H. AMANN'S SADDLE POINT REDUCTION AND FIXED POINT OF SYMPLECTIC DIFFEOMORPHISMS OF THE TORUS

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In this paper we study the existence of fixed points of symplectic diffeomorphisms of the torus $T^{2N}$ which are homologous to the identity (cf. app. 9 of [3]). It is known that such a diffeomorphism has at least $N+1$ geometrically different fixed points if it is not too far from the identity (cf. [6]). The main theorem here shows that the latter condition can be dropped.

At this point it should be mentioned that, as V.I. Arnold pointed out (app. 9 of [3]), this result gives another proof of H. Poincaré's geometric theorem.

In order to state the theorem, we recall first a definition. Let $g: T^{2N} \to T^{2N}$ be a symplectic diffeomorphism of class $C^1$. We say that $g$ is homologous to the identity if it can be connected to the identity diffeomorphism by a $C^1$-curve $g_t$ consisting of symplectic diffeomorphisms such that the field of velocities $g_t$ at each moment of time $t$ has a single-valued hamiltonian function. More precisely, there exists a function $K: T^{2N} \times [0,1] \to \mathbb{R}$ of class $C^2$, such that, if

$$\phi_K: T^{2N} \times [0,1] \to T^{2N}$$

is the flow of the hamiltonian vector field $X_K$ with initial conditions $(x,0)$, $x \in T^{2N}$, then

$$\phi_K(x,1) = g(x), \quad x \in T^{2N}.$$ 

Note that we do not impose a periodicity condition on $K$.

**THE MAIN THEOREM.** Such a diffeomorphism has at least $N+1$ geometrically different fixed points.

The strategy of the proof of this theorem is roughly as follows. In a standard way, the existence problem of periodic solution of the hamiltonian system
\[ \dot{q} = K_p(q,p,t), \quad p = -K_q(q,p,t), \quad \{q,p\} \in \mathbb{R}^N \times \mathbb{R}^N \]

(we consider \( K \) defined in \( \mathbb{R}^{2N} \) in a natural way), is transformed in an existence problem of solution of an equation of the form

\[ Au = F(u) \]

in a real Hilbert space \( H \), where \( A \) is a selfadjoint linear operator, and \( F \) is a potential operator, mapping \( H \) continuously into itself.

In [1], H. Amann shows that, with certain assumptions on the operator \( F \), it can only interact with finitely many eigenvalues of \( A \). Roughly speaking, this means that the original problem is reduced to the study of critical points of a functional \( \alpha \) defined in a finite-dimensional space that, in this particular case, is of the form \( \mathbb{R}^m \times \mathbb{R}^{2N} \).

Next, taking advantage of the symmetries of the problem (periodicity in \( \{q,p\} \)), the functional \( \alpha \) can be considered as defined in \( \mathbb{R}^m \times T^{2N} \). Finally, one can show with the aid of categorical arguments that \( \alpha \) has at least \( \text{cat}(T^{2N}) = N+1 \) different critical points that are solution of our problem.

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§ 1. H. AMANN'S SADDLE POINT REDUCTION.

In this section we follow almost word by word the exposition in [2].

(1.1). The basic hypothesis. \( H \) is a real Hilbert space with inner product \( \langle . , . \rangle \), and we identify \( H \) with its dual.

\[
\begin{align*}
A: \text{dom}(A) &\subset H \rightarrow H \text{ is a self-adjoint linear operator of discrete spectrum } \sigma(A). \\
\text{(A)} &\quad \text{There exist numbers } a < 0 < b \text{ such that } a, b \not\in \sigma(A) \text{ and} \\
&\quad \sigma(A) \cap (a, b) \text{ consists of at most finitely many eigenvalues of finite multiplicity.}
\end{align*}
\]

We denote by \( \lambda_1 < \lambda_2 < \ldots < \lambda_n \) the eigenvalues of \( A \) in \( (a, b) \), and by \( m(\lambda_j) \) the multiplicity of \( \lambda_j \).

\[
\begin{align*}
F: H &\rightarrow H \text{ is a continuous potential operator such that} \\
&\quad a \| u - v \|^2 \leq \langle F(u) - F(v), u - v \rangle \leq b \| u - v \|^2 \quad \forall \ u, v \in H.
\end{align*}
\]

