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HOMOTOPY STABILITY IN BANACH ALGEBRAS

Gustavo Corach

INTRODUCTION.

Let A be a (real or complex) Banach algebra with identity. Several algebraic objects associated to A are considered in this paper. Their common feature is that the general linear group $GL_n(A)$ acts over them defining fibre bundle structures.

In Section 1 we study the map $t_a: GL_n(A) \to U_{k,n}(A)$ defined by multiplication at left, where $U_{k,n}(A)$ is the set of left-invertible nxk-matrices on A ($k \leq n$) and a is a fixed element of $U_{k,n}(A)$; this map is a Serre fibration and, under some Hermite conditions [2,7], it follows that $U_{k,n}(A)$ is a Banach homogeneous space. In Section 2 we study the space $P_n(A)$ of idempotent matrices in $M_n(A)$; for each $p \in P_n(A)$ the map $u \mapsto upu^{-1}$ ($u \in GL_n(A)$) is a fibration. In Section 3 we relate the definition of Grassmannian manifold for Banach algebras, studied by Porta and Recht [10] to the more classical definition $M_{k,n}(A) = GL_n(A)/GL_k(A) \times GL_{n-k}(A)$; we show that, under some connectedness conditions, these spaces are homotopically equivalent. In Section 4 we prove that the sequences $\{\pi_i(GL_n(A))\}_n$, $\{\pi_i(P_n(A))\}_n$, $\{\pi_i(Grass(M_n(A)))\}_n$ and $\{\pi_i(M_{k,n}(A))\}_n$ stabilize under some "stable range conditions" introduced by Bass [1].

These results provide another link between algebraic K-theory and Banach algebra theory. As an application we show that the Grothendieck group $K_0(A)$, which may be presented (Karoubi [8]) as a direct limit $\lim P_{2n}(A)/GL_{2n}(A)$, can be identified with $P_{2n}(A) / GL_{2n}(A)$, for n large enough. Our final result is a Banach algebra version of a theorem of Suslin [2,17]: for $n \geq 1$ and r-1 in the stable range of A, the neutral component of $GL_r(A_n)$ acts transitively on $U_{1,r}(A_n)$, where $A_n = A(I^n)$ in the Banach algebra of continuous maps from $[0,1]^n$ into A. We also prove the analogous of Bass-Quillen conjecture for Banach algebras ([10], p.XI) which is related to Serre's problem. The proof, which is easier than those of Suslin and Bass for the algebraic result, suggests that several problems from algebraic K-theory can be clarified in the Banach algebra category.

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1. UNIMODULAR MATRICES.

Let A be a ring with unit. For $k \leqslant n$ let $U_{k,n}(A)$ be the set of left invertible nxk matrices: $U_{k,n}(A) = \{a \in A^{n\times k}$; there exists $b \in A^{k\times n}$ with $b.a = 1 \in M_k(A)\}$.

If A is a Banach algebra $U_{k,n}(A)$ is an open subset of $A^{n\times k}$. In particular, $U_{k,n}(A)$ is arcwise connected if it is connected.

The action of $GL_n(A)$ on $U_{k,n}(A)$ defined by left multiplication gives, for each $a \in U_{k,n}(A)$, a mapping $t_a : GL_n(A) \to U_{k,n}(A)$ $t_a(\pi) = \sigma.a$. If A is a Banach algebra t_a is a Serre fibration [2], so it induces an exact homotopy sequence [7]

(1.1) $$\dots \to \pi_i(S_a,1) \to \pi_i(GL_n(A),1) \to \pi_i(U_{k,n}(A),a) \to \pi_{i-1}(S_a,1) \to \dots$$

where S_a is the *stabilizer* of a by the action of $GL_n(A)$:

$$S_a = \{\sigma \in GL_n(A) \ ; \ \sigma.a = a\} .$$

1.2. REMARK. In general t_a is not surjective. When $GL_n(A)$ acts transitively on $U_{k,n}(A)$, A is called (n,k)-*Hermite*. See [4] for a closer study of these Banach algebras.

