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THE TAYLOR POLYNOMIAL AS BEST LOCAL APPROXIMATION IN RECTANGLES

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SUMMARY. Let R_ε be the rectangle $0 \leqslant x_1 \leqslant \varepsilon^{\alpha_1}$, $0 \leqslant x_2 \leqslant \varepsilon^{\alpha_2}, \ldots,$ $0 \leqslant x_m \leqslant \varepsilon^{\alpha_m}$ where $\alpha_i \geqslant 1$ and $\min\limits_{1\le i\le m} \alpha_i = 1$. Given f in $L^2(R_1)$, set $P_\varepsilon(f)$ for the L^2 projection onto the algebraic polynomials of degree not greater than n, where the projection is taken on R_ε. If $f \in C^{n_\alpha+1}$ with $n_\alpha > n \max\limits_{1\le i\le m} \alpha_i$, and Tf is the Taylor polynomial of f of degree n developed at x=0, then $P_\varepsilon f$ converges to Tf as $\varepsilon \to 0$. Where $\max\limits_{1\le i\le m} \alpha_i < (n+1)n^{-1}$, the rectangles R_ε can be replaced by a family $\{F_\varepsilon\}$ regular with respect to $(\alpha_1,\ldots,\alpha_m)$, and we have a similar result with the best L^p approximation on the sets F_ε. Throught this note we shall use Peano-like derivatives.

1. INTRODUCTION.

The notion of best local approximation may roughly be described as follows. Let f be a real-valued function defined on the unit cube Q in $\mathbf{R}^m$, $Q = \{x \,/\, 0 \leqslant x_i \leqslant 1\}$, and assume that f is in a normed linear space X with norm $\| \ \|$. Let V be a subset of X, and suppose we wish to best approximate f by elements of V near a point $x \in Q$, say x=0. Then we consider a family of regions $\{R_\varepsilon\}_{0<\varepsilon\le 1}$ shrinking down to zero as $\varepsilon \to 0$, and we look for a $P_\varepsilon f \in V$ which minimizes the expression $\|(f-P)\chi_{R_\varepsilon}\|$ with $P \in V$, where χ_{R_ε} denotes the characteristic function of the set R_ε. If such P_ε exists it will be called a best approximation of f on the region R_ε. In case that $P_\varepsilon f$ tends to some $P_o f$ in V, as $\varepsilon \to 0$, we say that $P_o f$ is a best local approximation of f at x=0 by elements of V.

The general concept of best local approximation as stated above, was introduced by C.Chui et.al., and developed in a serie of papers, [3], [4], etc., where the underlying results are one-dimensional and

V is taken as the class of algebraic polynomials, generalized polynomials, or quasi-rational approximation. Recently, higher dimensional results have been obtained in [1], [2].

The purpose of this note is to call the attention to the fact that in higher dimensions it is important how the family $\{R_\varepsilon\}$ shrinks to zero, even considering as approximating class the algebraic polynomials. It will follow, using the techniques in [2], that if $\{R_\varepsilon\}$ is a regular family with respect to the balls centered at the origin, then the best local approximation of f with respect to this family is the Taylor polynomial. Since the balls can be changed for a regular family and still we get the Taylor polynomial, we see that the balls are good regions to obtain a local approximation.

We have a different situation if we take as $\{R_\varepsilon\}$ a one-parametric family of rectangles whose sides are parallel to the axis. We shall see that the Taylor polynomial is the best L^2 local approximation with respect to this family. Moreover, we prove that rectangles which are not too flat can be replaced by a regular family with respect to them. But in general not any affine transformation of rectangles will give a suitable family for local approximation.

This note is self-contained.

