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ON FINITE BASIC SETS IN METRIC SPACES

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ABSTRACT. We prove that for every non-finite, locally connected, metrizable, topological space there exists a compatible distance such that it is not possible to choose a finite subset B and uniquely determine the points of the space by their distances to the points in B. The proof makes use of Cantor's function which is shown to be subadditive.

1. INTRODUCTION.

Let (M,d) be a metric space. A subset B of M is called basic if and only if every point in M is uniquely determined by its distances to the points in B [1].

It has recently been shown [1] that: i) every compact connected Riemannian manifold (M,g) with the distance d naturally associated to g does admit a finite basic set; ii) a compact connected topological space M can be imbedded in a finite dimensional Euclidean space if and only if M is metrizable and admits a distance with a finite basic set.

For a given topological space there may exist many different compatible distances. We show that, under certain conditions, some of them do not admit finite basic sets.

For example: Let M be the real interval [0,1]; M with the usual distance admits {0} as a finite basic set; but no finite basic set exists for M with distance $d(x,y) = f(|x-y|)$, where f is the Cantor function. Clearly d is not compatible with any Riemannian metric on M; but d is compatible (Corollary 3.4) with the usual topology [2].

2. DEFINITIONS.

Let K be the Cantor set:

$$K = \{x : x = \sum_{i=1}^{\infty} x_i/3^i \ , \quad x_i \in \{0,2\}\}$$

(that is, $K \subset [0,1]$ is the set of the numbers admitting a ternary representation without digits 1).

We shall call Cantor's function the following one:

$f: [0,\infty) \to [0,1]$ and if $x = x_0 + \sum_{i=1}^{\infty} x_i/3^i$ with x_i in $\{0,1,2\}$ for $i > 0$, x_0 in $N \cup \{0\}$ then

i) if $x_0 > 0$ then $f(x) = 1$.

ii) if $x_0 = 0$ and for each i in N $x_i \neq 1$, then $f(x) = \sum_{i=1}^{\infty} x_i/2/2^i$.

iii) if $x_0 = 0$ and for some i in N $x_i = 1$, then
$f(x) = \sum_{i=1}^{I-1} x_i/2/2^i + 1/2^I$, where $I = \min\{i : x_i = 1\}$.

Note that we are using a ternary expansion for the fractional part of x and a binary expansion for f(x). Whenever x admits two ternary expansions, the above rule results in two expansions for f(x), both with the same value.

3. PROPERTIES OF CANTOR'S FUNCTION.

(F1) f is constant over every interval without points in K.

(F2) $f(x) = 0$ if and only if $x = 0$.

(F3) f is non-decreasing.

(F4) f is continuous.

(F5) f is subadditive.

The first four properties are well known. We shall prove the fifth one. (Theorem 3.3).

LEMMA 3.1. *For any* x,y *in* K, $f(x+y) \leq f(x) + f(y)$.

Proof. Let $x = \sum_{i=1}^{\infty} x_i/3^i$, $y = \sum_{i=1}^{\infty} y_i/3^i$ with x_i, y_i in $\{0,2\}$.
It follows that $x_i + y_i \in \{0,2,4\}$.

We consider two complementary cases:

Case 1: For each i in N, $x_i + y_i \neq 4$. In this case $\sum_{i=1}^{\infty} (x_i+y_i)/3^i$ is a ternary expansion for x+y; therefore
$f(x+y) = \sum_{i=1}^{\infty} (x_i+y_i)/2/2^i = f(x)+f(y)$.

Case 2: For some i in N, $x_i + y_i = 4$. Let J and I be:

$$J = \min\{j \in N: x_j = y_j = 2\}\ ,$$
$$I = \max(\{i \in N: x_i = y_i = 0\ ,\ i < J\} \cup \{0\}).$$

It follows that $0 \leqslant I < J < \infty$, $x_I = y_I = 0$, $x_J = y_J = 2$,
$i < I \Rightarrow x_i+y_i \leqslant 2$, $I < i < J \Rightarrow x_i+y_i = 2$.

(For example:
$$\begin{aligned} x &= 0\,.\,0\,0\,2\,0\,0\,0\,2\,0\,2\,0\,2\,2\,0\,2\,0\,2\,0\,2\,\ldots\ldots\\ y &= \underbrace{0\,.\,2\,0\,0\,2\,0}_{x_i+y_i\leqslant 2}\,\underset{I}{\underset{\uparrow}{0}}\,\underbrace{0\,2\,0\,2\,0\,0\,2}_{x_i+y_i=2}\,\underset{J}{\underset{\uparrow}{2}}\,2\,0\,0\,0\,\ldots\ldots \end{aligned}$$
$I = 6$; $J = 14$;).

