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EXTENSION OF CHARACTERS AND GENERALIZED SHILOV BOUNDARIES

Gustavo Corach and Alejandra Maestripieri

ABSTRACT. We prove in a very simple way Shilov extension theorem (if A is a commutative Banach algebra, h a character in the Shilov boundary of A and B a commutative superalgebra of A then h admits an extension to B which can be chosen in the Shilov boundary of B) and use this proof to extend similar results to higher dimensional Shilov boundaries.

1. INTRODUCTION.

Let A be a (complex commutative) Banach algebra with unity and X(A) the spectrum space of characters of A. The Shilov boundary $\Gamma_o(A)$ is a compact subset of X(A) which plays an important role in various problems, in particular in the search of existence of 1-dimensional analytic structure on X(A). For higher dimensions $\Gamma_o(A)$ seems to be too small and Basener [2] and Sibony [11] have defined a bigger subset of X(A), $\Gamma_n(A)$, which is an appropiate generalized Shilov boundary (see also [1,5,6,7,11] for more information on $\Gamma_n(A)$).
In this paper we are concerned with the following property of $\Gamma_o(A)$, called Shilov extension theorem: if A is a closed subalgebra of a Banach algebra B then every character in $\Gamma_o(A)$ admits an extension to some character k of B. Moreover k can be chosen in $\Gamma_o(B)$, by a result of Rickart [9,(3.3.25)]. It turns out that there is a very simple proof of these results which yields to a generalization when $\Gamma_o(A)$ is replaced by $\Gamma_n(A)$. For this, we use a characterization of $\Gamma_n(A)$ due to Tonev [11]. We would like to thank Professor Tonev for providing his preprint [11] and giving us very useful information concerning $\Gamma_n(A)$.

2. PRELIMINARIES.

In this paper *Banach algebra* means a complex commutative Banach algebra with unit. If A is a Banach algebra a *character* of A is a non-zero homomorphism h: A → C. The *spectrum* of A is the space

$X(A)$ of all characters of A; provided with the weak * topology $X(A)$ is a compact Hausdorff space. The *Gelfand transformation* $\hat{}$: $A \to C(X(A))$ is defined by $\hat{a}(\phi) = \phi(a)$, $a \in A$. For $a = (a_1,\ldots,a_n) \in A^n$ and $h \in X(A)$ we put $\hat{a} = (\hat{a}_1,\ldots,\hat{a}_n): X(A) \to C^n$, $h(a) = (h(a_1),\ldots,h(a_n)) = (\hat{a}_1(h),\ldots,\hat{a}_n(h)) = \hat{a}(h)$, $|h(a)| = (\sum_{i=1}^{n} |h(a_i)|^2)^{1/2}$ and $Z(a) = \{h \in X(A): h(a) = 0\}$. We denote by $U_n(A)$ the set of all $a \in A^n$ such that $Z(a)$ is empty. It is well-known that $a \in U_n(A)$ if and only if there exist $b_1,\ldots,b_n \in A$ such that $\sum_{i=1}^{n} a_i b_i = 1$. The *joint spectrum* of $a \in A^n$ is $\sigma_A(a)$ = image of $\hat{a}$; it is clear that $a \in U_n(A)$ if and only if $0 = (0,\ldots,0) \notin \sigma_A(a)$. A *boundary* for A is a closed subset F of $X(A)$ such that $\sup\{|h(a)|: h \in F\} = \sup\{|h(a)| : h \in X(A)\}$, $(a \in A)$ i.e. $\|\hat{a}\|_F = \|\hat{a}\|_{X(A)}$, $(a \in A)$. The *Shilov boundary* of A is the intersection of all boundaries, which is itself a boundary of A; it will be denoted by $\Gamma_o(A)$. For $n \geqslant 1$ $\Gamma_n(A)$ is defined as follows: for a closed subset F of $X(A)$ let A_F be the closure in $C(F)$ of $\{\hat{a}|F: a \in A\}$; then for $a \in A^n$ it can be shown that $X(A_{Z(a)}) = Z(a)$ so $\Gamma_o(A_{Z(a)}) \subset Z(a) \subset X(A)$ and we define the n *th-Shilov boundary* of A, $\Gamma_n(A)$, as the closure in $X(A)$ of $\cup\{\Gamma_o(A_{Z(a)}): a \in A^n\}$. We shall need the following characterization of $\Gamma_n(A)$, due to Tonev [11]: a closed subset F of $X(A)$ is a n-*boundary* of A if $\min\{|h(a)| ; h \in X(A)\} = \min\{|h(a)| ; h \in F\}$ for every $a \in U_{n+1}(A)$; Tonev's theorem says that the intersection of all n-boundaries is a n-boundary which coincides with $\Gamma_n(A)$.

3. THE RESULTS.

The first theorem extends slightly Shilov theorem, as it appears in [8, §13, Theorem 3'] and Rickart theorem [9, (3.3.25)].

