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ON A GENERAL BOSONIC FIELD THEORY

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ABSTRACT. We present here the lagrangian density that can be constructed using tensorial concomitants with the bosonic fields (spin 0, 1 and 2) without dimensional constants. We recognize several classical theories into it and we show it yields a non renormalizable theory.

1. INTRODUCTION

The ultimate purpose of this and the forthcoming papers is to construct, by geometrical methods, the covariant lagrangian density for a general theory - i.e. without any initial physical hypothesis - containing the spin 0, 1/2, 1, 3/2 and 2 fields and to see then what restrictions must be imposed to make the theory renormalizable or simply with a finite direct outcome. Motivations for our work are the well-known problems arising when ordinary matter is added to Einstein gravity. Facing these problems, two ways of improvement show themselves: the addition of R^2 terms to the Einstein action [1,2] or the inclusion of supersymmetries as is the case of Supergravity [3].

Supergravity models seem to be the low energy limit of the most promising scheme to describe the fundamental interactions: Superstrings. So we will seek for the constrains on the lagrangian of our general theory, necessary to obtain extended Supergravity or supersymmetric matter coupled to N=1 Supergravity. A study of this sort, but not with a powerful device as concomitants, was begun by van Nieuwenhuizen et al. for spin 5/2 theory [4].

In this first paper we shall only deal with bosonic massless fields for the sake of simplicity, leaving for future papers to include fermions. Thus we study, somehow, a general bosonic field theory.

The aim of this work is then to present a general theory that fullfills the conditions below:

i) The fields involved are the metric g_{ij}, the electromagnetic potential vector A_i and a scalar field φ. The latter will play different roles. In general the one of an ordinary field or eventually a constant (to introduce the gravitational constant as in Brans-Dicke theory).

ii) Dimensional constants are not allowed because thay introduce well known problems yielding generally non renormalizable theories.

iii) The lagrangian will contain only the derivatives that appear in eq.(1) and it will be linear in the second derivatives, for the sake of simplicity at this first stage of research. We introduce only A_i and φ first derivatives because we want second order field equations for these fields. On the contrary, we allow g_{ij} second derivatives because naturally we want General Relativity to be contained in our general theory. (Remember that the Hilbert action is a degenerate one, so the field equations are of second order, even if the lagrangian has the same order).

iv) Units: We set $c = \hbar = 1$. The action is dimensionless so to be able to construct the generating functional. Therefore,

$[S] = 1$ and $[g_{ij}] = 1$, then $[\mathbf{L}] = 1^{-4}$, $[\varphi] = [A_i] \doteq 1^{-1}$

and $[\chi] = 1^{-1}$ with $\chi^2 = 16\pi G$, G the Newtonian constant.

2. INCORPORATION OF THE ELECTROMAGNETIC FIELD AT THE AFFINE CONNECTION LEVEL

Let us review briefly some of the different attempts to build a unified field theory. The first was the one by A. Einstein, who identified the antisymmetric part of the metric tensor, namely $g_{[ij]}$, with the Maxwell tensor F_{ij} [5]. J.W.Moffat studied the reasons which prevent the success of this theory [6], [7] showing that $g_{[ij]}$ cannot represent photons and that it is rather an auxiliary field like those that frequently appear in supergravity.

An alternative way is to introduce the electromagnetic field in the affine connection of the space-time manifold. This was the way followed by H. Weyl [8] and recently by N. Batakis [9], [10].

About these theories we observe:

i) It is easy to show that if the scalar field is compelled to be able to have a kinetic term in the lagrangian, the most general affine connection, without dimensional constants, that can be written with the metric, the electromagnetic vector and the scalar field (which is used to restore the dimensional cosmological constant) leads to the Weyl connection plus a torsion term. The difficulties to find a physical interpretation to the Weyl theory are well known.

ii) Had φ other units, it could be interpretated as in Batakis' models. But then it would be impossible for φ to have a kinetic term, which is necessary if we like to think of it, in the future, as a propagating Higgs field.

Consequently, it seems to us that unification must be searched directly at the lagrangian level. We do not impose restrictions about the kind of coupling among the fields. The problem will be stated for any dimension, not necessary a natural one, in order to be able to use dimensional regularization at the quantum level.