The exact sequence of t_a is better understood when a is the nxk matrix e whose columns are the first k canonical vectors of A^n. In this case

$$S_e = L_{k,n}(A) = \{\begin{pmatrix} 1 & x \\ 0 & \sigma \end{pmatrix} \in GL_n(A) \ ; \ x \in A^{k\times(n-k)} \ , \ \sigma \in GL_k(A)\}$$

which is homeomorphic to the product $GL_k(A) \times A^{k\times(n-k)}$. Now, $A^{k\times(n-k)}$ being contractible, the exact sequence becomes

(1.3) $$\dots \to \pi_i(GL_k(A),1) \overset{\alpha}{\to} \pi_i(GL_n(A),1) \to \pi_i(U_{k,n}(A),e) \to \pi_{i-1}(GL_k(A),1) \to \dots$$

where a is induced by the inclusion $\sigma \to \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}$ of $GL_k(A)$ into $GL_n(A)$.

1.4. THEOREM. *If* A *is* (n,k)-*Hermite then* $U_{k,n}(A)$ *is homeomorphic to the Banach homogeneous space* $GL_n(A)/L_{k,n}(A)$.

Proof. Given a topological group G and a closed subgroup H the projection $p: G \to G/H$ is a (locally trivial) fibre bundle if it admits a local section at H [7]. In particular, if G is a Banach-Lie group p is always a fibre bundle. Thus, if A is (n,k)-Hermite, then $t_e: GL_n(A) \to U_{k,n}(A)$ is a fibre bundle and it induces a homeomorphism $\bar{t}_e: GL_n(A)/L_{k,n}(A) \to U_{k,n}(A)$.

1.5. THE COMMUTATIVE CASE. The image of t_e is open and closed [4] so A is (n,k)-Hermite if $U_{k,n}(A)$ is connected. Now, if A is a complex commutative Banach algebra, a simple application of the Novodvorski-Taylor theory shows that $U_{k,n}(A)$ is connected if and only if all maps from the spectrum X(A) of A into $U_{k,n}(\mathbf{C})$ (the Stieffel manifold of k-frames in $\mathbf{C}^n$) are null-homotopic. Thus, if X(A) is dominated by a compact space of (Lebesgue) dimension at most 2(n-k), $U_{k,n}(A)$ is connected [11,4].

1.6. Let X be a compact space and A(X) the Banach algebra of all (continuous) maps from X into A, with the sup norm. For every x in X the evaluation morphism $\varepsilon = \varepsilon_x: A(X) \to A$ $\varepsilon(f) = f(x)$ has an algebra section ; more precisely, the morphism $s: A \to A(X)$ $s(a) =$ = the constant map $x \mapsto a$, satisfies $\varepsilon \circ s = 1_A$. In general, an epimorphism of Banach algebras $\phi: A \to B$ induces a Serre fibration $\phi: U_{k,n}(A) \to U_{k,n}(B)$ which is a fibre bundle when ϕ admits an algebra section. In this case, if $\phi(a_o) = b_o$ and $F = \{a \in U_{k,n}(A);\ \phi(a_o) = b_o\}$, the homotopy sequence of the fibration $F \to U_{k,n}(A) \to$ $\to U_{k,n}(B)$ splits at each i and produces short exact sequences

$$0 \to \pi_i(F,a_o) \to \pi_i(U_{k,n}(A),a_o) \to \pi_i(U_{k,n}(B),b_o) \to 0 \ .$$

Returning to the situation $\phi = \varepsilon$, $a_o = b_o = e$, when $X = S^\ell$ and $i = 0$, we get

$$0 \to \pi_\ell(U_{k,n}(A),e) \to \pi_o(U_{k,n}(A(S^\ell)),e) \to \pi_o(U_{k,n}(A),e) \to 0$$

Thus, if $U_{k,n}(A)$ is connected there is a bijection between

$\pi_\ell(U_{k,n}(A),e)$ and the set of connected components of $U_{k,n}(A(S^\ell))$. This remark will be used in §4.

