ABSTRACT. We prove in a very simple way Shilov extension theorem (if $A$ is a commutative Banach algebra, $h$ a character in the Shilov boundary of $A$ and $B$ a commutative superalgebra of $A$ then $h$ admits an extension to $B$ which can be chosen in the Shilov boundary of $B$) and use this proof to extend similar results to higher dimensional Shilov boundaries.

1. INTRODUCTION.

Let $A$ be a (complex commutative) Banach algebra with unity and $X(A)$ the spectrum space of characters of $A$. The Shilov boundary $\Gamma_0(A)$ is a compact subset of $X(A)$ which plays an important role in various problems, in particular in the search of existence of 1-dimensional analytic structure on $X(A)$. For higher dimensions $\Gamma_0(A)$ seems to be too small and Basener [2] and Sibony [11] have defined a bigger subset of $X(A)$, $\Gamma_n(A)$, which is an appropriate generalized Shilov boundary (see also [1,5,6,7,11] for more information on $\Gamma_n(A)$). In this paper we are concerned with the following property of $\Gamma_0(A)$, called Shilov extension theorem: if $A$ is a closed subalgebra of a Banach algebra $B$ then every character in $\Gamma_0(A)$ admits an extension to some character $k$ of $B$. Moreover $k$ can be chosen in $\Gamma_0(B)$, by a result of Rickart [9,(3.3.25)]. It turns out that there is a very simple proof of these results which yields to a generalization when $\Gamma_0(A)$ is replaced by $\Gamma_n(A)$. For this, we use a characterization of $\Gamma_n(A)$ due to Tonev [11]. We would like to thank Professor Tonev for providing his preprint [11] and giving us very useful information concerning $\Gamma_n(A)$.

2. PRELIMINARIES.

In this paper Banach algebra means a complex commutative Banach algebra with unit. If $A$ is a Banach algebra a character of $A$ is a non-zero homomorphism $h: A \to \mathbb{C}$. The spectrum of $A$ is the space
X(A) of all characters of A; provided with the weak * topology X(A) is a compact Hausdorff space, The Gelfand transformation ^:
A → C(X(A)) is defined by \( \hat{\psi}(\phi) = \phi(a), a \in A \). For a = \((a_1, \ldots, a_n) \in A^n\) and \( h \in X(A) \) we put \( \hat{a} = (\hat{a}_1, \ldots, \hat{a}_n): X(A) \to \mathbb{C}^n, \ h(a) = \)
\( (h(a_1), \ldots, h(a_n)) = (\hat{a}_1(h), \ldots, \hat{a}_n(h)) = \hat{a}(h), \ |h(a)| = \)
\( (\frac{1}{2} |h(a_i)|)^{2} \) and \( Z(a) = \{ h \in X(A): h(a) = 0 \} \). We denote by \( U_n(A) \) the set of all \( a \in A^n \) such that \( Z(a) \) is empty. It is well-known that \( a \in U_n(A) \) if and only if there exist \( b_1, \ldots, b_n \in A \) such that \( \sum_{i=1}^{n} a_i b_i = 1 \). The joint spectrum of \( a \in A^n \) is \( \sigma_A(a) = \text{image of } \hat{a} \); it is clear that \( a \in U_n(A) \) if and only if \( (0, \ldots, 0) \notin \sigma_A(a) \).

A boundary for A is a closed subset \( F \) of \( X(A) \) such that \( \sup \{|h(a)|: h \in F\} = \sup \{|h(a)|: h \in X(A)\}, (a \in A) \) i.e. \( \|\hat{a}\|_F = \|\hat{a}\|_{X(A)}, (a \in A) \). The Shilov boundary of A is the intersection of all boundaries, which is itself a boundary of A; it will be denoted by \( \Gamma_0(A) \). For \( n \geq 1 \) \( \Gamma_n(A) \) is defined as follows: for a closed subset \( F \) of \( X(A) \) let \( A_F \) be the closure in \( C(F) \) of \( \{ \hat{a} | F: a \in A \} \); then for \( a \in A^n \) it can be shown that \( X(A_Z(a)) = Z(a) \) so \( \Gamma_0(A_Z(a)) \subseteq X(a) \) is \( X(A) \) and we define the \( n \)-th Shilov boundary of A, \( \Gamma_n(A) \), as the closure in \( X(A) \) of \( \cup \{ \Gamma_0(A_Z(a)) : a \in A^n \} \). We shall need the following characterization of \( \Gamma_n(A) \), due to Tonev [11]: a closed subset \( F \) of \( X(A) \) is a \( n \)-boundary of A if \( \min \{|h(a)|: h \in X(A)\} = \min \{|h(a)|: h \in F\} \) for every \( a \in U_{n+1}(A) \); Tonev's theorem says that the intersection of all \( n \)-boundaries is a \( n \)-boundary which coincides with \( \Gamma_n(A) \).