3. THE LAGRANGIAN DENSITY

Let **L** be a lagrangian concomitant of a metric tensor, the electromagnetic potential (i.e. in a precise mathematical language a covector), a scalar field and its derivatives up to the indicated order:

$$\mathbf{L} = L(g_{ij}\ ,\ g_{ij,h}\ ,\ g_{ij,hk}\ ,\ A_i\ ,\ A_{i,j}\ ,\ \varphi\ ,\ \varphi_{,i}) \qquad (1)$$

From condition iii) and the field dimensions iv), by a change of scale λ in **L**, we will have:

$$L(g_{ij}\ ,\ \lambda g_{ij,h}\ ,\ \lambda^2 g_{ij,hk}\ ,\ \lambda A_i\ ,\ \lambda^2 A_{i,j}\ ,\ \lambda\varphi\ ,\ \lambda^2\varphi_{,i}) =$$

$$= \lambda^4\ L(g_{ij}\ ,\ g_{ij,h}\ ,\ g_{ij,hk}\ ,\ A_i\ ,\ A_{i,j}\ ,\ \varphi\ ,\ \varphi_{,i})$$

Derivating four times with respect to λ, making $\lambda \to 0$ and applying the replacement theorem [11], we obtain:

$$\begin{aligned}
\mathbf{L} = {} & \Lambda_1^{ijhk} R_{ijhk}\, \varphi^2 + \Lambda_1^{ijhks} R_{ijhk}\, A_s\, \varphi \; + \\
& + \Lambda_2^{ijhks} R_{ijhk}\, \varphi_{,s} + \Lambda_1^{ijhkrs} R_{ijhk}\, A_r\, A_s \; + \\
& + \Lambda_2^{ijhkrs} R_{ijhk}\, A_{r;s} + \Lambda_1\, \varphi^4 \; + \\
& + \Lambda_1^{i}\, A_i\, \varphi^3 + \Lambda_2^{i}\, \varphi^2\, \varphi_{,i} + \Lambda_1^{ij}\, \varphi^2\, A_i\, A_j \; + \\
& + \Lambda_2^{ij}\, A_{i;j}\, \varphi^2 + \Lambda_3^{ij}\, \varphi\, \varphi_{,i}\, A_j \; + \\
& + \Lambda_4^{ij}\, \varphi_{,i}\, \varphi_{,j} + \Lambda_1^{ijh}\, \varphi\, A_i\, A_j\, A_h \; +
\end{aligned}$$

$$+ \Lambda_2^{ijh} \varphi A_i A_{j;h} + \Lambda_2^{ijhk} A_i A_j A_h A_k +$$

$$+ \Lambda_3^{ijhk} A_i A_j A_{h;k} +$$

$$+ \Lambda_4^{ijhk} A_{i;j} A_{h;k} . \qquad (2)$$

where $\Lambda^{\cdots\cdot}_{t} = \Lambda^{\cdots\cdot}_{t}(g_{ij})$ are tensorial densities, R_{ijhk} is the Riemannian tensor and ; indicates covariant derivation.

Taking into account the recent determination of the concomitants of the metric tensor for the non-degenerate case [12], [13], we have:

THEOREM. *If* **L** *satisfies eq.*(1) *and also* i), ii), iii) and iv) *then, for* n > 2, *the general lagrangian is:*

$$\mathbf{L} = \Lambda_1 \varphi^2 R + \Lambda_1^{ijhkrs} R_{ijhk} A_r A_s +$$

$$+ \Lambda_2^{ijhkrs} R_{ijhk} A_{r;s} + \Lambda_2 \varphi^4 +$$

$$+ \Lambda_1^{ij} \varphi^2 A_i A_j + \Lambda_2^{ij} \varphi^2 A_{i;j} +$$

$$+ \Lambda_3^{ij} \varphi \varphi_{,i} A_j + \Lambda_4^{ij} \varphi_{,i} \varphi_{,j} +$$

$$+ \Lambda_1^{ijh} \varphi A_i A_{j;h} + \Lambda_1^{ijhk} A_i A_j A_h A_k +$$

$$+ \Lambda_2^{ijhk} A_i A_j A_{h;k} + \Lambda_3^{ijhk} A_{i;j} A_{h;k} \qquad (3)$$

$$\Lambda_1^{ijhkrs} = \sqrt{-g} \{a_2 g^{ih} g^{jk} g^{rs} + a_3 g^{ir} g^{jk} g^{hs} +$$

$$+ a_4 g^{ih} g^{jr} g^{hs}\}$$

$$\Lambda_2^{ijhkrs} = \sqrt{-g} \{a_5 g^{ih} g^{jk} g^{rs} + a_6 g^{rh} g^{jk} g^{is} +$$