2. PROJECTIONS.

Let B a ring with unit and P(B) the subset of idempotents of B: $P(B) = \{b \in B;\ b^2 = b\}$. For each $b \in P(B)$, $B^\cdot$ acts on P(B) by inner automorphisms, that is b defines a map $\theta_b: B^\cdot \to P(B)$ $\theta_b(\sigma) = \sigma b \sigma^{-1}$. If B is a Banach algebra, each θ_b is an open map and its image M_b is open and closed in P(B); moreover, P(B) is a Banach manifold, θ_b defines a fibre bundle over M_b and there is a homotopy sequence

$$\dots \to \pi_i(R_b,b) \to \pi_i(B^\cdot,1) \to \pi_i(P(B),b) \to \pi_{i-1}(R_b,b) \to \dots$$

where $R_b = \{\sigma \in B^\cdot\ ;\ \sigma b = b\sigma\}$ [13,14].

If A is a Banach algebra and B is the algebra of all nxn-matrices on A, $B = M_n(A)$, let $P_n(A) = P(B)$. For b=1 (see §1)

$$R_b = R_{k,n}(A) = \{\begin{pmatrix}\sigma & x\\ 0 & \tau\end{pmatrix} \in GL_n(A)\ ;\ \sigma \in GL_k(A)\ ,\quad \tau \in GL_{n-k}(A)\ ,\quad x \in A^{k\times(n-k)}\}\ ,$$

which is clearly homeomorphic to the product $GL_k(A) \times GL_{n-k}(A) \times A^{k\times(n-k)}$.

Thus, the exact sequence of θ_e becomes

$$\dots \to \pi_i(GL_k(A),1) \to \pi_i(GL_{n-k}(A),1) \overset{\beta}{\to} \pi_i(GL_n(A),1) \to$$

$$\pi_i(P_n(A),e) \to \pi_{i-1}(GL_k(A),1) \to \pi_{i-1}(GL_{n-k}(A),1) \to \dots$$

where β is the homomorphism induced by the inclusion $(\sigma,\tau) \mapsto \begin{pmatrix}\sigma & 0\\ 0 & \tau\end{pmatrix}$ of $GL_k(A) \times GL_{n-k}(A)$ into $GL_n(A)$.

3. THE GRASSMANN MANIFOLD.

Porta and Recht [13] propose the following algebraic definition for the Grassmann manifold: given a ring A with unit, let P(A) be the set of idempotent elements of A, $P(A) = \{a \in A;\ a^2 = a\}$ and consider the equivalence relation $\sim$ on P(A) defined by $a \sim b \iff ab = b$ and $ba = a$. The Grassmannian of A is the set Grass(A) = = $P(A)/\sim$. When A is a Banach algebra and Grass A is given the quotient topology, then the following result holds [13, §3]:

3.1. THEOREM. *Let* β: $P(A) \to \mathrm{Grass}(A)$ *be the projection map. Then*

3.1.1. β *is an open map*;

3.1.2. Grass(A) *is paracompact*;

3.1.3. β *has a continuous global section*;

3.1.4. β *is a homotopy equivalence.*

In the particular case when A the algebra $M_n(\mathbf{R})$ of all nxn real matrices, Grass(A) can be identified with the classical Grassmannian $\bigcup_{0\le k\le n} G_{k,n}$, where $G_{k,n}$ is the set of all k-dimensional subspaces of $\mathbf{R}^n$. Moreover, the connected components of Grass(A) are, precisely, the $G_{k,n}$.

Let us study more closely the Grassmannian of $M_n(A)$ when A is a Banach algebra. For this, we identify $A^k \times A^{n-k}$ with A^n and consider the following subgroups of $GL_n(A)$:

$$L_{k,n}(A) = \{\begin{pmatrix} 1 & x \\ 0 & \sigma \end{pmatrix} \in GL_n(A) \quad ; \quad x \in A^{k\times(n-k)} \quad , \quad \sigma \in GL_{n-k}(A)\},$$

$$H_{k,n}(A) = \{\begin{pmatrix} \tau & 0 \\ 0 & 1 \end{pmatrix} \in GL_n(A) \quad ; \quad \tau \in GL_k(A)\}.$$

We set $M_{k,n} = M_{k,n}(A) = GL_n(A)/L_{k,n}(A) \times H_{k,n}(A)$. Observe that $H_{k,n}(A)$ is isomorphic to $GL_k(A)$ (as topological groups) and that $L_{k,n}(A)$ is isomorphic to the semidirect product $A^{k\times(n-k)} \times GL_{n-k}(A)$. This remark, the general theory of Banach-Lie homogeneous spaces [9] and a well-known result of Palais [12, Th.15] yield the following

3.2. PROPOSITION. a. $M_{k,n}(A)$ *is homeomorphic to* $M_{n-k,n}(A)$;
b. $M_{k,n}(A)$ *is homotopically equivalent to* $GL_n(A)/GL_k(A) \times GL_{n-k}(A)$.

