Revista de la
Unión Matemática Argentina
Volumen 32, 1986.

GENERALIZED OPERATOR RICCATI EQUATIONS WITH TWO POINT BOUNDARY CONDITIONS

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ABSTRACT. Boundary problems for generalized Riccati equations whose coefficients are time-dependant closed linear operators densely defined on a separable complex Hilbert space H are studied. Necessary and sufficient conditions for the existence of solutions are given.

1. INTRODUCTION.

This paper is concerned with the resolution problem of generalized Riccati operator equations with two point boundary conditions of the type

$$\left.\begin{aligned} dU/dt\ U(t) &= A(t)+B(t)U(t)-U(t)C(t)-U(t)D(t)U(t) \\ E\ U(b)-U(0)F &= G \end{aligned}\right\} \qquad (1.1)$$

This equation arises in control theory, [10], transport theory, [14], and filtering problems, [1]. The Cauchy problem for this equation has been studied in [8], when $C(t) = -B(t)^*$, being $B(t)^*$ the adjoint operator of $B(t)$. If H is finite-dimensional, the coefficient operators are time-invariant, and $E=F=I$, $G=0$, $C=-B^*$, problem (1.1) has been studied in [17]. In a recent paper, [16], we study the problem (1.1), when the coefficient operators which appear in the differential equation are time-invariant bounded linear operators on H. In the following we study the problem (1.1) for the time-dependant case, and the coefficient are closed linear operators densely defined on a complex separable Hilbert space H.

The key idea is to reduce the boundary problem to the resolution problem of an algebraic operator equation of Riccati type

$$M + NX - XP - XQX = 0 \qquad (1.2)$$

From conditions for the resolution problem (1.2), conditions for the resolution problem (1.1) are obtained. Solutions of (1.1) are expressed in terms of solutions of (1.2). We apply the results to the study of the existence problem of b-periodic solutions for the differential equation which appear in (1.1).

If S is a linear operator on H, with domain $D(S)$ we denote the numerical range of S by

$$\Theta(S) = \{z \in C \; ; \; (Sx,x) = z \; ; \; \|x\| = 1\}$$

and $\sigma(S)$ its spectrum. In accordance with the definition given in [8], when S is a closed operator on H and generates an analytic semigroup denoted by $\exp(tS)$, we define the fractional powers:

$S^{\alpha} = (S^{-\alpha})^{-1}$

$$S^{-\alpha} = \frac{e^{-i\pi\alpha}}{\Gamma(\alpha)} \int_0^{\infty} e^{tS}\, t^{\alpha-1}\, dt \qquad (\alpha > 0)$$

We wish to express our thanks to Professors Hendrisk Kuiper and Hiroki Tanabe for providing us with preprints and copies of their papers.

2. ASSUMPTIONS ON ABSTRACT EVOLUTION EQUATIONS.

For the sake of convenience, we state some results from the theory of abstract evolution equations which will be used in the sequel. The results of this section can be found in [7], [8], and [9]. In what follows, we denote by Σ a fixed closed angular domain $\Sigma = \{\lambda; \; |\lambda| \leq \delta + \pi/2\}$, $0 < \delta < \pi/2$. Let us consider the abstract evolution equation

$$d/dt\; u(t) = V(t)u(t) \quad , \quad 0 \leq t \leq b \; , \qquad (2.1)$$

in a Hilbert space W. $u = u(t)$ is a function on $[0,b]$ to W and $V(t)$ is a function on $[0,b]$ to the set of closed densely defined linear operators acting in W.

We first state the assumptions to be made in the theorems.

(H.1) For each $t \in [0,b]$, $V(t)$ is a densely defined, closed linear operator. The resolvent set $\rho(V(t))$ of $V(t)$ contains Σ. The resolvent of $V(t)$ satisfies

$$\|(\lambda I - V(t))^{-1}\| \leq M/|\lambda|$$

for any $\lambda \in \Sigma$ and $t \in [0,b]$, where M is a constant independent of λ and t.

(H.2) $(-V(t))^{-1}$, which is a bounded operator for each t, is continuously differentiable in $t \in [0,b]$ in the uniform operator topology.

(H.3) For any $\lambda \in \Sigma$ and $t \in [0,b]$, the following inequality holds

$$\left\| \frac{\partial}{\partial t} (\lambda I - V(t))^{-1} \right\| \leq \frac{N}{|\lambda|^{1-\alpha}}$$

where N and α are constants independent of t and α with $0 < \alpha < 1$.