We denote the potential of \( A \) by \( \phi \), that is, \( \phi \in C^1(H, \mathbb{R}) \) and \( \phi' = F \).
Let \( \{E_\lambda \mid \lambda \in \mathbb{R}\} \) be the spectral resolution of \( A \); \( P_\pm, P \in L(H) \) the orthogonal projections defined by

\[
P_- := \int_{\infty}^{\alpha} dE_\lambda, \quad P_+ := \int_{\alpha}^{\infty} dE_\lambda, \quad P := \int_{\alpha}^{\infty} dE_\lambda,
\]

respectively, and \( X := P_-(H), Y := P_+(H), Z := P(H) \).

It is clear that \( H = X \oplus Y \oplus Z \), and \( Z \) is finite-dimensional with

\[
dim Z = \sum_{j=1}^{n} m(\lambda_j).
\]

Next, one defines self-adjoint linear operators \( R \in L(H, X), S \in L(H, Y), T \in L(H, Z) \) by

\[
R := \int_{\alpha}^{\infty} (\alpha - \lambda)^{-1} dE_\lambda, \quad S := \int_{\infty}^{\alpha} (\lambda - \alpha)^{-1} dE_\lambda
\]

\[
T := \int_{\alpha}^{\infty} (\lambda - \alpha)^{-1} dE_\lambda = \sum_{j=1}^{n} (\lambda_j - \alpha)^{-1} P_j,
\]

respectively, where \( P_j \) denotes the orthogonal projection of \( H \) onto the eigenspace \( \ker(\lambda_j I - A) \) of \( \lambda_j \).

It is an immediate consequence of these definitions that \( R, S \) and \( T \) are pairwise commuting, \( R|X, S|Y, \) and \( T|Z \) are injective, and

\[
-R^2 + S^2 + T^2 = (A - \alpha I)^{-1}.
\]

Moreover, \(-(A - \alpha I)R^2 = P_-, (A - \alpha I)S^2 = P_+, (A - \alpha I)T^2 = P\).

(1.2) Saddle point reduction. We let

\[
\tilde{\varphi}(u) = \varphi(u) - \frac{\alpha}{2} \|u\|^2 \quad \forall \ u \in H,
\]

and we define \( f \in C^1(X \times Y \times Z, \mathbb{R}) \) by

\[
f(x, y, z) := \frac{1}{2} (\|x\|^2 - \|y\|^2 - \|z\|^2) + \tilde{\varphi}(Rx + Sy + Tz)
\]

It is immediately verified that \((x, y, z)\) is a critical point of \( f \) iff \( Rx + Sy + Tz \) is a solution of \( Au = F(u) \). Moreover,

(1.3) PROPOSITION. There exists a globally Lipschitz continuous map \((x(\cdot), y(\cdot)) : Z \times X \times Y \) such that \((x(z), y(z))\) is the unique saddle point of \( f(\cdot, \cdot, z) : X \times Y \times \mathbb{R} \) for every \( z \in Z \). Thus the point \((x(z), y(z)) \in X \times Y \) is characterized by the "saddle point inequalities"
186

(1.4) \[ f(x(z),y,z) \leq g(z) \leq f(x,y,z), \quad (x,y,z) \in X \times Y \times Z, \]

where \[ g(z) := f(x(z),y(z),z), \quad \forall z \in Z \]

as well as by the fact that \((x(z),y(z))\) is, for every \(z \in Z\), the unique point \((x,y) \in X \times Y\) solving the system

(1.5) \[ 0 = x + rF(Rx + Sy + Tz) \]
(1.6) \[ 0 = -y + sF(Rx + Sy + Tz) \]

where \[ \tilde{F}(u) := F(u) - au, \quad \forall u \in H. \]

Moreover, \(g\) has a globally Lipschitz continuous derivative \(g' : Z \to Z\),
given by \[ g'(z) = -z + T\tilde{F}(Rx(z) + Sy(z) + Tz), \quad z \in Z. \]

Finally, \(z\) is a critical point of \(g\) iff \(Rx(z) + Sy(z) + Tz\) is a solution of \(Au = F(u)\).