$$+ a_7 g^{ir} g^{jk} g^{hs} + a_8 g^{ih} g^{rk} g^{js} +$$

$$+ a_9 g^{ih} g^{jr} g^{ks}\} + a_{10} \delta_4^n \varepsilon^{rjsk} g^{ih}$$

$$\Lambda_1 = a_1 \sqrt{-g} \qquad \Lambda_3^{ij} = a_{14} \sqrt{-g}\, g^{ij}$$

$$\Lambda_2 = a_{11} \sqrt{-g} \qquad \Lambda_4^{ij} = a_{15} \sqrt{-g}\, g^{ij}$$

$$\Lambda_{ij}^{1} = a_{12} \sqrt{-g}\, g^{ij} \qquad \Lambda_1^{ijh} = a_{16} \varepsilon^{ijh} \delta_3^n$$

$$\Lambda_2^{ij} = a_{13} \sqrt{-g}\, g^{ij} \qquad \Lambda_1^{ijhk} = a_{17} \sqrt{-g}\, g^{ij}\, g^{hk}$$

$$\Lambda_2^{ijhk} = \sqrt{-g}\, \{a_{18}\, g^{ij}\, g^{hk} + a_{19}\, g^{ih}\, g^{jk}\}$$

$$\Lambda_3^{ijhk} = \sqrt{-g}\, \{a_{20}\, g^{ij}\, g^{hk} + a_{21}\, g^{ih}\, g^{jk} + $$
$$+ a_{22}\, g^{ik}\, g^{jh}\} + a_{23}\, \delta_4^n\, \varepsilon^{ijhk}$$

where a_t *are constants and* n *the dimension of the space-time manifold.*

4. STUDY OF THE GENERAL LAGRANGIAN AT THE CLASSICAL LEVEL

Now we can reobtain several well known classical theories from this lagrangian choosing different values for the constants:

a) General Relativity:

General Relativity may be writen avoiding dimensional constants as a Brans-Dicke theory [14]. This theory is supported by the idea of Mach that inertia ought to arise from accelerations with respect to the general mass distribution of the universe. The inertial masses of elementary particles would not be fundamental constants but represent their interaction with some cosmic field. As particle masses are measured through their accelerations in the gravitational field - with the Newtonian constant G being a factor - one may conclude that G must be related to the average value of a scalar field which would connect the strength of gravitation with the matter content of the universe. The other known integer spin fields g_{ij} and A_j transport long range forces. It is natural, then, to suspect that the same may be for the scalar field φ [15].

The simplest generally covariant field equation for the scalar field is

$$\Box \varphi = 4\, \pi\, b\, T_M^{ij}$$

where b is a coupling constant and T_M^{ij} is the matter energy-momentum tensor of the universe - i.e. everything but gravitation and φ field -. Brans and Dicke suggested that the correct field equations for gravitation are obtained by replacing χ^{-1} by φ and including an energy-momentum tensor ${T_\varphi}^{ij}$ for the scalar field. So

$$R^{ij} - \frac{1}{2}\, g^{ij}\, R = -\varphi^2 \{T^{ij} + {T_\varphi}^{ij}\}$$

All of this may be derived from a lagrangian density

$$L = -\varphi^2 R + 4\, w\, g^{ij}\, \varphi_{,i}\, \varphi_{,j}$$

being w a numerical constant. This lagrangian is obtained from our eq.(3) taking $a_1 = -1$; $a_{15} = -4w$ and all other coefficients equal to zero. Finally, Einstein equation is obtained taking φ^2 to be χ^{-2}.

b) Maxwell-Einstein:

The electromagnetic field is minimally coupled to the gravitational field replacing all partial derivatives which appear in its formulation in Minkowskian space-time by covariant ones. This is to require that if J^i is the current four vector, F^{ij} is the field strength tensor and the equations for the electromagnetic field in Minkowskian space-time are

$$\partial_i F^{ij} = -J^j$$

$$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0$$

Then when one defines F^{ij} and J^i in general coordinates they must reduce to their previous expressions in locally inertial coordinates and they must behave as tensors under general coordinate transformations.