(H.4) $d(-V(t))^{-1}$ is Hölder continuous in $t \in [0,b]$ in the uniform topology.

Under the hypothesis (H.1)-(H.4), there exists a fundamental operator U(t,s) for the equation (2.1) which satisfies

$$\left\| \frac{\partial}{\partial t} U(t,s) \right\| = \| V(t)U(t,s) \| \leq \frac{C}{t-s} \quad ,$$

and consequently, a Cauchy problem for equation (2.1) has only one strongly continuous differentiable solution (see the proof and examples in [7]).

If $\omega \in R$, $0 < \delta < \pi/2$, we denote $S_{\omega,\partial} = \{z \in C;\ |\arg(z-w)| \leq \partial + \pi/2\}$ and $\Sigma_{\omega\,\delta}$ is the closure of the complement of $S_{\omega,\delta}$ in the complex plane C. Are also sufficient conditions for ensuring the existence of a fundamental operator, the following hypothesis:

(H.5) V(t) is a densely defined, closed linear operator and $D(V(t)) \cap D(V(s))$ is dense for any s,t in [0,b].

(H.6) $\Theta(V(t)) \subset \Sigma_{\omega\,\delta}$, $\omega < 0$, $0 < \delta < \pi/2$, $t \in [0,b]$.

(H.7) There exists a, $0 < a < 1$, and a constant C_V such that

$$\| V(t) - V(v))V^{-1}(s) \| \leq C_V \ |t-v|^a$$

uniformly for all $0 \leq t,\ v,\ s \leq b$.

From hypothesis (H.5)-(H.7) and lemma 7 of [8], the domains D(V(t)) are independent of t, and there exists a fundamental solution U(t,s) for equation (2.1), (see [8],[9]).

3. ON THE RESOLUTION OF THE BOUNDARY PROBLEM.

The first result is a necessary condition for the resolution problem (1.1).

THEOREM 1. *Let* U(t) *be a solution of* (1.1), *where the coefficient operators* A(t), B(t), C(t), D(t) *for all* t *in the interval* [0,b] *and* E, F *and* G *are bounded linear operators on* H. *Suppose that the following conditions are satisfied:*

(i) *The operator function* t $\dashrightarrow$ A(t)+B(t)U(t), *generates a funda-*

mental operator.

(ii) *The operator function* $t \dashrightarrow \begin{bmatrix} C(t) & D(t) \\ A(t) & B(t) \end{bmatrix}$, *generates a fundamental operator* $U(t,s)$.

Then $U_o = U(0)$ *satisfies the algebraic operator equation*

$$M + NX - XP - XQX = 0 \tag{3.1}$$

where

$$\left.\begin{matrix} M = E\,U_3(b,0) - G\,U_1(b,0) \quad , \quad P = F\,U_1(b,0) \\ N = E\,U_4(b,0) - G\,U_2(b,0) \quad , \quad Q = F\,U_2(b,0) \end{matrix}\right\} \tag{3.2}$$

$$U(t,s) = \begin{bmatrix} U_1(t,s) & U_2(t,s) \\ U_3(t,s) & U_4(t,s) \end{bmatrix} \tag{3.3}$$

Proof. From the hypothesis (i), there exists only one solution of the following problem

$$d/dt\ Y(t) = (C(t)+D(t)U(t))Y(t) \ ; \ Y(0) = I$$

If we define $Z(t) = U(t)Y(t)$, computing it follows that

$$d/dt\ Z(t) = (A(t)+B(t)U(t))Y(t) \quad ; \quad Z(0) = U(0) = U_o.$$

Thus $\begin{bmatrix} Y(t) \\ Z(t) \end{bmatrix}$, satisfies $\begin{bmatrix} Y(0) \\ Z(0) \end{bmatrix} = \begin{bmatrix} I \\ U_o \end{bmatrix}$, and

$$d/dt \begin{bmatrix} Y(t) \\ Z(t) \end{bmatrix} = \begin{bmatrix} C(t) & D(t) \\ A(t) & B(t) \end{bmatrix} \begin{bmatrix} Y(t) \\ Z(t) \end{bmatrix} = W(t) \begin{bmatrix} Y(t) \\ Z(t) \end{bmatrix} \tag{3.4}$$

From hypothesis (ii), and the properties of a fundamental operator, it follows that

$$\begin{bmatrix} Y(t) \\ Z(t) \end{bmatrix} = U(t,0) \begin{bmatrix} I \\ U_o \end{bmatrix} = \begin{bmatrix} U_1(t,0)+U_2(t,0)U_o \\ U_3(t,0)+U_4(t,0)U_o \end{bmatrix} \tag{3.5}$$