As a consequence of b., the exact sequence of the fibration $L_{k,n} \times H_{k,n} \to GL_n \to GL_n/GL_k \times GL_{n-k}$ becomes

$$\begin{aligned} &\ldots \to \pi_i(GL_{n-k}(A),1) \times \pi_i(GL_k(A),1) \to \pi_i(GL_n(A),1) \\ (3.3)\quad & \\ &\to \pi_i(M_{k,n}(A),\bar{1}) \to \pi_{i-1}(GL_{n-k}(A),1) \times \pi_{i-1}(GL_k(A),1) \to \ldots \end{aligned}$$

In connection with (1.1) we have

3.4. PROPOSITION. *If* A *is* (n-k)-*Hermite*, t_e *induces a principal lo-*

cally trivial fibre bundle $U_{k,n}(A) \to M_{k,n}(A)$ *whose fibers are homeomorphic to* $GL_k(A)$. *In particular* $M_{k,n}(A)$ *is homeomorphic to* $U_{k,n}(A)/GL_k(A)$.

3.5. REMARK. Let X be a compact space and A(X) the algebra of A-valued continuous maps on X. It is easy to prove that $GL_m(A(X))$ is isomorphic to $C(X,GL_m(A))$ (as topological groups). In the same way we get a homeomorphism from $M_{k,n}(A(X))$ onto $C(X,M_{k,n}(A))$. If $M_{k,n}(A)$ is connected, from the fibration properties of the evaluation maps $M_{k,n}(A(X)) \to M_{k,n}(A)$ we can prove that $\pi_i(M_{k,n}(A))$ is in a bijective correspondence with the set $[S^i,M_{k,n}(A)]$ (cf. [4,2.4]). This is particularly useful when A is a complex commutative algebra, for in this case

$$[S^i,M_{k,n}(A)] \leftrightarrow [S^i \times X(A),M_{k,n}(C)] \ .$$

3.6. PROPOSITION. *For each* $k \leq n$ $M_{k,n}(A)$ *is homeomorphic to a union of connected components of* $\mathrm{Grass}(M_n(A))$. *Moreover, if* $U_{k,n}(A)$ *is connected,* $M_{k,n}(A)$ *is (homeomorphic to) a connected component of* $\mathrm{Grass}(M_n(A))$.

Proof. Consider the composition $GL_n(A) \xrightarrow{\theta_e} P_n(A) \xrightarrow{\beta} \mathrm{Grass}(M_n(A))$. It is easy to see that, if σ has the form $\sigma = \begin{pmatrix} \tau & x \\ 0 & \rho \end{pmatrix}$ with $\tau \in GL_k(A)$, $\rho \in GL_{n-k}(A)$ and $x \in A^{k\times(n-k)}$, then $\beta(\sigma e \sigma^{-1}) = \beta(e)$. Thus we get a map $\overline{\theta}_e : M_{k,n}(A) \to \mathrm{Grass}(M_n(A))$. But θ_e and $\beta \circ \theta_e$ both are open maps and their images are closed in $P_n(A)$ and $\mathrm{Grass}(M_n(A))$, respectively. The result follows, then, by (3.4).

3.7. COROLLARY. If $U_k(A^n)$ is connected

$$\pi_i(M_{k,n}(A),\overline{e}) = \pi_i(\mathrm{Grass}(M_n(A)),\beta(e)) \ ,$$

where $\overline{e}$ denotes the image of $e \in U_k(A^n)$ in $M_{k,n}(A)$ by the fibre map of (3.3).

4. STABILIZATION.

Let A be a ring with unit. The stable rank of A, denoted by sr(A), is the least integer n such that, for every ${}^{t}a \in U_{1,n+1}(A)$ there exist elements $x_1, x_2, \dots, x_n$ in A with ${}^{t}(a_1+x_1a_{n+1}, a_2+x_2a_{n+1}, \dots$ $\dots, a_n+x_na_{n+1}) \in U_{1,n}(A)$.

