambda$$

$$T := \int_{\alpha}^{\beta} (\lambda-\alpha)^{-\frac{1}{2}} dE_\lambda = \sum_{j=1}^{n} (\lambda_j-\alpha)^{-\frac{1}{2}} P_j \quad ,$$

respectively, where P_j denotes the orthogonal projection of H onto the eigenspace $\ker(\lambda_j I-A)$ of λ_j.

It is an immediate consequence of these definitions that R,S and T are pairwise commuting, $R|X$, $S|Y$, and $T|Z$ are injective, and $-R^2 + S^2 + T^2 = (A-\alpha I)^{-1}$. Moreover $-(A-\alpha I)R^2 = P_-$, $(A-\alpha I)S^2 = P_+$, $(A-\alpha I)T^2 = P$.

(1.2) *Saddle point reduction.* We let

$$\tilde{\Phi}(u) = \Phi(u) - \frac{\alpha}{2}\|u\|^2 \qquad \forall\ u \in H \ ,$$

and we define $\quad f \in C^1(X \times Y \times Z , \mathbf{R})$ by

$$f(x,y,z) := \frac{1}{2}(\|x\|^2 - \|y\|^2 - \|z\|^2) + \tilde{\Phi}(Rx+Sy+Tz)$$

It is immediately verified that (x,y,z) is a critical point of f iff $Rx+Sy+Tz$ is a solution of $Au = F(u)$. Moreover,

(1.3) PROPOSITION. *There exists a globally Lipschitz continuous map* $(x(.),y(.)): Z \to X \times Y$ *such that* $(x(z),y(z))$ *is the unique saddle point of* $f(.,.,z): X \times Y \to \mathbf{R}$ *for every* $z \in Z$. *Thus the point* $(x(z),y(z)) \in X \times Y$ *is characterized by the "saddle point inequalities"*

(1.4) $f(x(z),y,z) < g(z) < f(x,y(z),z) \qquad (x,y,z) \in X \times Y \times Z$,

where $g(z) := f(x(z),y(z),z) \quad \forall\, z \in Z$

as well as by the fact that $(x(z),y(z))$ *is, for every* $z \in Z$, *the unique point* $(x,y) \in X \times Y$ *solving the system*

(1.5) $$0 = x + R\tilde{F}(Rx+Sy+Tz)$$

(1.6) $$0 = -y + S\tilde{F}(Rx+Sy+Tz)$$

where $\tilde{F}(u) := F(u) - \alpha u \quad \forall\, u \in H.$

Moreover, g has a globally Lipschitz continuous derivative $g':Z \to Z$, given by $g'(z) = -z + T\tilde{F}(Rx(z) + Sy(z) + Tz) \quad z \in Z.$

Finally, z is a critical point of g iff $Rx(z) + Sy(z) + Tz$ is a solution of $Au = F(u)$.

Now, we set $T^{-1} := (T|Z)^{-1} \in L_s(Z) := \{B \in L(Z) | B = B^*\}$, $a := -g \circ T^{-1} \in C^1(Z,\mathbf{R})$ and $u(z) := v(z) + z$ where

(1.7) $$v(z) := Rx(T^{-1}z) + Sy(T^{-1}z) \quad \forall\, z \in Z.$$

Then, by (1.6), $u(.): Z \to H$ is Lipschitz continuous, and $u(z) \in \mathrm{dom}(A) \quad \forall\, z \in Z.$

Moreover, a has a globally Lipschitz continuous derivative, given by $a' = -T^{-1} \circ g' \circ T^{-1}$ and z is a critical point of a iff u(z) is a solution of $Au = F(u)$.

(1.8) LEMMA (cf. [1], [2]). *For every* $z \in Z$

$$a(z) = \frac{1}{2} \langle Au(z),u(z)\rangle - \Phi(u(z))$$

or $$a(z) = \frac{1}{2} \langle Az,z\rangle + \frac{1}{2} \langle Av(z),v(z)\rangle - \Phi(u(z))$$

and $$a'(z) = Az - PF(u(z)).$$

In the following lemma we state a technical remark, which will be useful.