Now, we set \(T^{-1} := (T|Z)^{-1} \in L_b(Z) := \{B \in L(Z) | B = B^*\}\),
a := \(-g \circ T^{-1} \in C^1(Z,\mathbb{R})\) and \(u(z) := v(z) + z\) where

(1.7) \[ v(z) := Rx(T^{-1}z) + Sy(T^{-1}z), \quad \forall z \in Z. \]

Then, by (1.6), \(u(.) : Z \to H\) is Lipschitz continuous, and \(u(z) \in \text{dom}(A) \quad \forall z \in Z.\)

Moreover, \(a\) has a globally Lipschitz continuous derivative, given by \(a' = -T^{-1} \circ g' \circ T^{-1}\) and \(z\) is a critical point of \(a\) iff \(u(z)\) is a solution of \(Au = F(u)\).

(1.8) LEMMA (cf. [1], [2]). For every \(z \in Z\)

\[ a(z) = \frac{1}{2} \langle Au(z), u(z) \rangle - \phi(u(z)) \]
or \[ a(z) = \frac{1}{2} \langle Az, z \rangle + \frac{1}{2} \langle Av(z), v(z) \rangle - \phi(u(z)) \]

and \[ a'(z) = Az - Pf(u(z)). \]

In the following lemma we state a technical remark, which will be useful.

(1.9) LEMMA. If \(F\) is globally bounded, i.e. there exists \(M > 0\) such that \(\|F(u)\| \leq M < +\infty\) \(\forall u \in M\), then the mapping \(v\) defined in (1.7) is also globally bounded.

Proof. The equations (1.5) and (1.6) characterizing \((x(z),y(z))\), \(z \in Z\), can be rewritten as

\[ 0 = (I - aR^2)x(z) + RF(Rx(z) + Sy(z) + Tz) \]
0 = \((-I - \alpha S^2) y(z) + SF(Rx(z) + Sy(z) + Tz)\)

Therefore, there exists a constant \(C > 0\) such that

\[ \| (I - \alpha R^2) x(z) \|, \| (I - \alpha S^2) y(z) \| \leq C \quad \forall \ z \in Z \]

Now, it is easily verified that \((I - \alpha R^2)|X\) and \((-I - \alpha S^2)|Y\) are linear isomorphisms, and the assertion follows immediately.

\section*{§ 2. Reduction under Symmetry Assumptions.}

We assume henceforth that \(0 \in \sigma(A)\) and we let \(Z = Z_0 \oplus Z_1\) where 
\(Z_0 := \ker(A) = P_0(H)\).

The definition of \(T\) implies that \(T|Z_0 = |\alpha|^{-2} I\) that is,

\[ T(z_0) = |\alpha|^{-2} z_0 = \gamma z_0, \quad \forall \ z_0 \in Z_0 \text{ where } \gamma := |\alpha|^{-2} \text{.} \]

\((2.1)\). Let \(z \in Z\). Setting \(z = z_0 + z_1, \ z_0 \in Z_0\) and \(z_1 \in Z_1\) we have \(\langle A z, z \rangle = \langle A(z_0 + z_1), z_0 + z_1 \rangle = \langle A z_1, z_1 \rangle\).

\((2.2)\). Let now \(\overline{z_0} \in Z_0\) be fixed and assume that \(\Phi\) is invariant under translation by \(\overline{z_0}\). More precisely, \(\Phi(u + \overline{z_0}) = \Phi(u) \ \forall \ u \in H\).

It follows that \(F\) is also \(\overline{z_0}\)-invariant, i.e.,

\[ F(u + \overline{z_0}) = F(u) \quad \forall \ u \in H \]

The equations that characterize the point

\[ (x(z + \gamma^{-1} \overline{z_0}), y(z + \gamma^{-1} \overline{z_0})), \ z \in Z \]

are

\[ 0 = x - \alpha R^2 x + RF(Rx + Sy + Tz + \overline{z_0}) \]
\[ 0 = y - \alpha S^2 y + SF(Rx + Sy + Tz + \overline{z_0}) \]

because \(T(\gamma^{-1} \overline{z_0}) = \overline{z_0}\). Hence, from \((2.3)\), the latter equations can be written

\[ 0 = x - \alpha R^2 x + RF(Rx + Sy + Tz) \]
\[ 0 = y - \alpha S^2 y + SF(Rx + Sy + Tz) \]

Consequently,

\[ (2.4) \quad (x(z + \gamma^{-1} \overline{z_0}), y(z + \gamma^{-1} \overline{z_0})) = (x(z), y(z)) \quad z \in Z. \]

Moreover, from \((1.7)\) and \((2.4)\), for every \(z \in Z\),

\[ v(z + \overline{z_0}) = Rx(T^{-1}(z + \overline{z_0}) + Sy(T^{-1}(z + \overline{z_0}) = \]
Thus, we have proved that \( v \) is invariant under translation by \( \mathbf{z}_0 \).

