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A GENERALIZATION OF THE FUNDAMENTAL FORMULA FOR CYLINDERS

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1. ABSTRACT.

Using the G_n-invariant measure on the set of the convex infinite cylinders congruent to Z_q that touch a convex body K, we can generalize the fundamental kinematic formula for cylinders in the sense of Hadwiger in [2].

We obtain a bilinear combination of the "Quermassintegrale" of K and Z_q, with coefficients depending on the first n moments of a non-negative Borel measurable function.

2. INTRODUCTION.

Hadwiger, [2], found a generalization of the kinematic fundamental formula for convex bodies, extending the domain of integration to the whole group of motions in E_n, G_n. He proved that this integral is again a bilinear combination of the "Quermassintegrale" of the convex bodies.

On the other hand, in [1] we find the definition of n+1 functionals on convex bodies, which include the "Quermassintegrale" as particular cases.

In this paper we generalize the fundamental kinematic formula for cylinders in the same sense that Hadwiger in [2]. We show that we obtain a bilinear combination of the "Quermassintegrale" of the convex body and the infinite convex cylinder. This generalization includes [1] and [2] as particular cases.

The proof is essentially based on the existence of a tangential measure on the set of infinite convex cylinders congruent to a fixed one Z_q ([3]) and results much easier.

3. NOTATION.

Let E_n be the n-dimensional euclidean space with unit sphere Ω_n. λ_n is the Lebesgue-measure in E_n and $\omega_n = \lambda_n(\Omega_n)$. We denote by K_n the set of all convex, compact and non-empty subsets K in E_n, $K \in K_n$ is called a convex body. For $K \in K_n$, $W_i^n(K)$ are the "Quermassintegrale" of K (i = 0,...,n). If $\delta \geqslant 0$, K_δ is the parallel body in the distance δ of K, and the following Steiner formula holds ([4], p.220-221):

$$\lambda_n(K_\delta) = \sum_{j=0}^{n} \binom{n}{j} W_j^n(K) \delta^j$$

$$W_i^n(K_\delta) = \sum_{j=0}^{n-i} \binom{n-i}{j} W_{i+j}^n(K) \delta^j \quad . \tag{3.1}$$

Now we define infinite convex cylinders as in [4], p.270. Let O be a fixed point in E_n and let L_{n-q} be a (n-q)-plane through O. Let D be a bounded convex body in L_{n-q}. For each point x in D we consider the q-plane orthogonal to L_{n-q} through x. The union of all such L_q is the cylinder Z_q. The q-planes L_q are the generators and D a normal cross section of Z_q. As in [4], p.272, we take

$$W_i^n(Z_q) = \begin{cases} W_i^{n-q}(D) & 0 \leqslant i \leqslant n-q \\ 0 & n-q < i \leqslant n \end{cases} \tag{3.2}$$

We will now explain briefly the fundamental kinematic formula for cylinders ([4], p.272). Let $Z(D)_q$ be the set of all cylinders congruent to Z_q and let γ_D be the normalized G_n-invariant measure on $Z(D)_q$ ([5], p.106). If $K \in K_n$, A is the set of cylinders of $Z(D)_q$ that intersect K, that is $A = \{Z \in Z(D)_q \ / \ Z \cap K \neq \emptyset\}$.

We denote with $\gamma_D(K)$, the measure of A in the sense of γ_D, $\gamma_D(K) = \int_A d\gamma_D$, and the fundamental formula for cylinders ([4], p.272) holds:

$$\gamma_D(K) = \frac{1}{\omega_n} \sum_{t=0}^{n-q} \binom{n-q}{t} W_{t+q}^n(K) \, W_{n-t-q}^n(Z_q) \tag{3.3}$$

4. GENERALIZATION OF THE FUNDAMENTAL FORMULA (3.3).

Let $f: (0,\infty) \to [0,\infty)$ be an arbitrary Borel measurable function for which holds:

i) $f(0) \neq 0$

ii) The first n moments

(4.1) $$M_k(f) = \int_0^\infty f(r)\, r^k\, dr \qquad (k = 0,\dots,n-1)$$

are finite.

If we denote with $r = d(K,Z)$ the distance between the convex body K and the cylinder Z, we make the following integral

(4.2) $$R_q(f,K,Z_q) = \frac{\omega_n}{\omega_{n-q}} \int f(r)\, d\gamma_D \quad ,$$

where we integrate over the whole space $Z(D)_q$.

The existence of this integral is assured by the existence of the moments of f (4.1).

PROPOSITION. *If* $R_q(f,K,Z_q)$ *is as in* (4.2), *the following fundamental formula holds*

(4.3) $$R_q(f,K,Z_q) = \frac{1}{\omega_{n-q}} \sum_{t=0}^{n-q} \sum_{s=0}^{t} C_s(f) \binom{n-q}{t-s} \binom{s+n-t-q}{s} W^n_{t+q}(K)\, W^n_{s+n-t-q}(Z_q)$$

with $C_0(f) = f(0)$, $C_s(f) = s\, M_{s-1}(f)$ $(s = 1,\dots,n)$.

