Revista de la
Unión Matemática Argentina
Volumen 32, 1986.

NOTE ON THE WEIGHTED POINTWISE ERGODIC THEOREM

María Elena Becker

ABSTRACT. In this paper we exhibit a class of complex sequences for which the Weighted Pointwise Ergodic Theorem holds.

1. INTRODUCTION.

Let (Ω,A,μ) be a probability space and let C be the group of automorphisms of (Ω,A,μ); $T \in C$ if $T: \Omega \to \Omega$ is a bijection which is bimeasurable and preserves μ. For each $T \in C$ and $1 \leqslant p \leqslant \infty$ we denote U_T the operator on $L^p(\Omega) = L^p(\Omega,A,\mu)$, $U_Tf = f\circ T$, for $f \in L^p(\Omega)$. We denote by N the set of all nonnegative integers. Now let T be a continuous linear operator on $L^p(\Omega)$ for some $1 \leqslant p < \infty$. Let $a = (a_n)$ be a sequence of complex numbers.

DEFINITION 1.1. We say that $a = (a_n)$ is a good weight in L^p for T (relative to the Weighted Pointwise Ergodic Theorem) if for every $f \in L^p(\Omega)$

$$\lim_n \frac{1}{n} \sum_{j=0}^{n-1} a_j T^j f(\omega) \quad \text{exists } \mu\text{-a.e.}$$

In the case $T \in C$, we say that a is a good weight for T in L^1, or simply that a is a good weight for T if a is a good weight in L^1 for the operator U_T induced by T. We have (see [1] and [4])

THEOREM 1.2. *Let $a = (a_n)$ be a bounded complex sequence. The following assertions are equivalent:*

(i) *a is a good weight in L^1 for every ergodic $T \in C$.*

(ii) *a is a good weight in L^1 for every $T \in C$.*

(iii) *a is a good weight in L^1 for every Dunford-Schwartz operator.*

DEFINITION 1.3. A bounded complex sequence $a = (a_n)$ is said to be a good universal weight if a is a good weight in L^1 for every Dunford-Schwartz operator (equivalently, by Theorem 1.2, for every $T \in C$ ergodic).

DEFINITION 1.4. Let $a = (a_n)$ be a complex sequence. For $1 \leq p < \infty$ define $\|a\|_p$ by

$$\|a\|_p^p = \limsup_n \frac{1}{n} \sum_{k=0}^{n-1} |a_k|^p ,$$

and let $\ell(p) = \{a / \|a\|_p < \infty\}$. We also define $\ell(\infty)$ as the space of all bounded complex sequences and $\|a\|_\infty = \sup_k |a_k|$ for $a \in \ell(\infty)$.

We also say that $a = (a_n)$ has a mean if $\lim_n \frac{1}{n} \sum_{j=0}^{n-1} a_j$ exists.

We have (see [1])

LEMMA 1.5. Let $a(k)$, $k \in N$, and a be complex sequences such that each $a(k)$ has a mean. Suppose that $\|a(k)-a\|_1 \to 0$ as $k \to \infty$. Then a has a mean.

Finally, we consider the space S of complex sequences $a = (a_n)$ such that

$$\gamma_a(k) = \lim_n \frac{1}{n} \sum_{j=0}^{n-1} a_{j+k} \cdot \overline{a_j}$$

exists for each $k \in N$.

For all the information that we shall need about S, we refer to [2].

A.Bellow and V.Losert proved (see [2]) the following result

THEOREM 1.6. *Let D be the set of all $a \in S \cap \ell(\infty)$ satisfying the following conditions:*

(1) *The spectral measure σ_a corresponding to a is discrete.*

(2) *The amplitude $\lim_n \frac{1}{n} \sum_{j=0}^{n-1} a_j \, \overline{z}^j$ exists for all complex number z such that $|z| = 1$.*

Then every $a \in D$ is a good universal weight.

In fact more is true. Let $T \in C$ ergodic. For each $f \in L^1(\Omega)$ there exists a set $\Omega_f \subset \Omega$ of probability one such that

$\lim \frac{1}{n} \sum_{j=0}^{n-1} a_j \, \overline{f}(T^j w)$ *exists for any $a \in D$ and any $w \in \Omega_f$.*

Throughout this paper we will denote by D_1 the class of all bounded complex sequences $a = (a_n)$ satisfying the following properties:

(1) a has a mean.

(2) The sequence $b = (b_n)$ such that $b_n = a_n - a_{n+1}$ is in D, where D is the class of Theorem 1.6.

In the next section we shall prove that D_1 is strictly larger than D and that every $a \in D_1$ is a good universal weight.

2. STATEMENTS AND PROOFS.

We start with the following lemma

LEMMA 2.1. *Let a be a complex sequence. If $a \in D$ then $a \in D_1$.*

The proof is easy and we omit it.

The following example shows that D is strictly contained into D_1.