Maxwell electromagnetism plus relativity can be obtained from our eq.(3) making $a_1 = -1/2$; $a_{22} = -a_{21} = 2\pi/137$; all other $a_i = 0$ and thinking of φ^2 as χ^{-2}.

c) Weyl theory:

Weyl unified field theory [8] states that, to be able to characterize the physical state of the world at a certain point of it by means of numbers, one must not only refer the neighbourhood of this point to a coordinate system, but must also fix the units of measure. This implies that the metrical structure is not only determined by the quadratic form

$$g_{ij}\, dx^i\, dx^j$$

but by a linear form

$$\phi_j\, dx^j$$

too. Thus, when performing the parallel displacement of a vector along a closed curve, its variation is written taking into account both the curvature and the tensor of "distance curvature"

$$f_{ik} = \partial_k \phi_i - \partial_i \phi_k$$

and it may be identified, always following Weyl, with the Maxwell electromagnetic field tensor F^{ij} if ϕ^i is the potential vector A^i.

Given the linear and the quadratic forms, the affine relationship Γ^r_{ik} for the space-time manifold is:

$$\Gamma^r_{ik} = \frac{1}{2} g^{rs} (\partial_k g_{is} + \partial_i g_{ks} - \partial_s g_{ik}) + \\ + \frac{\lambda}{2} g^{rs} (g_{is} A_k + g_{ks} A_i - g_{ik} A_s).$$

The field equations are obtained using a variational principle, once fixed the unit of measure - i.e. when the special gauge has been chosen -, from

$$V = \frac{1}{2} \sqrt{-g}\, R + C \sqrt{-g}\, F^{ij} F_{ij} + \frac{\lambda}{2} \sqrt{-g}\, (1 - 3 A_i A^i) \qquad (4)$$

C being a number and λ the cosmological constant. This is the classical Maxwell-Einstein theory of electromagnetism and gravitation but for a small cosmological term.

The dimensions of the lagrangian of the Weyl theory of eq.(4) are 1^{-2} in gravitational units: $c = \chi = 1$, $[\lambda] = 1^{-2}$. To obtain it from eq.(3) we must write it in the natural system of units: $c = \hbar = 1$, $[\chi] = 1$ and $[\lambda] = 1$. As the electromagnetic vector in natural units is $\sim 1^{-1}$, this may be obtained making $A_k \to A_k/\chi$, χ appearing again through the usual identification

$$\varphi^{-1} \sim \chi$$

Thus eq.(4) becomes:

$$V = \frac{1}{2} \varphi^2 \sqrt{-g}\, R + a \sqrt{-g}\, F_{ij} F^{ij} + \sqrt{-g}\, \frac{\lambda}{2} \varphi^4 - \frac{3}{2} \sqrt{-g}\, \lambda \varphi^2 A_i A^i$$

which is obtained from eq.(3) taking

$$a_1 = 1/2 \qquad a_{11} = \lambda/2$$

$$a_{12} = -\frac{3}{2}\lambda \qquad a_{21} = -a_{22} = 2a$$

and all other coefficients equal to zero.

d) Scalar field lagrangians:

It is widely accepted that the lagrangian for the massless scalar autointeracting field in curved space-time is

$$L = \xi \varphi^2 \sqrt{-g}\, R + \sqrt{-g}\, g^{ij} \varphi_{,i} \varphi_{,j} + \sqrt{-g}\, \lambda \varphi^4$$

So we must take $a_{11} = \lambda$; $a_1 = \xi = 0$ or $a_1 = \xi = 1/6$ for minimal or conformal coupling respectively; $a_{15} = 1$ and all other coefficients a_i equal to zero. With two different scalar fields we could also add the $\chi^2 R$ term to obtain General Relativity coupled to the matter scalar field.

Besides all these familiar terms, whose physical role can be seen immediately, we have other ones that introduce non-minimal interactions:

e) Terms a_2 to a_9 couple the Maxwell field to the Riemannian tensor.

f) a_{16} is null in n=4 and a_{23} is a total divergence.

g) Terms a_{12}, a_{13}, a_{14}, a_{15}, a_{17}, a_{18} and a_{20} are interactions among the three fields and its derivatives with no Riemann tensor present. In particular, A^4 terms added to the Maxwell lagrangian give non causal modes of propagation [16].

5. THE QUANTUM LEVEL

When one tries to obtain conclusions from a physical theory, this theory must either yield results with physical meaning without making corrections, or one could be able to give sense to meaningless ones via a renormalization method. In quantum field theory this means to have non-divergent results for the elements of the S-matrix or to have infinite but renormalizable ones. The leading problem about this is the ultraviolet divergence which arises in evaluating the quantum correction to the propagators and vertices in the Feynman diagrams.