Since $U(t)$ satisfies the boundary condition associated with (1.1), from the expression of $Z(t)$ it follows that

$$EZ(b) = E\,U(b)Y(b) = (G+U_oF)Y(b)$$

and from (3.5)

$$E(U_3(b,0)+U_4(b,0)U_o) = (G+U_oF)(U_1(b,0)+U_2(b,0)U_o) \tag{3.6}$$

$$(EU_3(b,0)-GU_1(b,0))+(EU_4(b,0)-GU_2(b,0)U_o-U_oF\,U_1(b,0)-U_oFU_2(b,0)U_o = 0$$

From (3.2) the result is proved.

The following theorem is a reciprocal one of theorem 1.

THEOREM 2. *Let* U_o *be a solution of equation* (3.1) *with coefficient given by* (3.2). *If hypothesis* (ii) *of theorem* 1 *is satisfied and*

(iii) $U_1(t,0)+U_2(t,0)U_o$ *is invertible for all* t *in* [0,b],

then (1.1) *is solvable, and a solution is given by the expression*

$$U(t) = (U_3(t,0)+U_4(t,0)U_o)(U_1(t,0)+U_2(t,0)U_o)^{-1} \qquad (3.7)$$

Proof. From hypothesis (ii) of theorem 1, there exists a fundamental solution of problem (3.4). Now, we define $Y(t) = U_1(t,0)+U_2(t,0)U_o$, and $Z(t) = U_3(t,0)+U_4(t,0)U_o$, for all t in [0,b]. Thus U(t) given by (3.7) can be expressed by $U(t) = Z(t)Y(t)^{-1}$. It is easy to check that the operator function $t \dashrightarrow \begin{bmatrix} Y(t) \\ Z(t) \end{bmatrix}$ satisfies (3.4) with the initial condition $\begin{bmatrix} Y(0) \\ Z(0) \end{bmatrix} = \begin{bmatrix} I \\ U_o \end{bmatrix}$. By differentiation, it follows that

$$d/dt\ U(t) = \{d/dt\ Z(t)\}(Y(t))^{-1} - Z(t)(Y(t))^{-1}\{d/dt Y(t)\}(Y(t))^{-1} =$$

$$= A(t) + B(t)U(t) - U(t)C(t) - U(t)D(t)U(t)$$

with $U(0) = U_o$. Since U_o is a solution of (1.2) then satisfies (3.6) and postmultiplying this equation by $(Y(b))^{-1}$, it follows that $EU(b) - U(0)F = G$.

The following corollary contains as a particular case theorem 2 of [16], when the coefficient are time-invariant linear operators on H.

COROLLARY 1. Let us consider problem (1.1), where $A(t) = A$, $B(t) = B$, $C(t) = C$, $D(t) = D$ are densely defined closed linear operators on H. If the operator

$$W = \begin{bmatrix} C & D \\ A & B \end{bmatrix}$$

generates a fundamental operator $U(t,s) = \begin{bmatrix} U_1(t,s) & U_2(t,s) \\ U_3(t,s) & U_4(t,s) \end{bmatrix}$,

such that U_o is a solution of (1.2) and

$$U_1(t,0) + U_2(t,0)U_o \quad \text{is invertible for all } t \text{ in } [0,b]$$

then U(t) given by (3.7) is a solution of (1.1).

Proof. Is immediate from theorem 2.

REMARK 1. If A,B,C and D are bounded linear operators on H, then operator W of corollary 1 generates the fundamental operator defined by $U(t,s) = \exp(W(t-s))$, and from corollary 1, it follows theorem 2 of [16].

COROLLARY 2. (Lyapunov equations). Let us consider the following boundary problem

$$\left.\begin{aligned} d/dt\, U(t) = A(t) + B(t)U(t) - U(t)C(t) \\ EU(b) - U(0)F = G \end{aligned}\right\} \qquad (3.8)$$

