ABSTRACT. In this paper we exhibit a class of complex sequences for which the Weighted Pointwise Ergodic Theorem holds.

1. INTRODUCTION.

Let $(\Omega,\mathcal{A},\mu)$ be a probability space and let $C$ be the group of automorphisms of $(\Omega,\mathcal{A},\mu)$; $T \in C$ if $T: \Omega \to \Omega$ is a bijection which is bimeasurable and preserves $\mu$. For each $T \in C$ and $1 \leq p < \infty$ we denote $U_T$ the operator on $L^p(\Omega) = L^p(\Omega,\mathcal{A},\mu)$, $U_Tf = f \circ T$, for $f \in L^p(\Omega)$. We denote by $N$ the set of all nonnegative integers. Now let $T$ be a continuous linear operator on $L^p(\Omega)$ for some $1 \leq p < \infty$. Let $a = (a_n)$ be a sequence of complex numbers.

DEFINITION 1.1. We say that $a = (a_n)$ is a good weight in $L^p$ for $T$ (relative to the Weighted Pointwise Ergodic Theorem) if for every $f \in L^p(\Omega)$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} a_j T^j f(\omega)$$

exists $\mu$-a.e.

In the case $T \in C$, we say that $a$ is a good weight for $T$ in $L^1$, or simply that $a$ is a good weight for $T$ if $a$ is a good weight in $L^1$ for the operator $U_T$ induced by $T$. We have (see [1] and [4])

THEOREM 1.2. Let $a = (a_n)$ be a bounded complex sequence. The following assertions are equivalent:

(i) $a$ is a good weight in $L^1$ for every ergodic $T \in C$.

(ii) $a$ is a good weight in $L^1$ for every $T \in C$.

(iii) $a$ is a good weight in $L^1$ for every Dunford-Schwartz operator.

DEFINITION 1.3. A bounded complex sequence $a = (a_n)$ is said to be a good universal weight if $a$ is a good weight in $L^1$ for every Dunford-Schwartz operator (equivalently, by Theorem 1.2, for every $T \in C$ ergodic).
DEFINITION 1.4. Let \( a = (a_n) \) be a complex sequence. For \( 1 < p < \infty \) define \( \|a\|_p \) by
\[
\|a\|_p^p = \lim_{n \to \infty} \sup \left\{ \frac{1}{n} \sum_{k=0}^{n-1} |a_k|^p \right\},
\]
and let \( \ell(p) = \{a/\|a\|_p < \infty\} \). We also define \( \ell(\infty) \) as the space of all bounded complex sequences and \( \|a\|_\infty = \sup_k |a_k| \) for \( a \in \ell(\infty) \).

We also say that \( a = (a_n) \) has a mean if \( \lim \frac{1}{n} \sum_{j=0}^{n-1} a_j \) exists. We have (see [1])

LEMMA 1.5. Let \( a(k), k \in \mathbb{N} \), and \( a \) be complex sequences such that each \( a(k) \) has a mean. Suppose that \( \|a(k)\|_1 < \infty \) as \( k \to \infty \). Then \( a \) has a mean.

Finally, we consider the space \( S \) of complex sequences \( a = (a_n) \) such that
\[
\gamma_a(k) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} a_j a_{j+k}^* \text{ exists for each } k \in \mathbb{N}.
\]

For all the information that we shall need about \( S \), we refer to [2]. A.Bellow and V.Losert proved (see [2]) the following result

THEOREM 1.6. Let \( D \) be the set of all \( a \in S \cap \ell(\infty) \) satisfying the following conditions:

(1) The spectral measure \( \sigma_{a} \) corresponding to \( a \) is discrete.
(2) The amplitude \( \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} a_j z^j \) exists for all complex numbers \( z \) such that \( |z| = 1 \).

Then every \( a \in D \) is a good universal weight.

In fact more is true. Let \( T \in C \) ergodic. For each \( f \in L^1(\Omega) \) there exists a set \( \Omega_f \subset \Omega \) of probability one such that
\[
\lim \frac{1}{n} \sum_{j=0}^{n-1} a_j \mathcal{F}(T^j w) \text{ exists for any } a \in D \text{ and any } w \in \Omega_f.
\]

Throughout this paper we will denote by \( D_1 \) the class of all bounded complex sequences \( a = (a_n) \) satisfying the following properties:

(1) \( a \) has a mean.
(2) The sequence \( b = (b_n) \) such that \( b_n = a_n \cdot a_{n+1} \) is in \( D \), where \( D \) is the class of Theorem 1.6.

In the next section we shall prove that \( D_1 \) is strictly larger than \( D \) and that every \( a \in D_1 \) is a good universal weight.
2. STATEMENTS AND PROOFS.

We start with the following lemma

**LEMMA 2.1.** Let \( a \) be a complex sequence. If \( a \in D \) then \( a \in D_1 \).

The proof is easy and we omit it.