We say a theory is renormalizable when we can absorbe the infinities by means of an adequate redefinition of fields and parameters, after having added counterterms to the original lagrangian density of the theory. And it is well known that a necessary condition for a theory to be renormalizable is that the number of primitively divergent diagrams for an interaction term must be finite [17]. In order to satisfy this condition, the interaction terms must be superficially convergent. But if we write the metric

$$g_{ij} = \eta_{ij} + h_{ij}$$

as usually to have gravitons and because of the pecualiar role it plays being at the same time the spin two field and the metric of space-time, all terms in the lagrangian of eq.(3) become superficially divergent, exception made of:

$$a_{16}\ \delta^n_3\ \varepsilon^{ijh}\ \varphi\ A_i\ A_{j;h}$$

$$a_{23}\ \delta^n_4\ \varepsilon^{ijhk}\ A_{i;j}\ A_{h;k}$$

which we have already disregarded.

So we conclude that this general lagrangian - which allows all possible kind of interaction terms among the three fields, without dimensional constant - is not renormalizable.

Still it remains another possibility to obtain meaningful results when evaluating the diagrams: it may be that infinities coming from one contribution just cancel with the divergence arising from other ones as it actually is in supergravity theories.

We cannot answer the question about finiteness of the theory that lagrangian of eq.(3) describes, without evaluating its counter-terms. For the moment we can observe, most from the remarks Deser and van Nieuwenhuizen have made [18], that finiteness should not be expected. Using background field method and dimensional regularization, they studied the form of the counterterms for General Relativity free of sources and for several of its couplings: to a massless scalar field [19], to the electromagnetic field [18], and so on. All of them, described by subsets of the terms appearing in our eq.(3), showed to be non-renormalizable nor finite. They found for the scalar field coupled minimally to the gravitational field that the one loop counterterm was

$$\Delta \mathbf{L} = \frac{1}{\varepsilon}\,\frac{203}{80}\,\sqrt{-g}\ R$$

For Maxwell-Einstein, also at the one loop level, they found

$$\Delta \mathbf{L}^{ME} = \frac{1}{\varepsilon}\,\frac{137}{60}\,\sqrt{-g}\ R^{ij}\ R_{ij}$$

They also evaluated the $\Delta \mathbf{L}^{\gamma}$ counterterm for the photon loop in the background metric , obtaining

$$\int d^4x\ \Delta \mathbf{L}^{\gamma} = \frac{1}{\varepsilon}\,\frac{1}{10}\ I_2$$

The same for the other contributions $\sim R^2$ for the graviton loop $\Delta \mathbf{L}^h$

$$\int d^4x\ \Delta \mathbf{L}^h = \frac{1}{\varepsilon}\left\{\frac{7}{20}\ I_2 + \frac{5}{14}\ I_o\right\}$$

and for the scalar field loop $\Delta \mathbf{L}^{\varphi}$

$$\int d^4x \, \Delta L^{\varphi} = \frac{1}{\varepsilon} \{ \frac{1}{120} I_2 + \frac{1}{144} I_o \}$$

with

$$I_o = \int d^4x \sqrt{-g} \, R^2$$

$$I_2 = \int d^4x \sqrt{-g} \, (R_{ij} R^{ij} - \frac{1}{3} R^2)$$

and $\frac{1}{\varepsilon} = \frac{1}{8\pi^2} \frac{1}{n-4}$.

As all counterterms are positive, there is not cancellation in the studied cases nor the theories turn out to be finite. So, our conclusion is that it does not seem probable that cancellations occur, using only bosonic fields, and we do not believe this may be improved allowing all class of interactions. Anyhow, we will give a conclusive answer to this problem in a forthcoming paper.

CONCLUSIONS

We have constructed, by means of tensorial concomitants, a general lagrangian density that contains several classical theories, only impossing to it as restrictions not to have dimensional constants and we have shown it is not renormalizable.

In supergravity theories, the presence of spin 3/2 fields - plus supersymmetry - seems to be the cause of cancellations that yield finiteness. Supersymmetries let fermions and bosons "rotate" into each other and impose drastic limitations on the interaction terms that might be present in the lagrangian density and on the form of the counterterms. All of this suggests us the use of spinorial concomitants in the search of a very general lagrangian containing also spin 1/2 and 3/2 fields, to study later which invariances one has to ask to obtain acceptable quantum theories.

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