If the operator functions $t \dashrightarrow B(t)$ and $t \dashrightarrow C(t)$ are generators of fundamental operators $U_B(t,s)$ and $U_C(t,s)$, respectively, $A(t)$ is bounded for all t in $[0,b]$ and

$$\begin{aligned} &N = E\, U_B(b,0) \quad ; \quad P = F\, U_C(b,0) \\ &M = -G\, U_C(b,0) + E \int_0^b U_B(b,s)A(s)U_C(s,0)ds \end{aligned} \qquad (3.9)$$

then problem (3.8) is solvable, if and only if, the equation $M + NX - XP = 0$ is solvable. Moreover the relationship between solutions of both problems is given by

$$U(t) = U_B(t,0)U_o U_C(0,t) + \int_0^t U_B(t,s)A(s)U_C(s,t)ds \qquad (3.10)$$

Proof. It is easy to show that the operator

$$U(t,s) = \begin{bmatrix} U_C(t,s) & 0 \\ \int_s^t U_B(t,v)A(v)U_C(v,s)dv & U_B(t,s) \end{bmatrix} \qquad (3.11)$$

is a fundamental operator of system (3.4) with $D(t) = 0$. If $U(t)$ is a solution of (3.8) and if $Y(t)$ is the only solution of

$$d/dt\, Y(t) = C(t)Y(t) \quad ; \quad Y(0) = I \quad ; \quad t \in [0,b]$$

then $Z(t) = X(t)Y(t)$ satisfies $d/dt\, Z(t) = (A(t) + B(t)X(t))Y(t)$, for all t in $[0,b]$. Thus $\begin{bmatrix} Y(t) \\ Z(t) \end{bmatrix}$ satisfies (3.4), with $D(t) = 0$, and $\begin{bmatrix} Y(0) \\ Z(0) \end{bmatrix} = \begin{bmatrix} I \\ X(0) \end{bmatrix}$. In analougous way to the proofs of theorems 1 and 2, with $D(t) = 0$, the result is proved. In fact, the hypothesis (iii) of theorem 2 is satisfied since $U_1(t,0) = U_C(t,0)$ and $U_2(t,0) = 0$; thus (3.8) is solvable, if and only if, the equation $M+NX-XP = 0$ is solvable. Substituting expressions $U_3(t,0)$ and $U_4(t,0)$ of (3.7) by the correspondent blocks given by (3.11), the result is concluded.

In the previous section, the resolution problem (1.1) has been reduced to an algebraic operator equation (3.1). This equation appears in control problems, [2], [11], [15], and the quadratic eigenvalue problem [5]. It has been studied in different contexts, [3],

[6], [12].

4. APPLICATIONS.

In this section we apply some last results for obtaining b-periodic solutions of generalized operator Riccati equations. Let us consider a generalized operator Riccati equation of the type

$$d/dt\, U(t) = A(t)+B(t)U(t)-U(t)C(t)-U(t)D(t)U(t) \qquad (4.1)$$

The following fact gives sufficient conditions for the existence of b-periodic solutions of (4.1).

THEOREM 3. *With the hypothesis of theorem 2, when* E=F=I, G=0 *and the coefficient functions* A(t), B(t), C(t) *and* D(t) *are* b-*periodics, the operator function* U(t) *given by* (3.7) *is a* b-*periodic solution of* (4.1).

Proof. This is a consequence of theorem 2, for obtaining a solution in [0,b]. Extending b-periodically this solution, the result is proved.

THEOREM 4. *Let us consider* (4.1) *with* D(t) = 0. *If* A(t), B(t) *and* C(t) *are* b-*periodic* ($b > 0$) *operator functions which satisfy the hypothesis of corollary* 2; *if* N , M *and* P *are given by* (3.9) *and*

$$\sigma_{\partial}(N) \cap \sigma_{\pi}(P) = \emptyset \qquad (4.2)$$

where $\sigma_{\partial}(N) = \{\lambda \in C;\ \lambda - N$ *is not onto*$\}$ *and* $\sigma_{\pi}(P)$ *is the approximate point spectrum of* P, *then there exists a solution* U_0 *of* M+NX-XP = 0 *and* U(t), *given by* (3.10), *is a* b-*periodic solution of* (4.1), *with* D(t) = 0.

Proof. From hypothesis (4.2) and theorem 5, p.1387, [4], there exists a solution U_0 of equation M+NX-XP = 0. Now, from corollary 2, U(t) given by (3.10) is a solution in [0,b]. Extending b-periodically U(t), the result is concluded.

COROLLARY 3. Substituting hypothesis (4.2) in theorem 4 by

$$\sigma(N) \cap \sigma(P) = \emptyset \qquad (4.3)$$

there exists only one b-periodic solution of (4.1) with D(t) = 0.

Proof. It is a consequence of theorem 4 and Rosemblun's theorem, [13], p.8.

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Recibido en octubre de 1985.