Therefore $\quad x \leqslant \sum_{i=1}^{I-1} x_i/3^i + 1/3^I \quad , \quad y \leqslant \sum_{i=1}^{I-1} y_i/3^i + 1/3^I$

and
$$x+y \leqslant \sum_{i=1}^{I-1} (x_i+y_i)/3^i + 2/3^I.$$

Using the definition of f and the fact that f is non-decreasing,
$$f(x+y) \leqslant \sum_{i=1}^{I-1} (x_i+y_i)/2/2^i + 1/2^I\ .$$

On the other hand,

$$f(x) + f(y) = \sum_{i=1}^{\infty} (x_i+y_i)/2/2^i \geqslant \sum_{i=1}^{I-1} (x_i+y_i)/2/2^i + R\ ,$$

where $\quad R = \sum_{i=I}^{J} (x_i+y_i)/2/2^i.$

But $\quad R = (0+0)/2/2^I + \sum_{i=I+1}^{J-1} 2/2/2^i + (2+2)/2/2^J = 1/2^I$.

Hence $\quad f(x+y) \leqslant f(x) + f(y)$.

LEMMA 3.2. *For any* x *in* [0,1] *there exists* x' *in* K *such that* $x \leqslant x'$ *and* $f(x) = f(x')$.

Proof. Of course, if $x \in K$ we choose $x' = x$; if $x \notin K$ any ternary expansion of x will have some 1's. Let $I = \min\{i \in N: x_i = 1\}$.

We choose $\quad x' = \sum_{i=1}^{I-1} x_i/3^i + 2/3^I.$

THEOREM 3.3. (Subadditivity of Cantor's function). *For any* x,y *in* $[0,\infty)$, $f(x+y) \leqslant f(x) + f(y)$.

Proof. We consider two cases.

Case I: x,y in [0,1]. Let $x',y' \in K$ such that (Lemma 3.2):

$x \leqslant x'$, $f(x) = f(x')$, $y \leqslant y'$, $f(y) = f(y')$. Then $f(x+y) \leqslant$ $\leqslant f(x'+y') \leqslant f(x') + f(y') = f(x) + f(y)$. (The first inequality holds because f is non decreasing; the second one because of Lemma 3.2).

Case II: $x > 1$ or $y > 1$. It follows $f(x) = 1$ or $f(y) = 1$; also $x+y > 1$. Hence $1 = f(x+y) \leqslant f(x) + f(y)$.

Properties (F2) to (F5) allow us to claim that:

COROLLARY 3.4. If (M,d) is a metric space so is $(M, f\circ d)$ and both distances d and $f\circ d$ induce the same topology on M. (See for example [3], page 153).

4. SOME PROPERTIES OF METRIZABLE AND LOCALLY CONNECTED SPACES.

Let (M,d) be a locally connected metric space and f the Cantor function as defined above.

We know that finite subsets of metrizable spaces are closed; and connected components of open subsets of locally connected spaces are open. Therefore,

(M1) Every connected component of an open subset of M is either unitary or infinite.

(M2) If $C \subset M$ is infinite and connected then any non-empty open subset of C is infinite.

LEMMA 4.1. *If* C *is a connected, open and infinite subset of* M *and* p *is a point of* M *then there exists a connected, open and infinite subset* C' *of* C *such that for any two* x,y *in* C', $f(d(x,p)) =$ $= f(d(y,p))$.

Proof. Let $d_p(x) = d(p,x)$; $d_p: M \to [0,\infty)$ is a continuous function, then the image $d_p(C)$ is a connected subset of $[0,\infty)$ and (by M1) it is either a unitary set or a non degenerate interval.

If $d_p(C)$ is unitary so is $f(d_p(C))$ and we can take $C' = C$.

If $d_p(C)$ is a non degenerate interval it will be possible to select a,b in R with $a < b$ and $(a,b) \subset d_p(C) - K$ (for K is closed and null); thus $f(a) = f(x) = f(b)$ for any x in (a,b).

Let $A = d_p^{-1}((a,b)) \cap C$. It holds: A is open, $d_p(A) = (a,b) \subset$ $\subset [0,\infty)-K$ and $f(d_p(A))$ is unitary. Let C' be one of the connected components of A; C' is open (for M is locally connected), infinite (because of M2) and connected.

LEMMA 4.2. *If* $C \subset M$ *is connected, open and infinite and* $B = \{p_1, p_2, \ldots, p_n\} \subset M$ *then there exists an infinite* $C' \subset M$ *such that for any two* q, q' *in* C' *and any* p_i *in* B, $f(d(p_i,q)) = f(d(p_i,q'))$.
(In other words, no finite basic set exists for $(M, f\circ d)$).

Proof. It suffices to apply n times Lemma 4.1.

THEOREM 4.3. *For any metrizable, connected and infinite space there exists a compatible distance such that any basic set is infinite.*

Proof. We shall give a method to select an appropriate distance d'. Let M be the space and d one of the compatible distances. By M_1, connected components of M can be either unitary or infinite. If every component is unitary we select $d'(x,y) = 1$ for any two different x,y in M, and $d'(x,x) = 0$ for any x in M. (In this case the only basic set will be M). On the contrary if some components are infinite, we select $d' = f\circ d$. By applying Lemma 4.2 to one of the infinite components of M we conclude that no finite basic set exists for (M,d').

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