That means that $R_q(f,K,Z_q)$ *is a bilinear combination of the "Quermassintegrale" of* K *and* Z_q. *The moments of* f *appear in the coefficients.*

Proof. We need some previous results. Let $Z(D,K)_q$ be the set of all cylinders congruent to Z_q that touch K (they have non-empty intersection with K, but can be separated weakly by an hyperplane). If A_δ is the set of cylinders congruent to Z_q that intersect K_δ and not K, $A_\delta = \{Z \in Z(D)_q \,/\, Z \cap K_\delta \neq \emptyset$ and $Z \cap K = \emptyset\}$, then we have $Z(D,K)_q = \lim_{\delta\to 0} A_\delta$.

In [3] we defined a G_n-invariant measure ϕ_D on $Z(D,K)_q$ which results natural as it verifies $\phi_D(Z(D,K)_q) = \lim_{\delta\to 0} \frac{1}{\delta} \gamma_D(A_\delta)$.

We also showed in [3] that $\phi_D(Z(D,K)_q)$ is a bilinear combination of the "Quermassintegrale" of K and Z_q:

(4.4) $$\phi_D(Z(D,K)_q) = \sum_{j=0}^{n-q-1} \binom{n-q}{j+1} \frac{j+1}{\omega_n} W^n_{n-j}(K)\, W^n_{j+1}(Z_q)$$

Now we may prove (4.3). Following (4.2),

$$R_q(f,K,Z_q) = \frac{\omega_n}{\omega_{n-q}} \int f(r)\, d\gamma_D.$$

With an obvious reformulation we reach to

(4.5) $$R_q(f,K,Z_q) = \frac{\omega_n}{\omega_{n-q}} [f(0)\ \gamma_D(K) + \int_0^\infty (f(r)\ \phi_D(Z(D,K_r)_q)\ dr]$$

where $Z(D,K_r)_q$ denotes the set of cylinders congruent to Z_q that touch K_r (with K_r the parallel body to K in the distance r).

Using (4.4) we obtain

$$\phi_D(Z(D,K_r)_q) = \sum_{j=0}^{n-q-1} \binom{n-q}{j+1} \frac{j+1}{\omega_n} W^n_{n-j}(K_r)\, W^n_{j+1}(Z_q)$$

and taking (3.1) into account, it is

$$\phi_D(Z(D,K_r)_q) = \sum_{j=0}^{n-q-1} \sum_{k=0}^{j} \frac{j+1}{\omega_n} \binom{n-q}{j+1} \binom{j}{k} W^n_{j+1}(Z_q)\, W^n_{n-j+k}(K)\, r^k .$$

Now we take $s = k+1$, $t = n+k-j-q$, and get

$$\phi_D(Z(D,K_r)_q) = \sum_{t=1}^{n-q} \sum_{s=1}^{t} \frac{s\, r^{s-1}}{\omega_n} \binom{n-q}{t-s} \binom{s+n-t-q}{s} W^n_{t+q}(K)\, W^n_{s+n-t-q}(Z_q).$$

Hence

(4.6) $$\int_0^\infty f(r)\ \phi_D(Z(D,K_r)_q)\ dr =$$

$$= \sum_{t=1}^{n-q} \sum_{s=1}^{t} \frac{s\, M_{s-1}(f)}{\omega_n} \binom{n-q}{t-s} \binom{s+n-t-q}{s} W^n_{t+q}(K)\, W^n_{s+n-t-q}(Z_q) .$$

If we now use formula (3.3) for γ_D and together with (4.6) replace them in (4.5), we reach the desired result.

5. SPECIAL CHOOSES FOR f, q AND D.

5.1 Special choose for f.

If we take $f(0) = 1$ and $f(r) = 0$ $(r > 0)$ then from (4.3) we obtain (3.6) except an irrelevant constant

$$R_q(f,K,Z_q) = \frac{\omega_n}{\omega_{n-q}} \gamma_D(K).$$

5.2 Special choose for q.

In the case q=0, that means D is a convex body in E_n, by straightforward calculations we get

$$R_0(f,K,D) = \frac{1}{\omega_n} \sum_{t=0}^{n} \sum_{s=0}^{t} C_s(f) \binom{n}{t-s} \binom{s+n-t}{s} W_t^n(K)\, W_{s+n-t}^n(Z_q).$$

Hence $R_0(f,K,D) = J(f;K,D)$, where $J(f;K,D)$ is the kinematic integral defined by Hadwiger in [2].

5.3 Special choose for D.

Finally, if we take D in L_{n-q} as a point P, Z_q results a q-plane in E_n and taking into account that $W_0^n(Z_q) = W_0^{n-q}(P) = \omega_{n-q}$ and $W_k^n(Z_q) = W_k^{n-q}(P) = 0 \quad 1 \leqslant k \leqslant n$, from (4.3) we obtain

$$R_q(f,K,P) = \sum_{t=0}^{n} C_t(f) \binom{n-q}{t} W_{t+q}^n(K).$$

That is $R_q(f,K,P) = P_q(f,K)$, where $P_q(f,K)$ are the functionals defined in [1].

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