(2.4) Let now \( m = \text{dim} \mathbf{Z}_0 \) and \( \mathbf{w}_1, \ldots, \mathbf{w}_m \) be linearly independent vectors in \( \mathbf{Z}_0 \). The integral linear combinations \( n_1 \mathbf{w}_1 + \ldots + n_m \mathbf{w}_m \) form a lattice \( \mathcal{O} \) in \( \mathbf{Z}_0 \), and the quotient space \( \mathbf{Z}_0 / \mathcal{O} \) is precisely the \( m \)-dimensional torus \( T^m \).

The following result is just a consequence of (2.3).

(2.5) PROPOSITION. If \( \Phi(u + w) = \Phi(u), \forall u \in H \) and \( \forall w \in \mathcal{O} \), there exists a functional \( f \in C^1(T^m \times \mathbf{Z}^1, \mathbb{R}) \) such that the diagram

\[
\begin{array}{ccc}
\mathbf{Z}_0 \times \mathbf{Z}_1 & \xrightarrow{p \times \text{id}} & T^m \times \mathbf{Z}_1 \\
\downarrow a & & \downarrow f \\
\mathbb{R} & \xrightarrow{f} & T^m \times \mathbf{Z}_1
\end{array}
\]

is commutative, where \( p: \mathbf{Z}_0 \to T^m \) is the canonical projection. Moreover, \( f \) has a globally Lipschitz continuous derivative (gradient) \( f' \).

Of course, we consider \( T^m \) with its natural Riemannian metric.

\section*{3. Existence of Critical Points of \( f \).}

In this section we give an existence theorem of critical points of \( f \) based upon Ljusternik-Schnirelmann theory. The basic assumptions are:

(a): \( \Phi \) is \( \mathcal{O} \)-invariant as in (2.5).

(b): \( |\Phi| \) and \( \|F\| = \|\Phi'\| \) are uniformly bounded.

Let \( \mathbf{Z}_1 = \mathbb{R}^n \). We denote with \( y \) (\( x \), respectively) a generic point of \( \mathbb{R}^n \) (\( T^m \), respectively). Then, the functional \( f \) can be written

\[
f(y, x) = \frac{1}{2} \langle Ay, y \rangle + \Phi(y, x), \quad (y, x) \in \mathbb{R}^n \times T^m
\]

where, from (1.7),

\[
\Phi(y, x) = \frac{1}{2} \langle Av(y, x), v(y, x) \rangle - \Phi(u(y, x)).
\]

The linear operator \( A \in L(\mathbb{R}^n) \) is symmetric, non-singular and of index \( (A) = 1 \). Moreover, \( \Phi \in C^1(\mathbb{R}^n \times T^m, \mathbb{R}) \) is, from (1.9), uniformly bounded. Also, from (1.7), \( \Phi'(y, x) = -d(\text{id} \times p) PF(u(y, x)) \). It follows that \( \Phi' \) is globally Lipschitz continuous and \( \|\Phi'\| \) is uniformly bounded.

Hence, we can assume that there exists a constant \( N > 0 \) such that
\[
\max \{ |\phi(y,x)|, \|\phi'(y,x)\| \} \leq N, \; (x,y) \in \mathbb{R}^n \times T^m.
\]

Finally, we assume that index \((A) = i < n\).

(3.1) **THEOREM.** The number of geometrically different critical points of \(f\) is at least \(m+1\).

We begin with some preparations. First, it can be observed that \(f\) satisfies Palais-Smale condition \((C)\) because

\[
\|f'(y,x)\| > \|Ay\| - \|\phi'(y,x)\| > \|Ay\| - N \to +\infty
\]

when \(\|y\| \to +\infty\). Hence, the set \(K := \) critical points of \(f\), is compact. Of course, we have assumed that \(n > 0\), because in case \(n = 0\) the theorem is a known fact.