3. THE RESULTS.

The first theorem extends slightly Shilov theorem, as it appears in [8, §13, Theorem 3'] and Rickart theorem [9, (3.3.25)].

Recall that the transpose of a (continuous unital) homomorphism of Banach algebras \( f: A \to B \) is the map \( f^*: X(B) \to X(A) \) defined by \( f^*(k) = k \circ f \).

THEOREM 1. Let \( f: A \to B \) be a homomorphism of Banach algebras. Then the following conditions are equivalent:
(1) \( \Gamma_0(A) \subset f^*(X(B)) \);
(2) \( \|\hat{a}\|_X(A) = \|f(a)\|_X(B) \) \( (a \in A) \);
(3) \( \Gamma_0(A) \subset f^*(\Gamma_0(B)) \).

Proof. Observe, first, that (2) means that \( f \) preserves the spectral radius, because if \( r_A(a) = \sup \{ |\lambda| : \lambda \in \sigma_A(a) \} \) then \( r_A(a) = \sup \{ |h(a)| : h \in X(A) \} = \|\hat{a}\|_X(A) \). Observe also that \( r_B(f(a)) \leq r_A(a) \) because \( \sigma_B(f(a)) \subset \sigma_A(a) \). Then \( \|\hat{a}\|_X(A) \geq \|f(a)\|_X(B) \) \( (a \in A) \) always holds.

(1) \( \Rightarrow \) (2) Given \( a \in A \), the continuous function \( \hat{a} \) attains its maximum absolute value on the Shilov boundary, so there exists \( h_0 \in \Gamma_0(A) \) such that \( |h_0(a)| = \|\hat{a}\|_X(A) \). By (1), \( h_0 = f^*(k_0) \) for some \( k_0 \in X(B) \), so we get \( \|f(a)\|_X(B) = \sup \{ |k(f(a))| : k \in X(B) \} \geq |k_0(f(a))| = |h_0(a)| = \|\hat{a}\|_X(A) \), which gives (2).

(2) \( \Rightarrow \) (3) It suffices to see that \( f^*(\Gamma_0(B)) \) is a boundary for \( A \). Given \( a \in A \), \( \|f(a)\|_X(B) \geq \|f(a)\|_X(B) \) by (2). But \( \|f(a)\|_X(B) = \|f(a)\|_X(B) \) for some \( k \in \Gamma_0(B) \) because \( f(a) \) attains its maximum absolute value on \( \Gamma_0(B) \). Thus \( \|f(a)\|_X(B) = \|f(a)\|_X(B) \leq \|f(a)\|_X(B) \), which means that \( \|\hat{a}\|_X(A) = \|\hat{a}\|_X(A) \). This gives (3). The implication (3) \( \Rightarrow \) (1) is obvious.

Corollary 1. If \( A \) is a closed subalgebra of a Banach algebra \( B \) then every character in \( \Gamma_0(A) \) admits an extension to a character in \( \Gamma_0(B) \).

Proof. It is well known that \( r_A(a) = r_B(a) = r_B(i(a)) \) if \( i : A \to B \) is the inclusion and \( a \in A \). Then \( \Gamma_0(A) \subset i^*(\Gamma_0(B)) \), which means precisely the assertion.

We recall that a subalgebra \( C \) of \( B \) is full if every \( c \in C \) which is invertible in \( B \) is also invertible in \( C \). In general, a homomorphism \( f : A \to B \) is full if its image is a full subalgebra of \( B \).

Corollary 2. Let \( f : A \to B \) be a full homomorphism of Banach algebras. Then \( \Gamma_0(A) \subset f^*(\Gamma_0(B)) \subset f^*(X(B)) \).

Proof. It is easy to see that \( f \) is full if and only if \( \sigma_A(a) = \sigma_B(f(a)) \) \( (a \in A) \). In particular, if \( f \) is full \( \|\hat{a}\|_X(A) = \|f(a)\|_X(B) \)
We proceed now to extend Theorem 1 to higher dimensional Shilov boundaries.

**THEOREM 2.** Let $f: A \to B$ be a homomorphism of Banach algebras and $n \geq 1$. Then the following conditions are equivalent:

1. $\Gamma_n(A) \subseteq f^*(\Gamma_n(B))$;
2. $\min\{|h(a)| : h \in X(A)\} = \min\{|k(f(a))| : k \in X(B)\}$ for $a \in U_{n+1}(A)$;
3. $\Gamma_{n-1}(A) \subseteq f^*(\Gamma_{n-1}(B))$.

**Proof.** Observe, firstly, that $\{h(a) : h \in X(A)\}$ always contains $\{k(f(a)) : k \in X(B)\}$ and then $\min\{|h(a)| : h \in X(A)\} \leq \min\{|k(f(a))| : k \in X(B)\}$ for $a \in U_{n+1}(A)$.