Recall that the *transpose* of a (continuous unital) homomorphism of Banach algebras $f: A \to B$ is the map $f^*: X(B) \to X(A)$ defined by $f^*(k) = k \circ f$.

THEOREM 1. *Let* $f: A \to B$ *be a homomorphism of Banach algebras. Then the following conditions are equivalent:*

(1) $\Gamma_o(A) \subset f^*(X(B))$;

(2) $\|\hat{a}\|_{X(A)} = \|\widehat{f(a)}\|_{X(B)}$ $(a \in A)$;

(3) $\Gamma_o(A) \subset f^*(\Gamma_o(B))$.

Proof. Observe, first, that (2) means that f preserves the spectral radius, because if $r_A(a) = \sup\{|\lambda| : \lambda \in \sigma_A(a)\}$ then $r_A(a) = \sup\{|h(a)|;\ h \in X(A)\} = \|\hat{a}\|_{X(A)}$. Observe also that $r_B(f(a)) \leqslant r_A(a)$ because $\sigma_B(f(a)) \subset \sigma_A(a)$. Then $\|\hat{a}\|_{X(A)} \geqslant \|\widehat{f(a)}\|_{X(B)}$ $(a \in A)$ always holds.

(1) ⇒ (2) Given $a \in A$, the continuous function $\hat{a}$ attains its maximum absolute value on the Shilov boundary, so there exists $h_o \in \Gamma_o(A)$ such that $|h_o(a)| = \|\hat{a}\|_{X(A)}$. By (1), $h_o = f^*(k_o)$ for some $k_o \in X(B)$, so we get $\|\widehat{f(a)}\|_{X(B)} = \sup\{|k(f(a))| : k \in X(B)\} \geqslant |k_o(f(a))| = |h_o(a)| = \|\hat{a}\|_{X(A)}$, which gives (2).

(2) ⇒ (3) It suffices to see that $f^*(\Gamma_o(B))$ is a boundary for A. Given $a \in A$, $\|\hat{a}\|_{X(A)} = \|\widehat{f(a)}\|_{X(B)}$ by (2). But $\|\widehat{f(a)}\| = |k(f(a))|$ for some $k \in \Gamma_o(B)$ because f(a) attains its maximum absolute value on $\Gamma_o(B)$. Thus $\|\hat{a}\|_{X(A)} = |k(f(a))| \leqslant \|\hat{a}\|_{f^*(X(B))}$, which means that $\|\hat{a}\|_{X(A)} = \|\hat{a}\|_{f^*(X(B))}$.
This gives (3). The implication (3) ⇒ (1) is obvious.

COROLLARY 1. *If* A *is a closed subalgebra of a Banach algebra* B *then every character in* $\Gamma_o(A)$ *admits an extension to a character in* $\Gamma_o(B)$.

Proof. It is well known that $r_A(a) = r_B(a) = r_B(i(a))$ if $i: A \to B$ is the inclusion and $a \in A$. Then $\Gamma_o(A) \subset i^*(\Gamma_o(B))$, which means precisely the assertion.

We recall that a subalgebra C of B is *full* if every $c \in C$ which is invertible in B is also invertible in C. In general, a homomorphism $f: A \to B$ is *full* if its image is a full subalgebra of B.

COROLLARY 2. *Let* $f: A \to B$ *be a full homomorphism of Banach algebras. Then* $\Gamma_o(A) \subset f^*(\Gamma_o(B)) \subset f^*(X(B))$.

Proof. It is easy to see that f is full if and only if $\sigma_A(a) = \sigma_B(f(a))$ $(a \in A)$. In particular, if f is full $\|\hat{a}\|_{X(A)} = \|\widehat{f(a)}\|_{X(B)}$

($a \in A$) and the theorem applies.

We proceed now to extend Theorem 1 to higher dimensional Shilov boundaries.

THEOREM 2. *Let* $f: A \to B$ *be a homomorphism of Banach algebras and* $n \geq 1$. *Then the following conditions are equivalent:*

(1) $\Gamma_n(A) \subset f^*(X(B))$;

(2) $\min\{|h(a)|: h \in X(A)\} = \min\{|k(f(a))|; k \in X(B)\}$ $(a \in U_{n+1}(A))$;

(3) $\Gamma_n(A) \subset f^*(\Gamma_n(B))$.

Proof. Observe, firstly, that $\{h(a): h \in X(A)\}$ always contains $\{k(f(a)): k \in X(B)\}$ and then $\min\{|h(a)|: h \in X(A)\} \leq \min\{|k(f(a))|: k \in X(B)\}$, $(a \in A^{n+1})$.

(1) $\Rightarrow$ (2) Let $a \in U_{n+1}(A)$. By Tonev's theorem $|\hat{a}|$ attains its minimum at some $h_o \in \Gamma_n(A)$. By (1) $h_o = k_o f$ for some $k_o \in X(B)$ and $\min\{|h(a)|: h \in X(A)\} = |h_o(a)| = |k_o(f(a))| \geq \min\{|k(f(a))| : k \in X(B)\}$. Together with the previous remark, this gives (2).