The following example shows that \( D \) is strictly contained into \( D_1 \).

**EXAMPLE.** Let \( \alpha \) and \( \beta \) be two real and nonnegative numbers.

For each \( k \in \mathbb{N} \) let \( I_k \) be the integer interval

\[ I_k = \{ n \in \mathbb{N} / 4^k \leq n < 4^{k+1} \} \]

We consider the family \( \{ I_k(j) \} \) of subintervals of \( I_k \), where

\[ I_k(j) = \{ n \in \mathbb{N} / 4^k + 3j 2^k \leq n < 4^k + 3(j+1)2^k \}, \ 0 \leq j \leq 2^k - 1. \]

We define the sequence \( a = (a_n) \) in the following way:

\[ a_n = \begin{cases} (-1)^j \alpha & \text{if } n \in I_k(j), \ k \text{ even} \\ (-1)^j \beta & \text{if } n \in I_k(j), \ k \text{ odd} \end{cases} \]

It is easy to see that \( a \) has a mean. In fact, let \( n \in \mathbb{N} \) and let \( k \) such that \( n \in I_k \). Thus,

\[ \left| \sum_{j=1}^{n} a_j \right| \leq 3.2^k \cdot \max(\alpha, \beta) \] and therefore \( \lim_{n} \frac{1}{n} \sum_{j=1}^{n} a_j = 0. \)

Now let \( b = (b_j) \) such that \( b_j = a_j - a_{j+1} \). If \( j \in I_k \), then \( b_j \neq 0 \) only for \( 2^k \) values of \( j \). It follows that

\[ \lim_{n} \frac{1}{n} \sum_{j=1}^{n} |b_j| = 0. \]

From this we immediately obtain that \( b \in D \).

We shall prove that \( a \notin D \) by showing that \( \frac{1}{n} \sum_{j=1}^{n} |a_j|^2 \) is not convergent. In fact, we have

\[ |a_j|^2 = \begin{cases} \alpha^2 & \text{if } j \in I_k, \ k \text{ even} \\ \beta^2 & \text{otherwise} \end{cases} \]

A simple calculus shows that the sequence \( |a|^2 = (|a_j|^2) \) has not a mean and therefore \( a \notin D \).

**THEOREM 1.2.** Every \( a \in D_1 \) is a good universal weight. Furthermore, let \( T \in C \) ergodic. For each \( f \in L^1(\Omega) \) there exists a set \( \Omega_f \subseteq \Omega \)
of probability one such that
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} a_j \mathcal{F}(T^j w) \] exists for any \( a \in D_1 \) and any \( w \in \Omega_f \).

**Proof.** Let \( T \in C \) ergodic. Let us consider the set of all functions \( h \in L^1(\Omega) \) which can be represented in the form
\[ h(w) = g(w) - g(T^{-1}w) \]
where \( g \) is a bounded function.
For any function \( h \) of this form, we have
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} b_j \mathcal{F}(T^j w) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} b_j \mathcal{F}(T^j w) + \frac{c}{n} \]
where \( c \) is a constant depending only on \( \|a\|_\infty \) and \( \|g\|_\infty \).
By Theorem 1.6 for each \( g \) there exists a set \( \Omega_g \) of probability one such that
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} b_j \mathcal{F}(T^j w) \] exists for any \( b \in D_1 \) and any \( w \in \Omega_g \).
We conclude that for each \( f \) in the linear span \( V \) of the functions \( h \) and 1, there is a set \( \Omega_f \) of full measure such that
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} a_j \mathcal{F}(T^j w) \] exists for any \( a \in D_1 \) and any \( w \in \Omega_f \).
It is not hard to prove that \( V \) is dense in \( L^1(\Omega) \). (see [2, pp.39]).
Now let \( f \in L^1(\Omega) \). Let \( f_k \in V \) be such that \( f_k \to f \) \( L^1 \). By (2) and the Individual Ergodic Theorem, we can find for each \( k \) a set \( \Omega_k \subset \Omega \) of probability one with the following properties:
\begin{itemize}
  \item[i)] \( w \in \Omega_k \Rightarrow \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} a_j \mathcal{F}_k(T^j w) \) exists for all \( a \in D_1 \).
  \item[ii)] \( \Omega_k = \bigcap_{k} \Omega_k \). For fixed \( w \in \Omega_f \) we consider the sequences \( c(k,w) = (a_j \mathcal{F}_k(T^j w)) \) and \( c(w) = (a_j \mathcal{F}(T^j w)) \).
\end{itemize}
We have \( \|c(k,w) - c(w)\|_1 \leq \|a\|_\infty \|f - f_k\|_{L^1(\Omega)} \), and by lemma 1.5 we deduce
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} a_j \mathcal{F}(T^j w) \] exists for each \( a \in D_1 \).
An application of Theorem 1.2 concludes the proof.
REFERENCES


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