In the following, we denote \(z\) a generic point of \(\mathbb{R}^n \times T^m\) and for \(r \in \mathbb{R}\), we set

\[
f^r := \{ z \in \mathbb{R}^n \times T^m | f(z) = r \}
\]

Moreover, the term isotopy always refers to a Lipschitz continuous isotopy. With obvious reduction on smoothness, the following results are proved in (6.5.5) and (6.6.2) of \([4]\).

(3.2) **PROPOSITION.** Let \(c > b \geq a\) be real numbers. If \(f\) has no critical values on the interval \([b,c]\), then the sets \(f^c\) and \(f^b\) are isotopic. Furthermore, the isotopy may be so chosen that the points of \(f^a\) are fixed.

(3.3) **PROPOSITION.** Let \(c\) be an isolated critical value of \(f\) and \(a < c, a, c \in \mathbb{R}\). Then for \(\varepsilon > 0\) sufficiently small there is a neighborhood \(U\) of the set \(K_c := \{ z \in \mathbb{R}^n \times T^m | f(z) = c, f'(z) = 0 \}\) and a deformation \(\{\zeta_t\}\) of \(f^{c+\varepsilon} \setminus U\) such that \(\zeta_1(f^{c+\varepsilon} \setminus U) \subseteq f^{c-\varepsilon}\). The deformation may be so chosen that the points of \(f^a\) are fixed.

Now, let \(a < 0 < b\) real numbers such that, \((3.4)\): \(f\) has no critical points on \(\mathbb{R}^n \times T^m \setminus f^b\); \((3.5)\): \(f\) has no critical points on \(f^a\) if index \((A) > 0\), or \(f^a = \emptyset\) if index \((A) = 0\). It is clear that such numbers exist.

We set \(X := f^b / f^a\) (\(X := f^b\) in case \(f^a = \emptyset\)) and \(\tau : f^b \to X\) the canonical projection. In order to estimate the number of critical points of \(f\) in \(f^b \setminus f^a\), we use an integer-valued function defined on the class of subsets of \(f^b\). We define

\[
n(\emptyset) = 0, \; n(B) = \text{cat}_{X}[\tau(B)] \text{ if } B \subseteq f^b \text{ and } B \neq \emptyset.
\]
In the following lemma we collect the properties of \( n \) which will be useful and that are easily proved.

(3.6) LEMMA.

(i) \( n(B) = 1 \) if \( B \) is a point of \( f^b \).

(ii) \( n(B) \geq n(C) \) if \( B \supseteq C \).

(iii) \( n(B \cup C) < n(B) + n(C) \).

(iv) \( n(B) = n(B_\epsilon) \) where \( B_\epsilon \) is isotopic to \( B \) by an isotopy that leaves fixed the set \( f^a \).

(v) There is a neighborhood \( U \) of \( B \), for every \( B \), such that \( n(U) = n(B) \).

(3.7) LEMMA. Let \( a < c < b \) be an isolated critical value of \( f \). Then for \( \epsilon > 0 \) sufficiently small \( n(K_c) \geq n(f^{c+\epsilon}) - n(f^{c-\epsilon}) \).

Proof. Let \( U \) be a neighborhood of \( K_c \) such that (3.3) and (3.6.v) are valid, then \( n(f^{c-\epsilon}) \geq n(f^{c+\epsilon} \setminus U) \). Hence

\[
n(f^{c+\epsilon}) = n(f^{c+\epsilon} \cup U) \leq n(f^{c+\epsilon} \setminus U) + n(U) \leq n(f^{c-\epsilon}) + n(K_c).
\]

Now, we can prove

(3.8) THEOREM. The number of geometrically different critical points of \( f \) in \( f^b \) is at least \( n(f^b) - 1 \) if \( f^a \neq \emptyset \), and \( n(f^b) \) if \( f^a = \emptyset \).

Proof. Without loss of generality, we may suppose that the number of critical points of \( f \) is finite. Then \( f \) has a finite number of critical values \( c_i \), \( i = 0, 1, \ldots, N \), such that \( a < c_0 < c_1 < \ldots < c_N < b \), so that we may write \( f^b \supset f^{c_N} \supset \ldots \supset f^{c_0} \supset f^a \).