(1.9) LEMMA. *If* F *is globally bounded, i.e. there exists* $M > 0$ *such that* $\|F(u)\| \leq M < +\infty \quad \forall\, u \in M$, *then the mapping* v *defined in* (1.7) *is also globally bounded.*

Proof. The equations (1.5) and (1.6) characterizing $(x(z),y(z))$, $z \in Z$, can be rewritten as

$$0 = (I - \alpha R^2)x(z) + RF(Rx(z) + Sy(z) + Tz)$$

$$0 = (-I - \alpha S^2)y(z) + SF(Rx(z) + Sy(z) + Tz)$$

Therefore, there exists a constant $C > 0$ such that

$$\|(I-\alpha R^2)x(z)\| , \|(-I-\alpha S^2)y(z)\| \leq C \quad \forall z \in Z$$

Now, it is easily verified that $(I-\alpha R^2)|X$ and $(-I-\alpha S^2)|Y$ are linear isomorphisms, and the assertion follows immediately.

§ 2. REDUCTION UNDER SYMMETRY ASSUMPTIONS.

We assume henceforth that $0 \in \sigma(A)$ and we let $Z = Z_0 \oplus Z_1$ where $Z_0 := \ker(A) = P_0(H)$.

The definition of T implies that $T|Z_0 = |\alpha|^{-\frac{1}{2}} I$ that is, $T(z_0) = |\alpha|^{-\frac{1}{2}} z_0 = \gamma z_0$, $\forall z_0 \in Z_0$ where $\gamma := |\alpha|^{-\frac{1}{2}}$.

(2.1). Let $z \in Z$. Setting $z = z_0 + z_1$, $z_0 \in Z_0$ and $z_1 \in Z_1$ we have $\langle Az,z\rangle = \langle A(z_0 + z_1) , z_0 + z_1\rangle = \langle Az_1 , z_1\rangle$.

(2.2). Let now $\bar{z}_0 \in Z_0$ be fixed and assume that Φ is invariant under translation by $\bar{z}_0$. More precisely, $\Phi(u+\bar{z}_0) = \Phi(u)$ $\forall u \in H$. It follows that F is also $\bar{z}_0$-invariant, i.e.,

(2.3) $$F(u + \bar{z}_0) = F(u) \quad \forall u \in H$$

The equations that characterize the point

$(x(z + \gamma^{-1} z_0), y(z + \gamma^{-1} z_0))$, $z \in Z$ are

$$0 = x - \alpha R^2 x + RF(Rx + Sy + Tz + \bar{z}_0)$$
$$0 = -y - \alpha S^2 y + SF(Rx + Sy + Tz + \bar{z}_0)$$

because $T(\gamma^{-1} \bar{z}_0) = \bar{z}_0$. Hence, from (2.3), the latter equations can be written

$$0 = x - \alpha R^2 x + RF(Rx + Sy + Tz)$$
$$0 = -y - \alpha S^2 y + SF(Rx + Sy + Tz)$$

Consequently,

(2.4) $$(x(z + \gamma^{-1} \bar{z}_0) , y(z + \gamma^{-1} \bar{z}_0)) = (x(z),y(z)) \quad z \in Z.$$

Moreover, from (1.7) and (2.4), for every $z \in Z$,

$$v(z + \bar{z}_0) = Rx(T^{-1}(z + \bar{z}_0) + Sy(T^{-1}(z + \bar{z}_0)) =$$

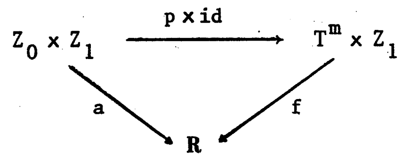
$= Rx(T^{-1}z + \gamma^{-1}\bar{z}_0) + Sy(T^{-1}z + \gamma^{-1}\bar{z}_0) = Rx(T^{-1}z) + Sy(T^{-1}z) = v(z).$

Thus, we have proved that v is invariant under translation by $\bar{z}_0$.