If no such integer exists we set $sr(A) = \infty$. Some results relating sr(A) to the topology of A may be found in [5], [6], [15]. We quote two of them, from which we shall deduce several consequences.

4.1. PROPOSITION. ([6]). *Let* A *be a Banach algebra. For every* n *and* k *such that* $n \geqslant sr(A) + k$, $U_{k,n}(A)$ *is connected.*

4.2. THEOREM. ([6, th.5.12]). *Let* X *be a compact space and* A *a Banach algebra. Then* $sr(A(X)) \leqslant d + sr(A)$, *where* d *is the (topological) dimension of* X.

(This result answers, partially, the questions 1.8 and 7.3 raised by Rieffel [15]).

From (1.3), (1.6), (4.1) and (4.2) we get

4.3. COROLLARY. Let A be a Banach algebra and $n \geqslant sr(A) + i+k$. Then

(i) $\pi_i(U_{k,n}(A), e)$ is trivial;

(ii) $\pi_i(GL_k(A),1) \cong \pi_i(GL_n(A),1) \cong \pi_i(GL(A),1) = K^{top}_{i+1}(A)$ ([9]).

Let us consider the commutative square

$$\begin{array}{ccc} GL_n(A) & \xrightarrow{\ i\ } & GL_{n+1}(A) \\ \downarrow{\scriptstyle \theta_e} & & \downarrow{\scriptstyle \theta_{e'}} \\ P_n(A) & \xrightarrow{\ j\ } & P_{n+1}(A) \end{array}$$

where $i(\sigma) = \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}$, $j(p) = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ and e' is the (n+1)xk matrix defined in the same way that e (§1). Comparing the homotopy sequences of θ_e and $\theta_{e'}$ (§2) and using 4.3.ii we get

4.4. COROLLARY. Let A be a Banach algebra and k, $n-k \geqslant sr(A)+i$. Then

$$\pi_i(P_n(A), e) \cong \pi_i(P_{n+1}(A), e').$$

(Observe that k appears implicitely in this formula, for e and e' depend on k).

Next, we use the preceding corollaries, 3.1.4 and 3.3 to obtain a similar result for the Grassmannian manifolds defined in §3:

4.5. COROLLARY. Let A be a Banach algebra. For k, n-k $\geqslant$ sr(A) + i it holds

(i) $\pi_i(\mathrm{Grass}(M_n(A)), \beta(e)) \cong \pi_i(\mathrm{Grass}(M_{n+1}(A)), \beta(e'))$

(ii) $\pi_i(M_{k,n}(A), \bar{e}) \cong \pi_i(M_{k,n+1}(A), \bar{e}')$.

The next result concerns the group $K_o(A)$. This is the Grothendieck group of the category of projective finitely generated left A-modules. Karoubi 8 has given an alternative description of $K_o(A)$: the action of $GL_n(A)$ on $P_n(A)$ defined in §2 allows us to consider a direct limit

$$(4.6) \qquad \widehat{P_2(A)} \to \widehat{P_4(A)} \to \dots \to \widehat{P_{2n}(A)} \xrightarrow{i_n} \widehat{P_{2n+2}(A)} \dots$$

where $\widehat{P_n(A)} = P_n(A)/GL_n(A)$ and i_n is induced by the mapping

$$p \mapsto \begin{pmatrix} p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Karoubi shows that $K_o(A)$ is isomorphic to $\varinjlim \{\widehat{P_{2n}(A)}, i_n\}$. We prove now that the sequence $\{\widehat{P_{2n}(A)}, i_n\}$ stabilizes if sr(A) is finite. More precisely