II. NOTATION AND RESULTS.

Let f be a Lebesgue measurable function defined on Q, and $E \subseteq Q$ a measurable set. We set

$$\|f\|_{p,E} = \left(\int_E |f(t)|^p \frac{dt}{|E|}\right)^{1/p} , \quad 0 < p < \infty ,$$

where $|E|$ denotes the Lebesgue measure of E. We always assume $|E| > 0$. The expression $\|f\|_{\infty,E}$ means the essential supremum on E. Thus, we have a norm on $L^p(Q)$ for $1 \leqslant p \leqslant \infty$ and a metric otherwise. Let $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_m)$ be an m-tuple of real numbers with $\alpha_i \geqslant 1$ and $\min\{\alpha_i\} = 1$. The non-homogeneous dilation $(\varepsilon^{\alpha_1}x_1,\varepsilon^{\alpha_2}x_2,\ldots,\varepsilon^{\alpha_m}x_m)$ is denoted by $\delta_\varepsilon(x)$, and set $R_\varepsilon = \delta_\varepsilon Q$. We say that a family $\{F_\varepsilon\}_{0<\varepsilon\leq 1}$ of measurable sets in $\mathbf{R}^m$ is a regular family with respect to α, if there exists a constant $c > 0$ such that for every ε the set F_ε is contained in R_ε and $|F_\varepsilon| \geqslant c\varepsilon^{|\alpha|}$, where $|\alpha| = \alpha_1 + \alpha_2 +\ldots+ \alpha_m$.

The real algebraic polynomials in m variables with degree not greater than n will be denoted by Π^n. Thus if $P \in \Pi^n$ we use the standard notation $P(x) = \sum_{|\beta|\le n} a_\beta x^\beta$, with $x^\beta = x_1^{\beta_1}, x_2^{\beta_2}, \dots, x_m^{\beta_m}$.

We shall often use the next inequality.

(2.1) *Let* $\{F_\varepsilon\}_{0<\varepsilon\le 1}$ *be a regular family with respect to* α *and* $p > 0$. *Then, there exists a constant* $c > 0$ *such that for every* $P \in \Pi^n$ *and* ε, *it follows that*

$$\|P\|_{\infty,Q} \leqslant c\varepsilon^{-n|\alpha|_\infty} \|P\|_{F_\varepsilon,p} \quad \textit{where} \quad |\alpha|_\infty = \max_{1\le i\le m} \alpha_i .$$

Note that in (2.1) the constant c is independent of $P \in \Pi^n$ and ε. Let $f \in L^p(Q)$, $0 < p$ and assume that there exist $Tf \in \Pi^n$ and a finite number R_f such that $\|f-Tf\|_{F_\varepsilon,p} \leqslant R_f\ \varepsilon^{n+1}$ for every $0 < \varepsilon \leqslant 1$. Then we say that Tf is a Taylor polynomial of degree n of f at the origin. If such is the case we write $f \in T^p_{n+1}$. By (2.1) we have the uniqueness of Tf if $|\alpha|_\infty < (n+1)n^{-1}$ and elementary examples show the lack of uniqueness for $|\alpha|_\infty > (n+1)n^{-1}$. Sometimes the class Π^n is replaced by a smaller one $\Pi^{n,\alpha}$ and we have uniqueness of $Tf \in \Pi^{n,\alpha}$ for every α, see [6].

Given $f \in L^p(Q)$ we call $B^p_\varepsilon(f) = B^p_{F_\varepsilon}(f)$ the set of all $P_\varepsilon f \in \Pi^n$ such that $\|f-P_\varepsilon f\|_{F_\varepsilon,p} = \inf_{P\in\Pi^n} \|f-P\|_{F_\varepsilon,p}$. A compactness argument together with (2.1) shows that $B^p_\varepsilon(f) \neq \emptyset$. For p outside the range $(1,\infty)$, $B^p_\varepsilon(f)$ may have more than one element. For example we can find an $f \in C^\infty$ such that $B^\infty_Q(f)$ has infinitely many quadratic polynomials, [5]. The next statement is an easy consequence of (2.1).