(2) $\Rightarrow$ (3) It suffices to show that $f^*(\Gamma_n(B))$ is a n-boundary for A. Let $a \in U_{n+1}(A)$. Then, by (2) $\min\{|h(a)|: h \in X(A)\} = \min\{|k(f(a))|: k \in X(B)\}$ and this minimum is attained at some $k_o \in \Gamma_n(B)$. This shows that $\min\{|h(a)|: h \in X(A)\} = \min\{|k(f(a))|: k \in \Gamma_n(B)\}$ $(a \in U_{n+1}(A))$, which means that $f^*(\Gamma_n(B))$ is a n-boundary.

COROLLARY. *Let* $f: A \to B$ *be a* n-*full homomorphism of Banach algebras, i.e. if* $f(a) \in U_n(B)$ *for some* $a \in A^n$ *then it must be* $a \in U_n(A)$. *Then* $\Gamma_{n-1}(A) \subset f^*(\Gamma_{n-1}(B)) \subset f^*(X(B))$.

Proof. Observe that f is n-full if and only if $\sigma_A(a) = \sigma_B(f(a))$ $(a \in A^n)$. It suffices to verify condition (2) of Theorem 2 (with n+1 replaced by n). Given $a \in U_n(A)$ let $h_o \in \Gamma_{n-1}(A)$ be such that $|h_o(a)| = \min\{|h(a)|: h \in X(A)\}$. The equality $\sigma_A(a) = \sigma_B(f(a))$ shows that $h_o(a) \in \sigma_B(f(a))$ so there exists $k_o \in X(B)$ such that $h_o(a) = k_o(f(a))$. In particular $\min\{|h(a)|: h \in X(A)\} = |k_o(f(a))| \geq \min\{|h(a)|: h \in X(A)\} = \min\{|k(f(a))|: k \in X(B)\} \geq \min\{|h(a)| : h \in X(A)\}$, as desired.

REMARKS. 1. It is false, in general, that every character in the n-th Shilov boundary of a closed subalgebra A of a Banach algebra B admits an extension to B. Indeed, if A is the disc algebra (the algebra of all continuous functions on the closed unit disc D which are analytic in the interior of D) and $B = C(T)$, where $T = \{z \in C: |z| = 1\}$, then the restriction $f \to f|T$ is an isometric isomorphism $r: A \to B$ onto a closed subalgebra (which we call again A) of B. The transpose r^* is the inclusion of $T = X(C(T))$ into $D = X(A)$. However, $\Gamma_1(A) = D$ and $r^*(X(B))$ does not contain $\Gamma_1(A)$. This shows that the condition (2) of Theorem 2 is strictly stronger than "$r_A(a) = r_B(f(a))$ $(a \in A)$".

2. The notion of n-full homomorphism, which appears in the study of the surjectivety of the transpose [4], is related to Carleson's theorem [3]: if $f_1,\ldots,f_n$ are bounded analytic functions on the open disc Δ (in symbols, $f_1,\ldots,f_n \in H^\infty$) satisfying

$$(*) \qquad |f_1(z)| + \ldots + |f_n(z)| \geq \delta \qquad (z \in \Delta)$$

for some $\delta > 0$, then there exist $g_1,\ldots,g_n \in H^\infty$ such that $f_1g_1 + \ldots + f_ng_n = 1$. In fact, condition (*) means that $(f_1,\ldots,f_n) \in U_n(BC(\Delta))$, where $BC(\Delta)$ is the algebra of all bounded continuous functions on Δ, whose spectrum is the Stone-Čech compactization of Δ, $\beta\Delta$. Carleson's theorem proves that the inclusion $i: H^\infty \to BC(\Delta)$ is n-full for $n \geq 1$. By our last corollary we get $\Gamma_n(H^\infty) \subset i^*(\beta\Delta)$ for $n \geq 1$.

3. With similar arguments to those used in Theorem 2 several results concerning the Shilov boundary can be generalized to Γ_n $(n \geq 1)$; for example Holladay's version of Rouché theorem [9, (3.3.22)] admits the following generalization:

THEOREM. *Let* A *be a Banach algebra,* $n \geq 1$ *and* $a,b \in A^n$. *Suppose that*

$$|h(a) - h(b)| < |h(a)| \quad (h \in \Gamma_n(A))$$

Then $a \in U_n(A)$ *if and only if* $b \in U_n(B)$.

The proof combines Tonev's characterization of $\Gamma_n(A)$, Rickart's proof and our Theorem 2.

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Gustavo Corach
Instituto Argentino de Matemática
Viamonte 1636
1055 Buenos Aires
Argentina

Alejandra Maestripieri
Departamento de Matemática
Facultad de Ciencias Exactas y Naturales
Ciudad Universitaria
1428 Buenos Aires
Argentina

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