From (3.7), for some \( \epsilon > 0 \) independent of \( i = 1, \ldots, N \), it is verified

\[
n(K_{c_i}) \geq n(f^{c_i+\epsilon}) - n(f^{c_i-\epsilon}).
\]

Adding this last inequality over \( i \), and using the fact that, from (3.2) and (3.6.iv),

\[
n(f^{c_i-\epsilon}) = n(f^{c_{i-1}+\epsilon}), \quad i = 1, \ldots, N
\]

we find

\[
\sum_{i=0}^{N} n(K_{c_i}) \geq n(f^{c_N+\epsilon}) - n(f^{c_0-\epsilon}).
\]

Also, for the same reason, \( n(f^{c_N+\epsilon}) = n(f^b) \). Finally, for the va-
value of \( n(f^{c_0-\epsilon}) \) there are two possibilities:

(a). \( f^a = \emptyset \), then \( f \) has an absolute minimum in \( c_0^a \) so that \( f^{c_0-\epsilon} = \emptyset \) and \( n(f^{c_0-\epsilon}) = 0 \).

(b). \( f^a \neq \emptyset \), then \( n(f^{c_0-\epsilon}) = n(f^a) = 1 \) because \( \tau(f^a) \) is a point of \( X \), and then, the theorem is proved.

We turn now to the problem of determining \( n(f^b) \). Assume first that index \((A) = i \geq 1 \). If \( \delta = 0 \) and \( g \in C^1(\mathbb{R}^n, \mathbb{R}) \) is defined by

\[
G(y) := \frac{1}{2} \langle Ay, y \rangle, \; y \in \mathbb{R}^n
\]

then we have

\[
H^p(g^b / g^a; \mathbb{R}) = \begin{cases} R & \text{if } p = i \\ 0 & \text{if } p \neq i \end{cases}
\]

As \( f^b = g^b \times T^m \) and \( f^a = g^a \times T^m \), Kunneth's formulae imply

\[
H^j(X; \mathbb{R}) = \begin{cases} 0 & \text{if } j < i \\ H^{j-i}(T^m, \mathbb{R}) & \text{if } j \geq i \end{cases}
\]

Therefore, \( \text{cuplong}(X) = m+1 \) and

(3.10) \( n(f^b) = \text{cat}(X) \geq \text{cuplong}(X) + 1 = m+2 \) (cf. [5]).

In the general case \((\delta \neq 0)\), the asymptotic behavior of \( f \) and \( g \) near infinity are similar. Then, for \(|a|, b\) sufficiently large, we can hope that \( f^b / f^a \) and \( g^b \times T^m / g^a \times T^m \) are homeomorphic spaces and (3.10) remains valid. In fact, we can prove

(3.11) THEOREM. If \(|a|, b\) are sufficiently large, there exists a diffeomorphism of class \( C^1 \) \( \psi: \mathbb{R}^n \times T^m \rightarrow \mathbb{R}^n \times T^m \) such that

\[ \psi(g^b \times T^m) = f^b \] \[ \psi(g^a \times T^m) = f^a. \]

Proof. Fix \( N > 0 \) such that

(3.12) \( \max \{ |\phi(z)|, \| \phi'(z) \| \} \leq \frac{1}{2} N \)

Then, if \( z = (y, x) \), we have

(3.13) \( \langle f(z), g(z) \rangle = \| Ay \|^2 + \langle Ay, \phi'(y, x) \rangle \geq \| Ay \|^2 - \| A \| \| y \| > \| y \| (\| A^{-1} \|^2 \| y \| - \| A \| N) \geq \frac{1}{2} \)

when \( \| y \| \) is sufficiently large, say \( \| y \| > C. \)
Now, let $c$ be a real number such that

$$c > \frac{1}{2} c^2 \|A\| + 4N$$

and set $a := -c$, $b := c$.