THEOREM 1. *Let* $f \in T^p_{n+1}$, $0 < p \leqslant \infty$. *Let* $\{F_\varepsilon\}_{0<\varepsilon\le 1}$ *be a regular family with respect to* α *with* $|\alpha|_\infty < (n+1)n^{-1}$. *Let* Tf *be the Taylor polynomial of degree* n *of* f. *Then, for every* $P \in B^p_\varepsilon(f)$ *we have*

$$\|P-Tf\|_{\infty,Q} \leqslant cR_f\varepsilon^{n+1-n|\alpha|_\infty}$$

where the constant c *depends on* Π^n, *the family* $\{F_\varepsilon\}$ *and* p.

We call n_α the minimum integer ℓ such that $\ell \geqslant n|\alpha|_\infty$. If $f \in T^p_{n_\alpha+1}$ the restriction to Π^n of a Taylor polynomial of degree n_α is uniquely determined and we shall call it the Taylor polynomial of f of degree n and again denoted by Tf.

THEOREM 2. *Let* $f \in T^2_{n_\alpha+1}$, *let* Tf *be the Taylor polynomial of* f *of degree* n, *and* $P_\varepsilon(f)$ *the* $L^2(R_\varepsilon)$ *projection of* f *onto* Π^n. *Then* $\|P_\varepsilon(f)-Tf\|_{\infty,Q}$ *tends to zero as* $\varepsilon \to 0$.

III. PROOFS.

The next Lemma is a general version of a similar one in [2]. The proof is included here for the sake of completeness.

LEMMA. *Let* $\{\mu_\varepsilon\}_{0<\varepsilon\leq 1}$ *be a family of measures on* Q *uniformly absolutely continuous with respect to the Lebesgue measure. Suppose that* $\mu_\varepsilon(Q) = 1$ *for every* ε. *Then there exists a constant* $c > 0$, $c = c(p,n)$ *such that for any* $P \in \Pi^n$ *and* $\varepsilon > 0$ *we have*

$$c\ \|P\|_{\infty,Q} \leqslant \left(\int_Q |P(x)|^p\, d\mu_\varepsilon(x)\right)^{1/p} \leqslant \|P\|_{\infty,Q}$$

In fact, suppose for the moment that for some $p > 0$, the statement does not hold. Then there exist sequences $\{P_\ell\}$ in Π^n and $\{\varepsilon_\ell\}$ such that for every ℓ we have $\|P_\ell\|_{\infty,Q} = 1$ and $\left(\int_Q |P_\ell(x)|^p\, d\mu_\varepsilon(x)\right)^{1/p} \leqslant$ $\leqslant 1/\ell$.

Now assume, by taking a subsequence, if necessary, that for some $P_o \in \Pi^n$, $\|P_\ell - P_o\|_{\infty,Q} \to 0$ as $\ell \to \infty$. Then $\|P_o\|_{\infty,Q} = 1$ and given $\varepsilon > 0$ for all ℓ large we have $\left(\int_Q |P_o(x)|^p\, d\mu_{\varepsilon_\ell}(x)\right)^{1/p} \leqslant \varepsilon$.

Given $\delta > 0$ we set $N_\delta = \{x \in Q \ /\ |P_o(x)| < \delta\}$. Then, for ℓ large we have $\varepsilon^p > \delta^p\,(1-\mu_{\varepsilon_\ell}(N_\delta))$. Since $P_o \neq 0$, $|N_\delta| \to 0$ as $\delta \to 0$, we arrive at a contradiction in the last inequality by observing that our measures are uniformly absolutely continuous. So the Lemma is proved.

In order to get (2.1), set $P^\varepsilon(t) = P(\delta_\varepsilon t)$ and $d\mu_\varepsilon(t) = \dfrac{\varepsilon^{|\alpha|}}{|F_\varepsilon|}\,\chi_{F_\varepsilon}(\delta_\varepsilon t)$.

Observe that $\sup_{0<\varepsilon\leq 1} \mu_\varepsilon(E) \to 0$ as $|E| \to 0$, if $\{F_\varepsilon\}$ is a regular family with respect to α.