EXAMPLE. Let α and β be two real and nonnegative numbers.
For each $k \in N$ let I_k be the integer interval
$I_k = \{n \in N / 4^k \leqslant n < 4^{k+1}\}$. We consider the family $\{I_k^{(j)}\}$ of subintervals of I_k, where

$$I_k^{(j)} = \{n \in N/4^k+3j2^k \leqslant n < 4^k+3(j+1)2^k\} \ , \quad 0 \leqslant j \leqslant 2^k-1.$$

We define the sequence $a = (a_n)$ in the following way:

$$a_n = \begin{cases} (-1)^j\alpha & \text{if} \quad n \in I_k^{(j)}, \quad k \text{ even} \\ (-1)^j\beta & \text{if} \quad n \in I_k^{(j)}, \quad k \text{ odd.} \end{cases}$$

It is easy to see that a has a mean. In fact, let $n \in N$ and let k such that $n \in I_k$. Thus,
$|\sum_{j=1}^{n} a_j| \leqslant 3.2^k.\max\{\alpha,\beta\})$ and therefore $\lim_n \frac{1}{n} \sum_{j=1}^{n} a_j = 0$.

Now let $b = (b_j)$ such that $b_j = a_j - a_{j+1}$. If $j \in I_k$, then $b_j \neq 0$ only for 2^k values of j. It follows that

$$\lim_n \frac{1}{n} \sum_{j=1}^{n} |b_j| = 0.$$

From this we immediately obtain that $b \in D$.
We shall prove that $a \notin D$ by showing that $\frac{1}{n} \sum_{j=1}^{n} |a_j|^2$ is not convergent. In fact, we have

$$|a_j|^2 = \begin{cases} \alpha^2 & \text{if} \quad j \in I_k, \quad k \text{ even} \\ \beta^2 & \text{otherwise.} \end{cases}$$

A simple calculus shows that the sequence $|a|^2 = (|a_j|^2)$ has not a mean and therefore $a \notin D$.

THEOREM 1.2. *Every $a \in D_1$ is a good universal weight. Furthermore, let $T \in C$ ergodic. For each $f \in L^1(\Omega)$ there exists a set $\Omega_f \subset \Omega$*

of probability one such that

$\lim_n \frac{1}{n} \sum_{j=0}^{n-1} a_j \overline{f}(T^j w)$ *exists for any* $a \in D_1$ *and any* $w \in \Omega_f$.

Proof. Let $T \in C$ ergodic. Let us consider the set of all functions $h \in L^1(\Omega)$ which can be represented in the form

$$(1) \qquad h(w) = g(w) - g(T^{-1}w)$$

where g is a bounded function.

For any function h of this form, we have

$$\frac{1}{n} \sum_{j=0}^{n-1} a_j \overline{h}(T^j w) = \frac{1}{n} \sum_{j=0}^{n-1} b_j \overline{g}(T^j w) + \frac{c}{n}$$

where c is a constant depending only on $\|a\|_\infty$ and $\|g\|_{L^\infty(\Omega)}$.

By Theorem 1.6 for each g there exists a set Ω_g of probability one such that

$\lim_n \frac{1}{n} \sum_{j=0}^{n-1} b_j \overline{g}(T^j w)$ exists for any $b \in D$ and any $w \in \Omega_g$.

We conclude that for each f in the linear span V of the functions h and 1, there is a set Ω_f of full measure such that

$$(2) \quad \lim_n \frac{1}{n} \sum_{j=0}^{n-1} a_j \overline{f}(T^j w) \text{ exists for any } a \in D_1 \text{ and any } w \in \Omega_f.$$

It is not hard to prove that V is dense in $L^1(\Omega)$. (see [3, pp.39]).

Now let $f \in L^1(\Omega)$. Let $f_k \in V$ be such that $f_k \xrightarrow{L^1} f$. By (2) and the Individual Ergodic Theorem, we can find for each k a set $\Omega_k \subset \Omega$ of probability one with the following properties:

i) $w \in \Omega_k \Rightarrow \lim_n \sum_{j=0}^{n-1} a_j \overline{f}_k(T^j w)$ exists for all $a \in D_1$.

ii) $w \in \Omega_k \Rightarrow \lim_n \frac{1}{n} \sum_{j=0}^{n-1} |f(T^j w) - f_k(T^j w)| = \|f - f_k\|_{L^1(\Omega)}$.

Let $\Omega_f = \bigcap_k \Omega_k$. For fixed $w \in \Omega_f$ we consider the sequences $c(k,w) = (a_j \overline{f}_k(T^j w))$ and $c(w) = (a_j \overline{f}(T^j w))$.

We have $\|c(k,w) - c(w)\|_1 \leq \|a\|_\infty \|f - f_k\|_{L^1(\Omega)}$, and by lemma 1.5 we deduce

$$\lim_n \frac{1}{n} \sum_{j=0}^{n-1} a_j \overline{f}(T^j w) \text{ exists for each } a \in D_1.$$

An application of Theorem 1.2 concludes the proof.

REFERENCES

[1] J.R.Baxter and J.H.Olsen, *Weighted and subsequential ergodic theorems*, Canad. J.Math. 35 (1983), 145-166.

[2] A.Bellow and V.Losert, *The Weighted Pointwise Ergodic Theorem and the Individual Ergodic Theorem along subsequences*, Trans. Amer. Math. Soc. 288 (1985), 307-345.

[3] A.Garsia, *Topics in almost everywhere convergence*, Markham, Chicago, 1970.

[4] V.A.Rohlin, *On the fundamental ideas of measure theory*, Amer. Math. Soc. Transl., Vol.10, N°1 (1962), 1-54.

Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires
Argentina.

Recibido en abril de 1986.