(2.4) Let now $m = \dim Z_0$ and $w_1,\dots,w_m$ be linearly independent vectors in Z_0. The integral linear combinations $n_1.w_1 + \dots + n_m.w_m$ form a lattice Ω in Z_0, and the quotient space $Z_0/_\Omega$ is precisely the m-dimensional torus T^m.

The following result is just a consequence of (2.3).

(2.5) PROPOSITION. *If* $\Phi(u+w) = \Phi(u)$, $\forall\, u \in H$ *and* $\forall\, w \in \Omega$, *there exists a functional* $f \in C^1(T^m \times Z_1, \mathbf{R})$ *such that the diagram*



is commutative, where $p\colon Z_0 \to T^m$ *is the canonical projection. Moreover,* f *has a globally Lipschitz continuous derivative* (*gradient*) f'.

Of course, we consider T^m with its natural Riemannian metric.

§ 3. EXISTENCE OF CRITICAL POINTS OF f.

In this section we give an existence theorem of critical points of f based upon Ljusternik-Schnirelmann theory. The basic assumptions are:

(a): Φ is Ω-invariant as in (2.5).

(b): $|\Phi|$ and $\|F\| = \|\Phi'\|$ are uniformly bounded.

Let $Z_1 = \mathbf{R}^n$. We denote with y (x, respectively) a generic point of $\mathbf{R}^n$ (T^m, respectively). Then, the functional f can be written $f(y,x) = \frac{1}{2}\langle Ay,y\rangle + \phi(y,x)$, $(y,x) \in \mathbf{R}^n \times T^m$ where, from (1.7),

$$\phi(y,x) = \frac{1}{2}\langle Av(y,x)\, , v(y,x)\rangle - \Phi(u(y,x)).$$

The linear operator $A \in L(\mathbf{R}^n)$ is symmetric, non-singular and of index (A) = i. Moreover, $\phi \in C^1(\mathbf{R}^n \times T^m, \mathbf{R})$ is, from (1.9), uniformly bounded. Also, from (1.7), $\phi'(y,x) = -d(\mathrm{id} \times p)\, PF\,(u(y,x))$. It follows that ϕ' is globally Lipschitz continuous and $\|\phi'\|$ is uniformly bounded.

Hence, we can assume that there exists a constant $N > 0$ such that

$$\max \{|\phi(y,x)|, \|\phi'(y,x)\|\} \leq N \ , \ (x,y) \in \mathbf{R}^n \times T^m.$$

Finally, we assume that index $(A) = i < n$.

(3.1) THEOREM. *The number of geometrically different critical points of* f *is at least* m+1.

We begin with some preparations. First, it can be observed that f satisfies Palais-Smale condition (C) because

$$\|f'(y,x)\| \geq \|Ay\| - \|\Phi'(y,x)\| \geq \|Ay\| - N \to +\infty$$

when $\|y\| \to +\infty$. Hence, the set K := critical points of f, is compact. Of course, we have assumed that $n > 0$, because in case n=0 the theorem is a known fact.

In the following, we denote z a generic point of $\mathbf{R}^n \times T^m$ and for $r \in \mathbf{R}$, we set

$$f^r := \{z \in \mathbf{R}^n \times T^m | f(z) \leq r\}$$

Moreover, the term isotopy always refers to a Lipschitz continuous isotopy. With obvious reduction on smoothness, the following results are proved in (6.5.5) and (6.6.2) of [4].