Thus, a change of variables and the Lemma yield

$$\|P^\varepsilon\|_{\infty,Q} \leqslant c\ \|P\|_{p,F_\varepsilon} \quad \text{for any} \quad P \in \Pi^n \quad \text{and} \quad \varepsilon > 0.$$

Moreover, the norm $\|P^\varepsilon\|_{\infty,Q}$ is equivalent to $\max_{|\beta|\leq n} \varepsilon^{\alpha\cdot\beta}|a_\beta|$, where

$P(x) = \sum_{|\beta|\le n} a_\beta x^\beta$ and $\alpha.\beta = \alpha_1\beta_1 + \alpha_2\beta_2 + \ldots + \alpha_m\beta_m$. Thus, we obtain

$$(3.2) \qquad \varepsilon^{\alpha.\beta}|a_\beta| \leqslant \| \sum_{|\beta|\le n} a_\beta x^\beta \|_{p,F_\varepsilon} \quad , \ |\beta| \leqslant n$$

Clearly, (3.2) implies (2.1).

REMARK 1. The inequality (2.1) remains valid if we consider polynomials $P(x) = \sum_{i=1}^{k} a_i u_i(x)$ in the sense of [1]. That is, we assume that each u_i is C^{n+1} in a neighborhood of zero, and the Wronskian determinant of the square matrix $(\partial^\beta u_i(0))$, $|\beta| \leqslant n$, $i = 1,\ldots,k$ being nonzero. Here, $k = \mathrm{card}\{\beta : |\beta| \leqslant n\}$. Then for $n|\alpha|_\infty < n+1$ and any ε small we have

$$\|P\|_{\infty,Q} < c\ \varepsilon^{-n|\alpha|_\infty} \|P\|_{p,F_\varepsilon}$$

$p > 0$ and $\{F_\varepsilon\}$ regular with respect to α.

In fact, by the Taylor theorem

$$P(x) = \sum_{|\beta|\leqslant n} \frac{1}{\beta!} A_\beta(0)x^\beta + \sum_{|\beta|=n+1} \frac{1}{\beta!} A_\beta(\xi)x^\beta = P_1(x) + R(x)$$

where $A_\beta(\xi) = \sum_{i=1}^{k} a_i \partial^\beta u_i(\xi)$.

Now, for a constant c, not necessarily the same on each ocurrence, we have

$$\|P\|_{p,F_\varepsilon} \leqslant c\varepsilon^{n+1} \max_{1<i<k} |a_i| \leqslant c\varepsilon^{n+1} \|P\|_{\infty,Q}$$

$$\|P_1\|_{p,F_\varepsilon} - c\ \|R\|_{p,F_\varepsilon} \leqslant c\ \|P\|_{p,F_\varepsilon}$$

and by (2.1), we have

$$\|P_1\|_{\infty,Q} \leqslant c\varepsilon^{-n|\alpha|_\infty} \|P_1\|_{p,F_\varepsilon} .$$

Since the vectors $(A_\beta(0))$ and (a_i) are related by a nonsingular matrix the norm $\|P_1\|_{\infty,Q}$ is equivalent to $\|P\|_{\infty,Q}$ and the remark follows.

Theorem 1 is a consequence of the definition of $B^p_\varepsilon(f)$ and (2.1). In fact, for $P \in B^p_\varepsilon(f)$ we have

$$\|P\text{-}Tf\|_{\infty,Q} \leqslant c\varepsilon^{-n|\alpha|_\infty} \|P\text{-}Tf\|_{p,F_\varepsilon} \leqslant c(p)\ \varepsilon^{-n|\alpha|_\infty} \|f\text{-}Tf\|_{p,R_\varepsilon} ,$$

and Theorem 1 follows.

We can not expect, in general, convergence of the best approximation polynomial $P_\varepsilon(f)$ to the Taylor polynomial when $|\alpha|_\infty > (n+1)n^{-1}$. Thus, if $|\alpha|_\infty > (n+1)n^{-1}$ it is easy to find a regular family $\{F_\varepsilon\}$ such that $\|P_\varepsilon(f)\|_{\infty,Q}$ tends to infinity for smooth functions f. If $|\alpha|_\infty = (n+1)n^{-1}$ the norm of $P_\varepsilon(f)$ will remain bounded, but in general, the polynomial $P_\varepsilon(f)$ will not converge to Tf. We shall give an example. First consider a further observation.