4.7. PROPOSITION. *Let* A *be a Banach algebra and* n $\geqslant$ sr(A). *Then*

$$\widehat{P_{2n}(A)} \cong \widehat{P_{2n+2}(A)}.$$

Let $X = P_{2n}(A)$, $Y = P_{2n+2}(A)$, $H = GL_{2n}(A)$ and $G = GL_{2n+2}(A)$. Observe that $\widehat{P_{2n}(A)} = X/H$ and $\widehat{P_{2n+2}(A)} = Y/G$ and that we have a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X/H & \xrightarrow{\bar{f}} & Y/G \end{array}$$

where $\bar{f}$ (class of p) = class of $\begin{pmatrix} p & & \\ & 1 & 0 \\ & 0 & 0 \end{pmatrix}$, for $p \in X$. By (4.2) and (4.3) $\pi_o(f): \pi_o(X) \to \pi_o(Y)$ and $\pi_o(i): \pi_o(H) \to \pi_o(G)$ are bijections and we have observed that, for each y in X (or Y), $U = \{\sigma q \sigma^{-1}\ ;\ \sigma \in H \text{ (or } G)\}$ is open and closed in X (or Y). Thus, the connected component of q is contained in U. A straightforward argument shows that $\bar{f}$ is, then, a bijection. The proof finishes just remarking that $\bar{f} = i_n$.

Our last result is a topological version of a theorem of Suslin

[3, Th.3], [17, Th.12.4]. Let A a commutative noetherian ring with Krull dimension d, let $A_n = A[t_1,\dots,t_n]$ the A-algebra of polynomials in n indeterminates. Suslin proved that, for every $n \geq 1$ the set of elementary matrices $E_r(A_n)$ acts transitively on $U_{1,r+1}(A_n)$ if $r \geq 1 + \max\{d,(d+n)/2\}$.

(Recall that, for a ring C, $E_r(C)$ is the subgroup of $GL_r(C)$ generated by the rxr-matrices $1+e^c_{ij}$, where $c \in C$ and $(e^c_{ij})_{k\ell} = c\delta_{ik}\delta_{j\ell}$). In a topological setting, $A[t_1,\dots,t_n]$ is replaced by $A(I^n)$ (the notation is like in 1.6), $E_r(C)$ by the neutral component $GL_r(C)_0$ of $GL_r(C)$ and the Krull dimension by the stable rank. More precisely we have the following

4.8. THEOREM. *Let* A *be a Banach algebra. Let* $A_n = A(I^n)$ *the algebra of continuous maps* $I^n = [0,1]^n \to A$. *Then, for every* $n \geq 1$, $GL_r(A_n)_o$ *acts transitively on* $U_{1,r}(A_n)$ *if* $r \geq sr(A) + 1$.

Proof. Recall that $t: GL_r(B)_o \to U_{1,r}(B)$, $t(\sigma) =$

$$= \sigma \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma_{11} \\ \vdots \\ \sigma_{n1} \end{pmatrix}$$ is a Serre fibration for every Banach algebra B (§1).

In particular, $GL_r(B)_o$ acts transitively on $U_{1,r}(B)$ if and only if $U_{1,r}(B)$ is connected. It is also known that the connected components of $U_{1,r}(B(X))$ are in a bijective correspondence with the set $[X,U_{1,r}(B)]$ of homotopy classes of maps $X \to U_{1,r}(B)$ [4,2.4]. Then, for B = A and $X = I^n$ we get that $U_{1,r}(A_n)$ is connected if and only if $[I^n,U_{1,r}(A)]$ is trivial and, I^n being contractible, this happens if and only if $U_{1,r}(A)$ is connected. But $U_{1,r}(A)$ is connected for $r \geq sr(A) + 1$ (see 4.1). This concludes the proof.

4.9. COROLLARY (of the proof). Let A be a Banach algebra. Then A_n is (m,k)-Hermite if and only if A is (m,k)-Hermite.

The result answers affirmatively the Banach algebra analogous of question (H) in [10] p.XI and, consequently, of Bass-Quillen conjecture. See the introduction of Lam's book for details.

REMARKS. 1. In the category of Banach algebras this result offers some advantages over that of Suslin: in fact, it holds even for noncommutative Banach algebras and the number sr(A) is smaller than max {sr(A), (sr(A)+n)/2} when n is large.

2. In the terminology of Rieffel [15], Theorem 4.8 says that the connected stable rank of A_n is a most sr(A)+1. In connection with that paper, it should be noted that left and right connected stable ranks, introduced in [15], actually coincide, for the spaces of left and right unimodular rows are homotopy equivalent [4]. This answers question 4.8 of [15].

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Instituto Argentino de Matemática
Viamonte 1636
1055 Buenos Aires
Argentina

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