(3.2) PROPOSITION. *Let* $c > b \geq a$ *be real numbers. If* f *has no critical values on the interval* $[b,c]$, *then the sets* f^c *and* f^b *are isotopic. Furthermore, the isotopy may be so chosen that the points of* f^a *are fixed.*

(3.3) PROPOSITION. *Let* c *be an isolated critical value of* f *and* $a < c$, $a,c \in \mathbf{R}$. *Then for* $\varepsilon > 0$ *sufficiently small there is a neighborhood* U *of the set* $K_c := \{z \in \mathbf{R}^n \times T^m | f(z) = c, f'(z) = 0\}$ *and a deformation* $\{\zeta_t\}$ *of* $f^{c+\varepsilon} \setminus U$ *such that* $\zeta_1(f^{c+\varepsilon} \setminus U) \subseteq f^{c-\varepsilon}$. *The deformation may be so chosen that the points of* f^a *are fixed.*

Now, let $a < 0 < b$ real numbers such that, (3.4): f has no critical points on $\mathbf{R}^n \times T^m \setminus f^b$; (3.5): f has no critical points on f^a if index $(A) > 0$, or $f^a = \emptyset$ if index $(A) = 0$. It is clear that such numbers exist.

We set $X := f^b/_{f^a}$ ($X := f^b$ in case $f^a = \emptyset$) and $\tau: f^b \to X$ the canonical projection. In order to estimate the number of critical points of f in $f^b \setminus f^a$, we use an integer-valued function defined on the class of subsets of f^b. We define

$$n(\emptyset) = 0 \quad , \quad n(B) = \text{cat}_X [\tau(B)] \text{ if } B \subset f^b \text{ and } B \neq \emptyset.$$

In the following lemma we collect the properties of n which will be useful and that are easily proved.

(3.6) LEMMA.

(i) $n(B) = 1$ *if* B *is a point of* f^b.

(ii) $n(B) \geqslant n(C)$ *if* $B \supset C$.

(iii) $n(B \cup C) \leqslant n(B) + n(C)$.

(iv) $n(B) = n(B_t)$ *where* B_t *is isotopic to* B *by an isotopy that leaves fixed the set* f^a.

(v) *There is a neighborhood* U *of* B, *for every* B, *such that* $n(U) = n(B)$.

(3.7) LEMMA. *Let* c, $a < c < b$ *be an isolated critical value of* f. *Then for* $\varepsilon > 0$ *sufficiently small* $n(K_c) \geqslant n(f^{c+\varepsilon}) - n(f^{c-\varepsilon})$.

Proof. Let U be a neighborhood of K_c such that (3.3) and (3.6.v) are valid, then $n(f^{c-\varepsilon}) \geqslant n(f^{c+\varepsilon} \setminus U)$. Hence

$$n(f^{c+\varepsilon}) = n(f^{c+\varepsilon} \cup U) \leqslant n(f^{c+\varepsilon} \setminus U) + n(U) \leqslant n(f^{c-\varepsilon}) + n(K_c).$$

Now, we can prove

(3.8) THEOREM. *The number of geometrically different critical points of* f *in* f^b *is at least* $n(f^b)-1$ *if* $f^a \neq \emptyset$, *and* $n(f^b)$ *if* $f^a = \emptyset$.

Proof. Without loss of generality, we may suppose that the number of critical points of f is finite. Then f has a finite number of critical values c_i, $i = 0,1,\dots,N$, such that $a < c_0 \leqslant c_1 \leqslant \dots \leqslant c_N < b$, so that we may write $f^b \supset f^{c_N} \supset \dots \supset f^{c_0} \supset f^a$.

From (3.7), for some $\varepsilon > 0$ independent of $i = 1,\dots,N$, it is verified

$$n(K_{c_i}) \geqslant n(f^{c_i+\varepsilon}) - n(f^{c_i-\varepsilon}). \tag{3.9}$$

Adding this last inequality over i, and using the fact that, from (3.2) and (3.6.iv),

$$n(f^{c_i-\varepsilon}) = n(f^{c_{i-1}+\varepsilon}), \quad i = 1,\dots,N$$

we find
$$\sum_{i=0}^{N} n(K_{c_i}) \geqslant n(f^{c_N+\varepsilon}) - n(f^{c_0-\varepsilon}).$$

Also, for the same reason, $n(f^{c_N+\varepsilon}) = n(f^b)$. Finally, for the value of $n(f^{c_0-\varepsilon})$ there are two possibilities:

(a). $f^a = \emptyset$, then f has an absolute minimum in c_0, so that $f^{c_0-\varepsilon} = \emptyset$ and $n(f^{c_0-\varepsilon}) = 0$.