Set $P_E(f)$ for the L^2 projection onto Π^n, where the L^2 norm is taken over the set E.

Let L be a nonsingular affine transformation of $\mathbf{R}^m$ onto $\mathbf{R}^m$. Then for $f \in L^2(Q)$ and measurable $E \subseteq Q$ we have

$$P_{LE}(f) = P_E(f \circ L) \circ L^{-1} \tag{3.3}$$

To obtain (3.3), we use a change of variables and the uniqueness of the L^2 projection. Thus, the L^p version of (3.3) is also true for $1 < p < \infty$.

(3.4) EXAMPLE. Set $\Pi^n = \Pi^1(x,y)$ and $E = \{(x,y) \;/\; 0 \leqslant y \leqslant c(x)\;,\; 0 \leqslant x \leqslant 1\}$, where c is a positive continuous function. Suppose for the moment that there exists c(x) such that $P_E(x^2)(x,y) = a_0 + a_1 x + a_2 y$ has the coefficient $a_2 \neq 0$. Now let $F_\varepsilon = \delta_\varepsilon E$ where $\delta_\varepsilon(x,y) = (\varepsilon x\;,\; \varepsilon^\alpha y)$. Then (3.3) yields $P_{F_\varepsilon}(x^2)(x,y) = \varepsilon^2(a_0 + a_1\varepsilon^{-1}x + a_2\varepsilon^{-\alpha}y)$. Therefore, $\|P_{F_\varepsilon}\|_{\infty,Q}$ tends to infinity for $\alpha > 2$ and if $\alpha = 2$, P_{F_ε} does not converge to the Taylor polynomial. It remains to find a function c(x) with the required conditions. But $a_2 \neq 0$ iff

$$\Delta(c) = \begin{vmatrix} \int_0^1 c(x)\,dx & \int_0^1 x\,c(x)\,dx & \int_0^1 x^2c(x)\,dx \\ \int_0^1 x\,c(x)\,dx & \int_0^1 x^2c(x)\,dx & \int_0^1 x^3c(x)\,dx \\ \int_0^1 c^2(x)\,dx & \int_0^1 x\,c^2(x)\,dx & \int_0^1 x^2c^2(x)\,dx \end{vmatrix} \neq 0$$

Now if $c \in \Pi^1(x)$, $c(x) = b_0 + b_1x$, the determinant $\Delta(c) = \Delta c(b_0,b_1)$ belongs to $\Pi^4(b_0,b_1)$. But $\Delta(c)$ is not the zero polynomial. So there exist (b_0,b_1) such that $\Delta(c) \neq 0$. Moreover we can choose a positive $c(x)$. Since for b_1 fixed $\Delta(c)(b_0,b_1) \in \Pi^4(b_0)$ then $\Delta c(b_0,b_1) \neq 0$ for all large b_0.

The next property for the L^2 projection P_Q will be used.

$$(3.5) \qquad P_Q(x^\beta)(x) = \sum_{\substack{|\gamma|\le n \\ \gamma\le\beta}} a_\gamma(\beta)\, x^\gamma .$$

Here $\gamma \leqslant \beta$ means that for each component holds $\gamma_i \leqslant \beta_i$.

In fact, let $V_0, V_1, \ldots, V_n$ be an orthonormal system in Π^n of $[0,1]$. Then $V_\gamma(x) = V_{\gamma_1}(x_1)\, V_{\gamma_2}(x_2) \ldots V_{\gamma_m}(x_m)$, $|\gamma| \leqslant n$ is an orthonormal system in Π^n of Q. Clearly, $V_\gamma(x) = \sum_{\sigma\le\gamma} a_\sigma^\gamma x^\sigma$. Now in the expression

$$P_Q(x^\beta) = \sum_{|\gamma|\le n} (x^\beta, V_\gamma)_{L^2(Q)}\, V_\gamma$$

we have $(x^\beta , V_\gamma)_{L^2(Q)} = 0$ if $\gamma \leqslant \beta$ does not hold.