When $z = (y,x) \in g^{-1}(a-4N,a+4N) \cup g^{-1}(b-4N,b+4N)$ it is verified that

$$\frac{1}{2} C^2 \|A\| < c-4N < |g(z)| < \frac{1}{2} \|A\| \|y\|^2.$$

Thus, $\|y\| > C$. In other words, (3.13) is verified in

$$g^{-1}(a-4N,a+4N) \cup g^{-1}(b-4N,b+4N).$$

On the other hand, (3.12) implies

$$(3.14) \; f^{-1}(b) \subset g^{-1}(a-N,b-N) \; , \; f^{-1}(a) \subset g^{-1}(a-N,a+N).$$

It is easily seen now that the gradient vector field $g'$ can be used in order to construct a diffeomorphism of class $C^1$

$$\alpha_b : g^{-1}(b-4N,b+4N) \to g^{-1}(b) \times (b-4N,b+4N)$$

such that $g = p_2 \circ \alpha_b$, where $p_2$ is projection onto the second factor.

Conditions (3.13) and (3.14) imply that $\alpha_b(f^{-1}(b))$ is the graph in $g^{-1}(b) \times (b-4N,b+4N)$ of a function $h : g^{-1}(b) \to (b-4N,b+4N)$ of class $C^1$, which verifies $|h(s)| < N$ $\forall s \in g^{-1}(b)$.

Now, let $\lambda_b : (b-4N,b+4N) \to [0,1]$ be a function of class $C^1$ such that

$$\lambda_b(b) = 1 \; \text{and} \; \text{supp}(\lambda_b) \subseteq [b-3N,b+3N],$$

$$|\lambda_b(t)| < \frac{1}{2} N \quad \forall \; t \in (b-4N,b+4N).$$

It is easily verified that the mapping $\beta_b$ defined by

$$\beta_b(s,t) = (s,t + \lambda_b(t) |h(s) - b|), \; (s,t) \in g^{-1}(b) \times (b-4N,b+4N)$$

is a diffeomorphism of class $C^1$ of $g^{-1}(b) \times (b-4N,b+4N)$ onto itself. Moreover, $\beta_b$ maps $g^{-1}(b) \times \{b\}$ onto $f^{-1}(b)$ and $\beta_b(s,t) = (s,t)$ if $|t - b| > 3N$.

Now, setting $\psi_b = \beta_b \circ \alpha_b$, we obtain a diffeomorphism of $g^{-1}(b-4N,b+4N)$ onto itself mapping $g^{-1}(b)$ onto $f^{-1}(b)$.

Analogously we construct $\alpha_a$, $\beta_a$ and $\psi_a$. Finally we define $\psi$ by the formulae
\[ \psi(z) = \begin{cases} 
\psi_b(z) & \text{if } z \in g^{-1}(b-4N,b+4N) \\
\psi_a(z) & \text{if } z \in g^{-1}(a-4N,a+4N) \\
z & \text{in the complementary case} 
\end{cases} \]

and the assertion follows immediately.

In a similar way it is proved that, when index \((A) = 0\) and consequently, \(f^a = \emptyset\) for a sufficiently large, the estimate \(n(f^b) \geq m+1\) is valid. Finally, if we collect all these preliminary results, we have proved theorem (3.1).

(3.15). COROLLARY. Let \(a\) and \(f\) be as in (2.5). Then, there exist at least \(m+1\) critical points of \(a, z_1, \ldots, z_{m+1}\), such that

\[ P_0(z_i) - P_0(z_j) \notin \Omega \quad \forall \ i, j = 1, \ldots, m+1, \text{ and } i \neq j. \]

§ 4. THE PROOF OF THE MAIN THEOREM.

We denote a generic point of \(R^{2N} = R^N \times R^N\) by \(x = (q,p)\), where \(q,p \in R^N\), and let \(\gamma : R^{2N} \to T^{2N}\) the canonical projection defined by \(\gamma((q,p)) = (e^{2\pi i q_1}, \ldots, e^{2\pi i q_N}, e^{2\pi i p_1}, \ldots, e^{2\pi i p_N})\).

Let \(\tilde{K} : T^{2N} \times [0,1] \to R\) be a function of class \(C^2\) and \(K : R^{2N} \times [0,1] \to R\ the function defined by \(K(x,t) = \tilde{K}(\gamma(x),t)\), \(\forall (x,t) \in R^{2N} \times [0,1]\).