(b). $f^a \neq \emptyset$, then $n(f^{c_0-\varepsilon}) = n(f^a) = 1$ because $\tau(f^a)$ is a point of X, and then, the theorem is proved.

We turn now to the problem of determining $n(f^b)$. Assume first that index (A) = i $\geqslant$ 1. If $\phi \equiv 0$ and $g \in C^1(\mathbf{R}^n,\mathbf{R})$ is defined by $G(y) := \frac{1}{2}\langle Ay,y\rangle$, $y \in \mathbf{R}^n$ then we have

$$H^p(g^b/g^a;\mathbf{R}) = \begin{cases} \mathbf{R} & \text{if } p=i \\ 0 & \text{if } p\neq i \end{cases}$$

As $f^b = g^b \times T^m$ and $f^a = g^a \times T^m$, Kunneth's formulae imply

$$H^j(X;R) = \begin{cases} 0 & \text{if } j < i \\ H^{j-i}(T^m,\mathbf{R}) & \text{if } j \geqslant i \end{cases}$$

Therefore, cuplong (X) = m+1 and

(3.10) $\quad n(f^b) = \mathrm{cat}(X) \geqslant \mathrm{cuplong}(X) + 1 = m+2$ (cf. [5]).

In the general case ($\phi \not\equiv 0$), the asymptotic behavior of f and g near infinity are similar. Then, for $|a|$, b sufficiently large, we can hope that f^b/f^a and $g^b \times T^m/g^a \times T^m$ are homeomorphic spaces and (3.10) remains valid. In fact, we can prove

(3.11) THEOREM. *If* $|a|$, b *are sufficiently large, there exists a diffeomorphism of class* C^1 $\psi: \mathbf{R}^n \times T^m \to \mathbf{R}^n \times T^m$ *such that* $\psi(g^b \times T^m) = f^b$ *and* $\psi(g^a \times T^m) = f^a$.

Proof. Fix $N > 0$ such that

(3.12) $$\max\{|\phi(z)|\ ,\ \|\phi'(z)\|\} \leqslant \frac{1}{2}N$$

Then, if $z = (y,x)$, we have

(3.13) $$\langle f(z),g(z)\rangle = \|Ay\|^2 + \langle Ay\ \phi'(y,x)\rangle \geqslant$$
$$\geqslant \|Ay\|^2 - \|A\|N\|y\| \geqslant \|y\|(\|A^{-1}\|^2\|y\| - \|A\|N) \geqslant \frac{1}{2}$$

when $\|y\|$ is sufficiently large, say $\|y\| \geqslant C$.

Now, let c be a real number such that

$$c \geq \frac{1}{2} C^2\|A\| + 4N \quad \text{and set} \quad a := -c \ , \ b := c.$$

When $z = (y,x) \in g^{-1}(a-4N,a+4N) \cup g^{-1}(b-4N,b+4N)$ it is verified that

$$\frac{1}{2} C^2\|A\| < c-4N \leq |g(z)| \leq \frac{1}{2} \|A\| \ \|y\|^2.$$

Thus, $\|y\| > C$. In other words, (3.13) is verified in

$$g^{-1}(a-4N,a+4N) \cup g^{-1}(b-4N,b+4N).$$

On the other hand, (3.12) implies

(3.14) $\quad f^{-1}(b) \subset g^{-1}(a-N,b-N) \quad , \quad f^{-1}(a) \subset g^{-1}(a-N,a+N).$

It is easily seen now that the gradient vector field g' can be used in order to construct a diffeomorphism of class C^1

$$\alpha_b: g^{-1}(b-4N,b+4N) \to g^{-1}(b) \times (b-4N,b+4N)$$

such that $g = p_2 \circ \alpha_b$, where p_2 is projection onto the second factor. Conditions (3.13) and (3.14) imply that $\alpha_b(f^{-1}(b))$ is the graph in $g^{-1}(b) \times (b-4N,b+4N)$ of a function $h: g^{-1}(b) \to (b-4N,b+4N)$ of class C^1, which verifies $|h(s)-b| \leq N \quad \forall \ s \in g^{-1}(b)$.