By using the form for $V_\gamma(x)$ and a change in the order of sums we get

$$P_Q(x^\beta)(x) = \sum_{\substack{|\gamma|\le n \\ \gamma\le\beta}} \Big(\sum_{\substack{\gamma\le\sigma\le\beta \\ |\sigma|\le n}} (x^\beta, V_\sigma)\, a_\gamma^\sigma \Big)\, x^\gamma ,$$

and (3.5) is proved.

We make a few comments on (3.5). This property implies that if $f \in L^2(Q)$ is independent of some set of variables then $P_Q f$ does not depend on the same set of variables. We can replace Q in (3.5) by $E = AQ+v$, $v \in \mathbf{R}^m$ and A a diagonal positive matrix and still have the same order in the sum. This follows at once by (3.3).

(3.6) EXAMPLE. Let E be the square with vertices $(\pm 1,0),(0,\pm 1)$ and P_E the projection on Π^n, polynomials in the variables x and y. Then there exists ℓ such that $P_E(x^\ell)$ depends of the two variables x and y. Thus (3.5) is not invariant by rotations. The example (3.6) could "be proved" by a long calculation or else we may use the next argument. Let $y = c(x)$, $-1 \leqslant x \leqslant 1$ the piecewise polynomial function which gives the upper boundary of E. We claim that $P_E(c^2)$ is a polynomial in Π^n which does depend on the variable y. Otherwise, $c^2(x) - P_E(c^2)(x)$ will be orthogonal to the subspace $M = \{x^i c^j(x) : i+j \leqslant n,\ j \text{ even}\}$, where the $L^2[0,1]$ is taken with the measure $c(x)dx$. But this is a contradiction since $c^2 \in \Pi^n$ and

$P_E(c^2) \neq 0$. Now approaching c^2 by polynomials in the variable x we get our claim.

Now we prove Theorem 2. First note that if $f \in T^2_{n_\alpha+1}$ by (2.1) and (3.2) the restriction to Π^n of the Taylor polynomial of degree n_α is uniquely determined. We write

$f(x) = Tf(x) + \sum_{n<|\beta|\leqslant n_1} a_\beta x^\beta + R(x)$ with $Tf \in \Pi^n$ and $\|R\|_{2,R_\varepsilon} \leqslant R_f \varepsilon^{n_\alpha+1}$.

Since $\|P_\varepsilon(R)\|_{2,R_\varepsilon} \leqslant 2\|R\|_{2,R_\varepsilon}$, by (2.1) and (3.2) $\|P_\varepsilon(R)\|_{\infty,Q} \to 0$ as $\varepsilon \to 0$.

On the other hand, by (3.2) and (3.5), we have

$$P_\varepsilon(x^\beta)(x) = \sum_{\substack{|\gamma|\leq n \\ \gamma\leq\beta}} \varepsilon^{\alpha.\beta-\alpha.\gamma} a_\gamma(\beta)\, x^\gamma$$

So if $|\beta| > n$ we have $P_\varepsilon(x^\beta) \to 0$, and the Theorem follows.

REMARK 2. Theorem 2 is no longer true if Q is replaced by a rotation of it. This follows from example (3.6).

REMARK 3. In Theorem 2 the sets R_ε can be replaced by $R^*_\varepsilon = [0,\phi_1(\varepsilon)] \times \ldots \times [0,\phi_n(\varepsilon)]$ where the positive functions ϕ_i behave like powers. That is, there exists $s > 0$ such that for every pair i,j we have $\phi_i^s(\varepsilon) = o(\phi_j(\varepsilon))$. Now we should assume a smoothness condition according to s and n.

It could be of some interest to obtain a similar result to Theorem 2 for $p \neq 2$.

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