Then, it is clear that there exists a constant \(C > 0\) such that

\[ \max_{(x,t)} \{|K(x,t)|, \|K_x(x,t)\|, \|K_{xx}(x,t)\|\} \leq C. \]

Moreover, \(K\) is invariant under translation by elements of the lattice

\[ \Omega = \{ \sum_{i=1}^{2N} n_i e_i \mid n_i \in Z \} \]

where \(\{e_i\}\) is the canonical basis of \(R^{2N}\).

We consider the existence problem of periodic solutions of the hamiltonian system

\[ q = K_p(q,p,t), \quad p = -K_q(q,p,t) \]

Setting \(H := L_2([0,1] ; R^{2N})\), we define a linear operator

\[ A : \text{dom}(A) \subseteq H \to H \]

\[ \text{dom}(A) := \{ u \in H^1([0,1] ; R^{2N}) \mid u(0) = u(1) \}, \text{ and } Au := -Ju = \{p, -q\} \]
where
\[
J := \begin{bmatrix}
0 & I_N \\
-I_N & 0
\end{bmatrix}
\]
is the standard symplectic structure on \(\mathbb{R}^{2N}\).

(4.3) **Lemma** (cf. [1]).

(i) \(A\) is self-adjoint, has closed range and a compact resolvent.

(ii) \(\sigma(A) = 2\pi \mathbb{Z}\), and each \(\lambda \in \sigma(A)\) is an eigenvalue of multiplicity \(2N\).

(iii) For each \(\lambda \in \sigma(A)\), the eigenspace \(\ker(\lambda I - A)\) is spanned by the orthogonal basis \(t + (\cos 2\pi \lambda t)e_k + (\sin 2\pi \lambda t)Je_k\), \(k = 1, \ldots, 2N\).

In particular, \(\ker(A) = \mathbb{R}^{2N}\), that is, it consists of the constant functions.

Now, \(F: \mathcal{H} \to \mathcal{H}\) is defined by \(F(u)(t) := K(u(t),t)\), \(t \in [0,1]\) and \(u \in \mathcal{H}\).

The assumptions imply that \(F\) is a continuous potential operator on \(\mathcal{H}\), the potential \(\Phi\) being given by
\[
\Phi(u) = \int_0^1 K(u(t),t) \,dt \quad \forall u \in \mathcal{H}.
\]

Clearly, classic periodic solutions of (4.2) are precisely the solutions \(u \in \text{dom}(A)\) of equation \(Au = F(u)\). Moreover, (4.1) and the mean value theorem imply that there exist constants \(\alpha, \beta \in \mathbb{R}\) such that hypothesis (F) of (1.1) is satisfied. Without loss of generality, we suppose \(\alpha < 0 < \beta\) and \(|\alpha|, \beta > 2\pi\). Then,

\[
(4.4) \quad \sigma(A) \cap (\alpha,0) \quad \text{and} \quad \sigma(A) \cap (0,\beta) \quad \text{are non vacuous.}
\]

Therefore, \(Z = \mathbb{R}^{2N} \oplus \mathbb{R}^n\) where \(\mathbb{R}^{2N} = \ker(A)\) and \(\mathbb{R}^n\) is the direct sum of the eigenspaces \(\ker(\lambda I - A)\), \(\lambda \in (\alpha,0) \cap \sigma(A)\) and \(\lambda \neq 0\). Moreover, \(A := A|\mathbb{R}^n\) is non-singular and, from (4.4), \(0 < \text{index}(A) < n\).

On the other hand, if \(w \in \Omega \subset \mathbb{R}^{2N}\), then
\[
\Phi(u+w) = \int_0^1 K(u(t)+w,t) \,dt = \int_0^1 K(u(t),t) \,dt = \Phi(u) \quad \text{because } K \text{ is } \Omega\text{-periodic.}
\]

We have seen that all conditions in order to apply theorem (3.1) and its corollary (3.15) are verified. Consequently, there exist at least \(2N+1\) periodic mappings \(u_1, \ldots, u_{2N+1}\) that are classic solutions of (4.2) such that \(P_0(u_1), \ldots, P_0(u_{2N+1})\) are pairwise inequivalent.
module the lattice $\Omega$.

This last condition implies clearly that $\gamma \cdot u_1, \ldots, \gamma \cdot u_{2N+1}$ are different periodic solutions of the hamiltonian vector field $X_K$, and the proof of the main theorem is finished.

REFERENCES