Now, let $\lambda_b: (b-4N,b+4N) \to [0,1]$ be a function of class C^1 such that

$$\lambda_b(b) = 1 \quad \text{and} \quad \text{supp}(\lambda_b) \subseteq [b-3N,b+3N] \ ,$$

$$|\lambda_b(t)| \leq \frac{1}{2} N \qquad \forall \ t \in (b-4N,b+4N).$$

It is easily verified that the mapping β_b defined by

$\beta_b(s,t) = (s,t + \lambda_b(t) \ |h(s) - b|) \ , \quad (s,t) \in g^{-1}(b) \times (b-4N,b+4N)$

is a diffeomorphism of class C^1 of $g^{-1}(b) \times (b-4N,b+4N)$ onto itself. Moreover, β_b maps $g^{-1}(b) \times \{b\}$ onto $f^{-1}(b)$ and $\beta_b(s,t) = (s,t)$ if $|t - b| \geq 3N$.

Now, setting $\psi_b = \beta_b \circ \alpha_b$, we obtain a diffeomorphism of $g^{-1}(b-4N,b+4N)$ onto itself mapping $g^{-1}(b)$ onto $f^{-1}(b)$. Analogously we construct α_a, β_a and ψ_a. Finally we define ψ by the formulae

$$\psi(z) = \begin{cases} \psi_b(z) & \text{if} \quad z \in g^{-1}(b-4N,b+4N) \\ \psi_a(z) & \text{if} \quad z \in g^{-1}(a-4N,a+4N) \\ z & \text{in the complementary case} \end{cases}$$

and the assertion follows immediately.

In a similar way it is proved that, when index (A) = 0 and consequently, $f^a = \emptyset$ for a sufficiently large, the estimate $n(f^b) \geq m+1$ is valid. Finally, if we collect all these preliminary results, we have proved theorem (3.1).

(3.15). COROLLARY. *Let* a *and* f *be as in* (2.5). *Then, there exist at least* m+1 *critical points of* a, $z_1,\dots,z_{m+1}$, *such that*

$$P_0(z_i) - P_0(z_j) \notin \Omega \quad \forall\, i,j = 1,\dots,m+1, \quad \textit{and} \quad i \neq j.$$

§ 4. THE PROOF OF THE MAIN THEOREM.

We denote a generic point of $\mathbf{R}^{2N} = \mathbf{R}^N \times \mathbf{R}^N$ by $x := \{q,p\}$, where $q,p \in \mathbf{R}^N$, and let $\gamma: \mathbf{R}^{2N} \to T^{2N}$ the canonical projection defined by $\gamma(\{q,p\}) = (e^{2\pi i q_1},\dots,e^{2\pi i q_N}, e^{2\pi i p_1},\dots, e^{2\pi i p_N})$.

Let $\tilde{K}: T^{2N} \times [0,1] \to \mathbf{R}$ be a function of class C^2 and $K: \mathbf{R}^{2N} \times [0,1] \to \mathbf{R}$ the function defined by $K(x,t) = \tilde{K}(\gamma(x),t)$, $\forall\, (x,t) \in \mathbf{R}^{2N} \times [0,1]$.