1. $\Rightarrow$ 2. Let $a \in U_{n+1}(A)$. By Tonev's theorem $|\hat{a}|$ attains its minimum at some $h_o \in \Gamma_n(A)$. By 1, $h_o = k_o f$ for some $k_o \in X(B)$ and $\min\{|h(a)| : h \in X(A)\} = |h_o(a)| = |k_o(f(a))| > \min\{|k(f(a))| : k \in X(B)\}$. Together with the previous remark, this gives 2.

2. $\Rightarrow$ 3. It suffices to show that $f^*(\Gamma_n(B))$ is a $n$-boundary for $A$. Let $a \in U_{n+1}(A)$. Then, by 2, $\min\{|h(a)| : h \in X(A)\} = \min\{|k(f(a))| : k \in X(B)\}$ for $a \in U_{n+1}(A)$, which means that $f^*(\Gamma_n(B))$ is a $n$-boundary.

**COROLLARY.** Let $f: A \to B$ be a $n$-full homomorphism of Banach algebras, i.e. if $f(a) \in U_n(B)$ for some $a \in A^n$ then it must be $a \in U_n(A)$. Then $\Gamma_{n-1}(A) \subseteq f^*(\Gamma_{n-1}(B)) \subseteq f^*(X(B))$.

**Proof.** Observe that $f$ is $n$-full if and only if $\sigma_A(a) = \sigma_B(f(a))$ for $a \in A^n$. It suffices to verify condition (2) of Theorem 2 (with $n+1$ replaced by $n$). Given $a \in U_n(A)$ let $h_o \in \Gamma_{n-1}(A)$ be such that $|h_o(a)| = \min\{|h(a)| : h \in X(A)\}$. The equality $\sigma_A(a) = \sigma_B(f(a))$ shows that $h_o(a) \in \sigma_B(f(a))$ so there exists $k_o \in X(B)$ such that $h_o(a) = k_o(f(a))$. In particular $\min\{|h(a)| : h \in X(A)\} = |k_o(f(a))| > \min\{|h(a)| : h \in X(A)\} = \min\{|k(f(a))| : k \in X(B)\} > \min\{|h(a)| : h \in X(A)\}$, as desired.
REMARKS. 1. It is false, in general, that every character in the n-th Shilov boundary of a closed subalgebra $A$ of a Banach algebra $B$ admits an extension to $B$. Indeed, if $A$ is the disc algebra (the algebra of all continuous functions on the closed unit disc $D$ which are analytic in the interior of $D$) and $B = C(T)$, where $T = \{ z \in \mathbb{C} : |z| = 1 \}$, then the restriction $f \mapsto f|T$ is an isometric isomorphism $r: A \to B$ onto a closed subalgebra (which we call again $A$) of $B$. The transpose $r^*$ is the inclusion of $T \times (C(T))$ into $D = X(A)$. However, $\Gamma_1(A) = D$ and $r^*(X(B))$ does not contain $\Gamma_1(A)$.

This shows that the condition (2) of Theorem 2 is strictly stronger than "$r_A(a) = r_B(f(a))$ ($a \in A$)."

2. The notion of n-full homomorphism, which appears in the study of the surjectivity of the transpose [4], is related to Carleson's theorem [3]: if $f_1, \ldots, f_n$ are bounded analytic functions on the open disc $\Delta$ (in symbols, $f_1, \ldots, f_n \in H^\infty$) satisfying

\[ (*) \quad |f_1(z)| + \ldots + |f_n(z)| > \delta \quad (z \in \Delta) \]

for some $\delta > 0$, then there exist $g_1, \ldots, g_n \in H^\infty$ such that $f_1g_1 + \ldots + f_ng_n = 1$. In fact, condition (*) means that $(f_1, \ldots, f_n) \in U_n(BC(\Delta))$, where $BC(\Delta)$ is the algebra of all bounded continuous functions on $\Delta$, whose spectrum is the Stone-Cech compactization of $\Delta$, $\beta \Delta$. Carleson's theorem proves that the inclusion $i: H^\infty \to BC(\Delta)$ is n-full for $n \geq 1$. By our last corollary we get $\Gamma_n(H^\infty) \subset i^*(\beta \Delta)$ for $n \geq 1$.

3. With similar arguments to those used in Theorem 2 several results concerning the Shilov boundary can be generalized to $\Gamma_n(n \geq 1)$; for example Holladay's version of Rouché theorem [9, (3.3.22)] admits the following generalization:

**THEOREM.** Let $A$ be a Banach algebra, $n \geq 1$ and $a, b \in A^n$. Suppose that

\[ |h(a) - h(b)| < |h(a)| \quad (h \in \Gamma_n(A)) \]

Then $a \in U_n(A)$ if and only if $b \in U_n(B)$.

The proof combines Tonev's characterization of $\Gamma_n(A)$, Rickart's proof and our Theorem 2.
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