Then, it is clear that there exists a constant $C > 0$ such that

$$\max_{(x,t)} \{|K(x,t)|, \|K_x(x,t)\|, \|K_{xx}(x,t)\|\} \leq C. \tag{4.1}$$

Moreover, K is invariant under translation by elements of the lattice

$$\Omega = \{\sum_{i=1}^{2N} n_i e_i \mid n_i \in \mathbf{Z}\}$$

where $\{e_i\}$ is the canonical basis of $\mathbf{R}^{2N}$.

We consider the existence problem of periodic solutions of the hamiltonian system

$$\dot{q} = K_p(q,p,t), \quad \dot{p} = -K_q(q,p,t) \tag{4.2}$$

Setting $H := L_2([0,1];\mathbf{R}^{2N})$, we define a linear operator

$$A: \mathrm{dom}(A) \subseteq H \to H$$

$\mathrm{dom}(A) := \{u \in H^1([0,1];\mathbf{R}^{2N}) \mid u(0) = u(1)\}$, and $Au := -J\dot{u} = \{\dot{p},-\dot{q}\}$

where $$J := \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$$

is the standard symplectic structure on $\mathbf{R}^{2N}$.

(4.3) LEMMA (cf. [1]).

(i) A *is self-adjoint, has closed range and a compact resolvent.*

(ii) $\sigma(A) = 2\pi\mathbf{Z}$, *and each* $\lambda \in \sigma(A)$ *is an eigenvalue of multiplicity* 2N.

(iii) *For each* $\lambda \in \sigma(A)$, *the eigenspace* $\ker(\lambda I - A)$ *is spanned by the orthogonal basis* $t \to (\cos 2\pi\lambda t)e_k + (\operatorname{sen} 2\pi\lambda t)Je_k$, $k = 1,\dots,2N$.

In particular, $\ker(A) = \mathbf{R}^{2N}$, that is, it consists of the constant functions.

Now, $F: H \to H$ is defined by $F(u)(t) := K_x(u(t),t)$, $t \in [0,1]$ and $u \in H$.

The assumptions imply that F is a continuous potential operator on H, the potential Φ being given by

$$\Phi(u) = \int_0^1 K(u(t),t)\, dt \quad \forall\ u \in H.$$

Clearly, classic periodic solutions of (4.2) are precisely the solutions $u \in \operatorname{dom}(A)$ of equation $Au = F(u)$. Moreover, (4.1) and the mean value theorem imply that there exist constants $\alpha,\beta \in \mathbf{R}$ such that hypothesis (F) of (1.1) is satisfied. Without loss of generality, we suppose $\alpha < 0 < \beta$ and $|\alpha|, \beta > 2\pi$. Then,

(4.4) $\sigma(A) \cap (\alpha,0)$ and $\sigma(A) \cap (0,\beta)$ are non vacuous.

Therefore, $Z = \mathbf{R}^{2N} \oplus \mathbf{R}^n$ where $\mathbf{R}^{2N} = \ker(A)$ and $\mathbf{R}^n$ is the direct sum of the eigenspaces $\ker(\lambda I - A)$, $\lambda \in (\alpha,\beta) \cap \sigma(A)$ and $\lambda \neq 0$. Moreover, $A := A|\mathbf{R}^n$ is non-singular and, from (4.4), $0 < \operatorname{index}(A) < n$.

On the other hand, if $w \in \Omega \subset \mathbf{R}^{2N}$, then

$$\Phi(u+w) = \int_0^1 K(u(t)+w,t)\, dt = \int_0^1 K(u(t),t)\, dt = \Phi(u)$$ because K is Ω-periodic.

We have seen that all conditions in order to apply theorem (3.1) and its corollary (3.15) are verified. Consequently, there exist at least 2N+1 periodic mappings $u_1,\dots,u_{2N+1}$ that are classic solutions of (4.2) such that $P_0(u_1),\dots,P_0(u_{2N+1})$ are pairwise inequivalent module the lattice Ω.

This last condition implies clearly that $\gamma\circ u_1,\ldots,\gamma\circ u_{2N+1}$ are different periodic solutions of the hamiltonian vector field X_K, and the proof of the main theorem is finished.

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